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On Generalized Eigenvalues of MAX Matrices to MIN Matrices and LCM Matrices to GCD Matrices

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Abstract

We determine, for every $n \geq 1$, the generalized eigenvalues of an $n \times n$ MAX matrix to the corresponding MIN matrix. We also show that a similar result holds for the generalized eigenvalues of an $n \times n$ LCM matrix to the corresponding GCD matrix when $n \leq 4$, but breaks down for $n > 4$. In addition, we prove Cauchy's interlacing theorem for generalized eigenvalues, and we conjecture an unexpected connection between the OEIS sequence A004754 and the appearance of -1 as a generalized eigenvalue in the LCM–GCD setting.

1 Introduction

Let $\mathbf{A}$ and $\mathbf{B}$ be complex Hermitian $n \times n$ matrices, and suppose that $\mathbf{B}$ is positive definite; that is, the conjugate transpose $\mathbf{A}^*$ and $\mathbf{A}$ are equal, and $\mathbf{x}^*\mathbf{B}\mathbf{x} > 0$ whenever $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$.

The generalized eigenvalue equation of $\mathbf{A}$ to $\mathbf{B}$ is

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x}, \quad \mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n. \tag{1}$$

Then λ is called a generalized eigenvalue ("g-eigenvalue" for short) of $\mathbf{A}$ to $\mathbf{B}$, and $\mathbf{x}$ is a corresponding generalized eigenvector ("g-eigenvector"). For more information, see, for example, Ghojogh, Karray, and Crowley [5]. They consider real symmetric matrices, yet all results extend naturally to complex Hermitian matrices.

It is actually enough that $\mathbf{B}$ is invertible in (1), and $\mathbf{A}$ can be arbitrary. However, the above assumptions are usually stated. Then all g-eigenvalues are real, and g-eigenvectors corresponding to distinct g-eigenvalues are orthogonal with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^*\mathbf{B}\mathbf{x}$.

The standard eigenvalues ("s-eigenvalues" for short) are widely studied. The g-eigenvalue equation (1) reduces to the s-eigenvalue equation

$$\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

However, this "quick and dirty solution" [5] does not have significant use. So, g-eigenvalues must be considered in a different way. This area has not been studied much in the literature.

Let

$$S = \{s_1, \ldots, s_n\}, \quad s_1 < \cdots < s_n,$$

be a set of positive real numbers. The $n \times n$ MAX matrix $\mathbf{M}_S$ and MIN matrix $\mathbf{N}_S$ on S are defined by

$$\mathbf{M}_S = (m_{ij}^S),\ m_{ij}^S = \max(s_i, s_j), \quad \mathbf{N}_S = (n_{ij}^S),\ n_{ij}^S = \min(s_i, s_j).$$

Then $\mathbf{N}_S$ is positive definite [10, Theorem 8.1]. Also, let

$$T = \{t_1, \ldots, t_n\}, \quad t_1 < \cdots < t_n,$$

be a set of positive integers. The $n \times n$ LCM matrix $\mathbf{L}_T$ and GCD matrix $\mathbf{G}_T$ on T are defined by

$$\mathbf{L}_T = (l_{ij}^T),\ l_{ij}^T = \operatorname{lcm}(t_i, t_j), \quad \mathbf{G}_T = (g_{ij}^T),\ g_{ij}^T = \gcd(t_i, t_j).$$

Also, $\mathbf{G}_T$ is positive definite [3, Theorem 2].

We study g-eigenvalues of $\mathbf{M}_S$ to $\mathbf{N}_S$ in Section 2, and those of $\mathbf{L}_T$ to $\mathbf{G}_T$ in Sections 3 and 4. Finally, we complete our paper with discussion in Section 5.

