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Integer Values of Generating Functions for a Type of Second-Order Linear Recurrence Sequence

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Abstract

Define an integer sequence $(A_n)_{n\geq 0}$ by setting $A_0 = a, A_1 = b$, and $A_{n+1} = pA_n + qA_{n-1}$ for all n. We consider the case $q = 1$ to explore the problem of finding all rational numbers x such that the generating function of (A_n) yields an integer when evaluated at x. We point out that we can divide the set of all x-values into families and find some families that always exist. Then we provide an algorithm to find all the families through a finite computation. Finally, we apply the algorithm to the special cases that $(a, b) = (0, 1)$ and $(a, b) = (1, 1)$.

1 Introduction

Consider a second-order linear recurrence sequence $(A_n)_{n\geq 0}$ defined by

$$\begin{cases} A_0 = a, A_1 = b; \\ A_{n+1} = pA_n + qA_{n-1}, \end{cases}$$

where a, b, p, and q are integers and $(a, b) \neq (0, 0), p, q \neq 0$.

The generating function of $(A_n)_{n\geq 0}$ is given by

$$A(x) = \sum_{n=0}^{\infty} A_n x^n.$$

We see that $A(x)$ is a formal power series in the formal power series domain over the rational field, commonly denoted by $\mathbb{Q}[[x]]$. We can calculate that

$$A(x) = \frac{a(1-px)+bx}{1-px-qx^2}.$$

Thus we can consider $A(x)$ as a rational function of x.

If $(a,b,p,q) = (0,1,1,1)$ or $(a,b,p,q) = (2,1,1,1)$, then $(A_n)_{n\geq 0}$ becomes the well-known Fibonacci sequence $(F_n)_{n\geq 0}$ (A000045) or the Lucas sequence $(L_n)_{n\geq 0}$ (A000032). Then the generating functions of $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are, respectively,

$$F(x) = \frac{x}{1-x-x^2} \text{ and } L(x) = \frac{2-x}{1-x-x^2}.$$

Now, for each second-order linear recurrence sequence $(A_n)_{n\geq 0}$, we set

$$\mu(A) = \{x \in \mathbb{Q} \mid A(x) \in \mathbb{Z}\}.$$

Finding all elements of $\mu(A)$ is an interesting research problem. When $(A_n)_{n\geq 0}$ becomes the Fibonacci sequence or the Lucas sequence, in 2015, Hong [2] showed that

$$\frac{F_n}{F_{n+1}} \in \mu(F) \cap \mu(L) \text{ and } \frac{L_n}{L_{n+1}} \in \mu(L)$$

for all non-negative integers n. He also conjectured whether his results were the only ones that led $F(x)$ and $L(x)$ to integer values. Pongsriiam [6] answered this question in 2017 by proving that

$$\mu(F) = \left\{ \frac{F_n}{F_{n+1}} \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\} \cup \left\{ -\frac{F_{n+1}}{F_n} \;\middle|\; n \in \mathbb{N} \right\}$$

and

$$\begin{aligned} \mu(L) = &\left\{ \frac{F_n}{F_{n+1}} \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\} \cup \left\{ -\frac{F_{n+1}}{F_n} \;\middle|\; n \in \mathbb{N} \right\} \\ &\cup \left\{ \frac{L_n}{L_{n+1}} \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\} \cup \left\{ -\frac{L_{n+1}}{L_n} \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\}. \end{aligned}$$

At the same time, Bulawa and Lee [1] discovered a result for the sequence $(R_n)_{n\geq 0}$ defined by

$$\begin{cases} R_0 = 0, R_1 = 1; \\ R_{n+1} = pR_n + qR_{n-1}, \end{cases}$$

where p, q are positive integers and p is divisible by q. They found all the values of $\mu(R)$ that are in the interval of convergence, where $R(x)$ is the generating function of $(R_n)_{n\geq 0}$. They also extended the results for the Lucas, Pell, and Pell-Lucas sequences.