All g-eigenvalues of $\mathbf{A}$ to $\mathbf{A}$ are trivially equal to one. We can therefore expect that, also in some nontrivial cases, the g-eigenvalues of $\mathbf{A}$ to $\mathbf{B}$ may be more accessible than the s-eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. We will see this in the case $\mathbf{A} = \mathbf{M}_S$, $\mathbf{B} = \mathbf{N}_S$, and also in the case $\mathbf{A} = \mathbf{L}_T$, $\mathbf{B} = \mathbf{G}_T$, where $T = \{1, \ldots, n\}$, $n \leq 4$. Recently, these matrices have been studied extensively (e.g., [1, 4, 6, 9, 10, 13]). These works discuss not only new results in this field but also applications to various other areas of mathematics—and, for instance, to computing [8], statistics [10], and signal processing [12].

2 MAX–MIN setting

We want to evaluate the g-eigenvalues of $\mathbf{M}_S$ to $\mathbf{N}_S$, i.e., the solutions λ to the equation $\det(\mathbf{M}_S - \lambda\mathbf{N}_S) = 0$.

We begin with $n = 2$. Let $S = \{a, b\}$, $0 < a < b$. Then

$$\det(\mathbf{M}_S - \lambda\mathbf{N}_S) = \begin{vmatrix} a - \lambda a & b - \lambda a \\ b - \lambda a & b - \lambda b \end{vmatrix} = a(b-a)\lambda^2 - b(b-a) = 0$$

if and only if

$$\lambda = \pm\sqrt{\frac{b}{a}}.$$

Our aim is to prove Theorem 1 below. However, because the general proof is not easily readable, we show the details only in the case $n = 4$. A careful reader will notice that we can proceed similarly for every integer $n > 2$.

Theorem 1. *The g-eigenvalues of $\mathbf{M}_S$ to $\mathbf{N}_S$, $n > 2$, are*

$$\lambda_1 = \sqrt{\frac{s_n}{s_1}},\ \lambda_2 = \cdots = \lambda_{n-1} = -1,\ \lambda_n = -\sqrt{\frac{s_n}{s_1}}. \tag{2}$$

Proof. Let $S = \{a, b, c, d\}$, $0 < a < b < c < d$. Then

$$\mathbf{M}_S = \begin{pmatrix} a & b & c & d \\ b & b & c & d \\ c & c & c & d \\ d & d & d & d \end{pmatrix}, \quad \mathbf{N}_S = \begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix}.$$

The matrix

$$\mathbf{M}_S + \mathbf{N}_S = \begin{pmatrix} 2a & a+b & a+c & a+d \\ a+b & 2b & b+c & b+d \\ a+c & b+c & 2c & c+d \\ a+d & b+d & c+d & 2d \end{pmatrix}$$

has rank 2 and nullity 2. Consequently, -1 is a g-eigenvalue of $\mathbf{M}_S$ to $\mathbf{N}_S$ with multiplicity 2. We show that the remaining g-eigenvalues are

$$\lambda = \pm\sqrt{\frac{d}{a}}.$$

Regardless of the sign of λ, we have (note that $d = \lambda^2 a$)

$$\det(\mathbf{M}_S + \lambda\mathbf{N}_S) = \begin{vmatrix} a+\lambda a & b+\lambda a & c+\lambda a & d+\lambda a \\ b+\lambda a & b+\lambda b & c+\lambda b & d+\lambda b \\ c+\lambda a & c+\lambda b & c+\lambda c & d+\lambda c \\ d+\lambda a & d+\lambda b & d+\lambda c & d+\lambda d \end{vmatrix}$$

$$= \begin{vmatrix} a+\lambda a & b+\lambda a & c+\lambda a & d+\lambda a \\ b-a & \lambda(b-a) & \lambda(b-a) & \lambda(b-a) \\ c-b & c-b & \lambda(c-b) & \lambda(c-b) \\ d-c & d-c & d-c & \lambda(d-c) \end{vmatrix} = \begin{vmatrix} \lambda a & b & c & \lambda^2 a \\ b-a & \lambda(b-a) & \lambda(b-a) & \lambda(b-a) \\ c-b & c-b & \lambda(c-b) & \lambda(c-b) \\ d-c & d-c & d-c & \lambda(d-c) \end{vmatrix}$$

$$+ \begin{vmatrix} a & \lambda a & \lambda a & \lambda a \\ b-a & \lambda(b-a) & \lambda(b-a) & \lambda(b-a) \\ c-b & c-b & \lambda(c-b) & \lambda(c-b) \\ d-c & d-c & d-c & \lambda(d-c) \end{vmatrix} =: D_1 + D_2.$$

Since $D_1 = D_2 = 0$, the claim follows. □

Remark 2. Theorem 1 holds also for $n = 2$. Then the chain $\lambda_2 = \cdots = \lambda_{n-1}$ is "empty".

Remark 3. If the ordering of $s_1, \dots, s_n$ is arbitrary, then (2) reads

$$\lambda_1 = \max_{i,j} \sqrt{\frac{s_i}{s_j}},\ \lambda_2 = \cdots = \lambda_{n-1} = -1,\ \lambda_n = -\max_{i,j} \sqrt{\frac{s_i}{s_j}}. \tag{3}$$

See also [10, Remark 2.1].

Remark 4. Theorem 1 applies also to the g-eigenvalues of $\mathbf{M}_S$ to $\mathbf{N}_{S'}$, where

$$S' = \{s'_1, \dots, s'_n\}, \quad s'_1 - s_1 = \cdots = s'_n - s_n.$$

3 LCM–GCD setting on $T = \{1, 2, \dots, n\}$

3.1 The case $n \le 4$

Let $T = \{1, 2, \dots, n\}$, let $\lambda_{n1} \ge \cdots \ge \lambda_{nn}$ be the g-eigenvalues of $\mathbf{L}_T$ to $\mathbf{G}_T$, and let $p_n(\lambda) = \det(\mathbf{L}_T - \lambda \mathbf{G}_T)$ be the g-characteristic polynomial. Then

$$p_1(\lambda) = 1 - \lambda, \quad \lambda_{11} = 1,$$

$$p_2(\lambda) = \begin{vmatrix} 1-\lambda & 2-\lambda \\ 2-\lambda & 2-2\lambda \end{vmatrix} = \lambda^2 - 2, \quad \lambda_{21} = \sqrt{2},\ \lambda_{22} = -\sqrt{2},$$

$$p_3(\lambda) = \begin{vmatrix} 1-\lambda & 2-\lambda & 3-\lambda \\ 2-\lambda & 2-2\lambda & 6-\lambda \\ 3-\lambda & 6-\lambda & 3-3\lambda \end{vmatrix} = -2(\lambda+1)(\lambda^2-6), \quad \lambda_{31} = \sqrt{6},\ \lambda_{32} = -1,\ \lambda_{33} = -\sqrt{6},$$

$$p_4(\lambda) = \begin{vmatrix} 1-\lambda & 2-\lambda & 3-\lambda & 4-\lambda \\ 2-\lambda & 2-2\lambda & 6-\lambda & 4-2\lambda \\ 3-\lambda & 6-\lambda & 3-3\lambda & 12-\lambda \\ 4-\lambda & 4-2\lambda & 12-\lambda & 4-4\lambda \end{vmatrix} = 4(\lambda+1)^2(\lambda^2-12),$$

$$\lambda_{41} = \sqrt{12},\ \lambda_{42} = \lambda_{43} = -1,\ \lambda_{44} = -\sqrt{12}.$$

3.2 The case $n > 4$

The g-eigenvalues in Section 3.1 suggest that there may also be values of $n > 4$ such that

$$\lambda_{n1} = \sqrt{m},\ \lambda_{n2} = \cdots = \lambda_{n,n-1} = -1,\ \lambda_{nn} = -\sqrt{m} \tag{4}$$

for some integer m. We examine this hypothesis and begin with the case $n = 5$. We have

$$\begin{aligned} p_5(\lambda) &= -16\lambda^5 - 48\lambda^4 + 528\lambda^3 + 2480\lambda^2 + 2880\lambda + 960 \\ &= -16(\lambda + 1)(\lambda^4 + 2\lambda^3 - 35\lambda^2 - 120\lambda - 60) \\ &=: -16(\lambda + 1)q(\lambda). \end{aligned}$$

Because $q(-1) = 24 \neq 0$, the multiplicity of $\lambda = -1$ is only one, falsifying (4). The g-eigenvalues are

$$\lambda_{51} = 6.4798,\ \lambda_{52} = -0.6118,\ \lambda_{53} = -1,\ \lambda_{54} = -3.3489,\ \lambda_{55} = -4.5191.$$

(These are approximations to four decimal places, similarly throughout the paper.) Interestingly, $\sqrt{42} = 6.4807$ is near to λ_{51}. If their difference were due to rounding errors, then the first equation in (4) would hold for $n = 5$, too. But $p_5(\sqrt{42}) = -3168\sqrt{42} + 20448$, showing that the difference is actual.

Moreover,

$$\lambda_{61} = 6.8501,\ \lambda_{62} = 2.5592,\ \lambda_{63} = -0.7419,\ \lambda_{64} = -1.3749,\ \lambda_{65} = -3.4396,\ \lambda_{66} = -5.8528.$$

Thus -1 is not a g-eigenvalue when $n = 6$.

These two cases already suffice to make it fairly clear that $\lambda_{n2} = \cdots = \lambda_{n,n-1} = -1$ does not hold for $n > 4$. However, we choose to verify this claim thoroughly, as it involves first proving and then applying Cauchy's interlacing theorem [7, Theorem 4.3.17] for g-eigenvalues—a result that is arguably of general interest. To that end, recall that there are two positive g-eigenvalues for $n = 6$. By Theorem 6 below, there are at least two positive g-eigenvalues for $n = 7$. Continuing in this way confirms the claim.

We conclude this section by exploring in which dimensions -1 occurs as a g-eigenvalue. Computer experiments covering the range $1 \leq n \leq 1000$ (with code provided in the Appendix) show that -1 is a g-eigenvalue if and only if

$$n = \overbrace{4, 5}^{2}, \overbrace{8, 9, 10, 11}^{2^2}, \overbrace{16, \ldots, 23}^{2^3}, \overbrace{32, \ldots, 47}^{2^4}, \overbrace{64, \ldots, 95}^{2^5}, \overbrace{128, \ldots, 191}^{2^6}, \overbrace{256, \ldots, 383}^{2^7}, 512, \ldots,$$

where the overbrace indicates the number of terms. This sequence is the same as the OEIS [11] sequence A004754 without the first term. Its description [11] raises an interesting conjecture.

Conjecture 5. Let $T = \{1, \ldots, n\}$, $n > 3$. Then -1 is a g-eigenvalue of $\mathbf{L}_T$ to $\mathbf{G}_T$ if and only if the binary representation of n begins with 10.

For example, $4 = (100)_2$, $5 = (101)_2$, $8 = (1000)_2$, $19 = (10011)_2$.

This OEIS sequence (a_n) satisfies [11]

$$a_{2^m+k} = 2^{m+1} + k, \quad m \geq 0,\ 0 \leq k < 2^m. \tag{5}$$

If, for example, $m = k = 3$, then the left-hand side equals $a_{8+3} = a_{11} = 19$, and the right-hand side equals $16 + 3 = 19$. An induction proof of Conjecture 5 can perhaps be found by using (5).

3.3 Cauchy's interlacing theorem for g-eigenvalues

Theorem 6. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be as in* (1), $n > 1$, *with first leading principal submatrices* $\mathbf{A}'$ *and respectively* $\mathbf{B}'$ (*obtained by removing the* n*th row and column*). *Let*

$$\lambda_1 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \cdots \geq \lambda'_{n-1}$$

be the g-eigenvalues of $\mathbf{A}$ *to* $\mathbf{B}$ *and, respectively, of* $\mathbf{A}'$ *to* $\mathbf{B}'$. *Then*

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n.$$

Proof. Let $\preceq$ denote the subspace inclusion. Because the Courant-Fischer theorem [7, Theorem 4.2.6] extends to g-eigenvalues [2, Theorem 3] (note the wrong ordering of max and min in its formulation), we have

$$\lambda_k = \max_{\substack{U \preceq \mathbb{C}^n \\ \dim U = k}} \min_{\mathbf{0} \neq \mathbf{x} \in U} \frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{B}\mathbf{x}}, \quad k = 1, \ldots, n, \tag{6}$$

and

$$\lambda'_k = \max_{\substack{V \preceq \mathbb{C}^{n-1} \\ \dim V = k}} \min_{\mathbf{0} \neq \mathbf{y} \in V} \frac{\mathbf{y}^*\mathbf{A}'\mathbf{y}}{\mathbf{y}^*\mathbf{B}'\mathbf{y}}, \quad k = 1, \ldots, n-1. \tag{7}$$

Let

$$\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}' \\ x_n \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}' & \mathbf{u} \\ \mathbf{u}^* & a_{nn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}' & \mathbf{v} \\ \mathbf{v}^* & b_{nn} \end{pmatrix}.$$

If $x_n = 0$, then

$$\frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{B}\mathbf{x}} = \frac{(\mathbf{x}')^*\mathbf{A}'\mathbf{x}'}{(\mathbf{x}')^*\mathbf{B}'\mathbf{x}'},$$

so, for $k = 1, \ldots, n-1$,

$$\lambda'_k = \max_{\substack{V \preceq \mathbb{C}^{n-1} \\ \dim V = k}} \min_{\substack{\mathbf{0} \neq \mathbf{x}' \in V \\ x_n = 0}} \frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{B}\mathbf{x}} =: M.$$

Since

$$\left\{ \min_{\substack{\mathbf{0} \neq \mathbf{x}' \in V \\ x_n = 0}} \frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{B}\mathbf{x}} : V \preceq \mathbb{C}^{n-1}, \dim V = k \right\} \subseteq \left\{ \min_{\mathbf{0} \neq \mathbf{x} \in U} \frac{\mathbf{x}^*\mathbf{A}\mathbf{x}}{\mathbf{x}^*\mathbf{B}\mathbf{x}} : U \preceq \mathbb{C}^n, \dim U = k \right\},$$

it follows that

$$M \leq \max_{\substack{U \preceq \mathbb{C}^n \\ \dim U = k}} \min_{\mathbf{0} \neq \mathbf{x} \in U} \frac{\mathbf{x}^* \mathbf{A} \mathbf{x}}{\mathbf{x}^* \mathbf{B} \mathbf{x}}. \tag{8}$$

Now, by (7), (8), and (6),

$$\lambda_k' \leq \lambda_k.$$

To find a reverse inequality, we change the ordering of max and min in the generalized Courant-Fischer theorem:

$$\lambda_{k+1} = \min_{\substack{U \preceq \mathbb{C}^n \\ \dim U = n-k}} \max_{\mathbf{0} \neq \mathbf{x} \in U} \frac{\mathbf{x}^* \mathbf{A} \mathbf{x}}{\mathbf{x}^* \mathbf{B} \mathbf{x}}, \quad k = 0, \ldots, n-1,$$

and

$$\lambda_k' = \min_{\substack{V \preceq \mathbb{C}^{n-1} \\ \dim V = n-k}} \max_{\mathbf{0} \neq \mathbf{y} \in V} \frac{\mathbf{y}^* \mathbf{A}' \mathbf{y}}{\mathbf{y}^* \mathbf{B}' \mathbf{y}}, \quad k = 1, \ldots, n-1.$$

By a simple modification of the previous argument, we obtain

$$\lambda_k' \geq \lambda_{k+1},$$

completing the proof. □

4 LCM–GCD setting on some $T \neq \{1, 2, \ldots, n\}$

4.1 The cases $n = 2, 3$

First, let $T = \{u, v\}$, $0 < u < v$. Studying $\{u/d, v/d\}$ if $d = \gcd(u, v) > 1$, we can assume that $\gcd(u, v) = 1$. Then

$$\det(\mathbf{L}_T - \lambda \mathbf{G}_T) = \begin{vmatrix} u - \lambda u & uv - \lambda \\ uv - \lambda & v - \lambda v \end{vmatrix} = (uv - 1)(\lambda^2 - uv), \quad \lambda_1 = \sqrt{uv},\ \lambda_2 = -\sqrt{uv}.$$

Next, let $T = \{1, u, v\}$, where $1 < u < v$ and $\gcd(u, v) = 1$. Then

$$\det(\mathbf{L}_T - \lambda \mathbf{G}_T) = \begin{vmatrix} 1 - \lambda & u - \lambda & v - \lambda \\ u - \lambda & u - \lambda u & uv - \lambda \\ v - \lambda & uv - \lambda & v - \lambda v \end{vmatrix}$$
$$= (u + v - uv - 1)\lambda^3 + (u + v - uv - 1)\lambda^2 + (u^2v^2 + uv - u^2v - uv^2)\lambda + u^2v^2 - u^2v$$
$$- uv^2 + uv = (u-1)(v-1)(\lambda + 1)(\lambda^2 - uv), \quad \lambda_1 = \sqrt{uv},\ \lambda_2 = -1,\ \lambda_3 = -\sqrt{uv}.$$

More generally, let $T = \{u, v, w\}$, where $1 < u < v < w$ and $\gcd(u, v) = \gcd(u, w) = \gcd(v, w) = 1$. It seems that we do not get pretty results. If, for example, $T = \{2, 3, 5\}$, then

$$\det(\mathbf{L}_T - \lambda \mathbf{G}_T) = \begin{vmatrix} 2-2\lambda & 6-\lambda & 10-\lambda \\ 6-\lambda & 3-3\lambda & 15-\lambda \\ 10-\lambda & 15-\lambda & 5-5\lambda \end{vmatrix} = -22\lambda^3 - 38\lambda^2 + 420\lambda + 900,$$

$$\lambda_1 = 4.5128,\ \lambda_2 = -2.3027,\ \lambda_3 = -3.9371.$$

4.2 The case $T = \{1, p, \ldots, p^{n-1}\}$, $p \in \mathbb{P}$

In this case,

$$\mathbf{L}_T = \begin{pmatrix} 1 & p & p^2 & \cdots & p^{n-1} \\ p & p & p^2 & \cdots & p^{n-1} \\ p^2 & p^2 & p^2 & \cdots & p^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p^{n-1} & p^{n-1} & p^{n-1} & \cdots & p^{n-1} \end{pmatrix} = \mathbf{M}_T, \quad \mathbf{G}_T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & p & p & \cdots & p \\ 1 & p & p^2 & \cdots & p^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & p & p^2 & \cdots & p^{n-1} \end{pmatrix} = \mathbf{N}_T.$$

By Theorem 1, the g-eigenvalues of $\mathbf{L}_T$ to $\mathbf{G}_T$ are

$$\lambda_1 = p^{\frac{n-1}{2}},\ \lambda_2 = \cdots = \lambda_{n-1} = -1,\ \lambda_n = -p^{\frac{n-1}{2}}.$$

4.3 Reordering does not matter

We noted in Remark 3 that reordering S in the MAX–MIN setting only changes (2) to (3), so all g-eigenvalues remain unchanged. We now show that reordering T in the LCM–GCD setting also preserves the g-eigenvalues. More generally, let $\mathbf{X} = (x_{ij})$ be a complex square matrix of order n. Given a permutation σ of $(1, \ldots, n)$, define

$$\mathbf{X}_\sigma = (x_{ij}^\sigma), \quad x_{ij}^\sigma = x_{\sigma(i),\sigma(j)},$$

and let $\mathbf{P}_\sigma$ denote the permutation matrix corresponding to σ. Since

$$\mathbf{X}_\sigma = \mathbf{P}_\sigma \mathbf{X} \mathbf{P}_\sigma \quad \text{and} \quad \det \mathbf{P}_\sigma = \pm 1,$$

we have

$$\det \mathbf{X}_\sigma = \det \mathbf{X},$$

implying the claim.

5 Discussion

Above, we first examined the g-eigenvalues of MAX matrices to MIN matrices. The results reveal distinct structural patterns: the g-eigenvalues are completely characterized and consist of one positive value, several values -1 (with multiplicity zero when $n = 2$), and one negative value. This regularity highlights an underlying symmetry and robustness in the generalized eigenstructure of these matrices.

We then turned to the g-eigenvalues of LCM matrices to GCD matrices on $T = \{1, \ldots, n\}$. This case is more intricate. While the above pattern holds for $n \leq 4$, it breaks down for $n > 4$. In the course of verifying this, we proved a generalization of Cauchy's interlacing theorem for eigenvalues—namely, the corresponding theorem for g-eigenvalues.

A surprising observation is the emergence of connection to OEIS sequence A004754 in Conjecture 5. If proven, this would provide a novel bridge between matrix theory and number theory, offering a new insight into exploring spectral properties of matrices through binary representations of integers.

We concluded our study by considering sets $T \neq \{1, \ldots, n\}$ to demonstrate that certain configurations in the LCM–GCD setting exhibit a MAX–MIN structure. We also showed that, in general, reordering S and T does not affect the g-eigenvalues, thereby reinforcing the robustness of these matrices under permutations.

From a computational perspective, determining generalized eigenvalues poses significant challenges, as it typically requires finding the roots of high-degree characteristic polynomials. As n increases, these polynomials become difficult to construct and numerically unstable to solve. However, in constructing the sequence in Conjecture 5, these difficulties can largely be avoided: it is not necessary to form or factorize $p_n(\lambda) = \det(\mathbf{L}_T - \lambda \mathbf{G}_T)$ explicitly. Instead, one can directly compute $p_n(-1) = \det(\mathbf{L}_T + \mathbf{G}_T)$. This approach is computationally lighter, numerically more stable, and sufficient to verify whether -1 is a g-eigenvalue.

Appendix

```python
import numpy as np
import math
from scipy.linalg import eig

def gcd_matrix(n):
    """Construct an n x n matrix with entries gcd(i, j)."""
    M = np.zeros((n, n), dtype=int)
    for i in range(1, n+1):
        for j in range(1, n+1):
            M[i-1, j-1] = math.gcd(i, j)
    return M

def lcm_matrix(n):
    """Construct an n x n matrix with entries lcm(i, j)."""
```

```
    M = np.zeros((n, n), dtype=int)
    for i in range(1, n+1):
        for j in range(1, n+1):
            M[i-1, j-1] = math.lcm(i, j)
    return M

def find_n_with_minus_one(tol=1e-5, max_n=1000):
    """
    For n = 1 to max_n, computes the generalized eigenvalues for Ax =
       lambda Bx, where A is the LCM matrix and B is the GCD matrix.
       Returns a list of n for which -1 appears as a generalized
       eigenvalue (within a tolerance tol).
    """
    n_list = []
    for n in range(1, max_n+1):
        A = lcm_matrix(n).astype(float)
        B = gcd_matrix(n).astype(float)
        # Compute g-eigenvalues of A to B:
        eigenvalues, _ = eig(A, B)
        # Convert real g-eigenvalues to real values:
        eigenvalues = np.real_if_close(eigenvalues, tol=tol)
        # Check if some g-eigenvalue is equal to -1:
        if any(np.isclose(ev, -1, atol=tol) for ev in eigenvalues):
            n_list.append(n)
    return n_list

if __name__ == '__main__':
    result = find_n_with_minus_one(tol=1e-5, max_n=1000)
    print("Dimensions n for which -1 appears as a g-eigenvalue:")
    print(result)
```

Listing 1: Python code to examine Conjecture 5.

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