Another generalization was published by Knapp [3] in 2024 when he constructed an algorithm to find all elements of $\mu(G)$, where $G(x)$ is the generating function of the Gibonacci sequences $(G_n)_{n\geq 0} : G_0 = a, G_1 = b, G_{n+1} = G_n + G_{n-1}$. Afterwards, Knapp et al. [4] also studied the sequence $(a_n)_{n\geq 0}$ of the form:

$$\begin{cases} a_0 = 0, a_1 = 1; \\ a_{n+1} = pa_n + qa_{n-1}, \quad \text{if } n \geq 1, \end{cases}$$

and indicated the cases of (p, q) in which they could provide a constructive method to find all the elements of $\mu(a)$.

In this article, we study the second-order linear recurrence sequences $(A_n)_{n\geq 0}$ defined by

$$\begin{cases} A_0 = a, A_1 = b; \\ A_{n+1} = pA_n + A_{n-1}, \end{cases}$$

where a, b, p are integers and $(a, b) \neq (0, 0), p \neq 0$. It turns out that the values x that make the generating function of $(A_n)_{n\geq 0}$ an integer come in families. Moreover, the elements of each family are ratios of consecutive terms of a sequence given in the following theorem:

Theorem 1. *Let $A(x)$ be the generating function of $(A_n)_{n\geq 0}$, and r, s be relatively prime integers. Assume that $A(r/s)$ is an integer. Define the sequence $(A_n^{(r,s)})_{n\in\mathbb{Z}}$ by setting:*

$$\begin{cases} A_0^{(r,s)} = r, A_1^{(r,s)} = s; \\ A_{n+1}^{(r,s)} = pA_n^{(r,s)} + A_{n-1}^{(r,s)}, \quad \text{if } n > 1; \\ A_n^{(r,s)} = A_{n+2}^{(r,s)} - pA_{n+1}^{(r,s)}, \quad \text{if } n < 0. \end{cases}$$

Then $A\big(A_n^{(r,s)}/A_{n+1}^{(r,s)}\big)$ is an integer whenever $A_{n+1}^{(r,s)} \neq 0$.

From Theorem 1, we present some families of rational values x that make $A(x)$ an integer.

Theorem 2.

(i) *$A\big(A_n^{(0,1)}/A_{n+1}^{(0,1)}\big)$ are integers for all $n \neq -1$, where we define $(A_n^{(0,1)})$ as in Theorem 1.*

(ii) *$A\big(A_n^{(-b,a)}/A_{n+1}^{(-b,a)}\big)$ are integers whenever $A_{n+1}^{(-b,a)} \neq 0$, where $(A_n^{(-b,a)})$ is defined as in Theorem 1.*

The next theorem shows all the values that make $A(x)$ an integer.

Theorem 3. *Set C by*

$$C = \begin{cases} \sqrt{(abp^3 + (a^2 - b^2)p^2)/(p^2 + 4)}, & \text{if } abp^3 + (a^2 - b^2)p^2 \geq 0; \\ \sqrt{-abp + b^2 - a^2}, & \text{if } abp^3 + (a^2 - b^2)p^2 < 0, \end{cases}$$

and find all integers M such that $|M| \leq C$ and the equation

$$M^2 + (p^2 + 2)MN + N^2 = abp^3 + (a^2 - b^2)p^2$$

has an integer solution N. For each value M, let k be a nonzero integer solution of the equation

$$M^2 = (kp - ap + b)^2 - 4k(a - k).$$

For each k, set

$$x = \frac{ap - kp - b \pm \sqrt{(kp - ap + b)^2 - 4k(a - k)}}{2k}.$$

Then $A(x)$ is an integer, and by Theorem 1, x generates a family of elements of $\mu(A)$. All families of x can be found by this way, apart from those given in Theorem 2.

After that, we consider some more typical sequences. The former is the case $(a, b) = (0, 1)$, where we give another proof of a special case of Theorem 1.1 in the paper of Bulawa and Lee [1], as well as Theorem 1.7 in the paper of Knapp, Lemos, and Neumann [4]. The latter is the case $(a, b) = (1, 1)$, where we provide our corollary.

2 Proofs of Theorems 1 to Theorem 3

In this section, we consider the sequence $(A_n)_{n\geq 0}$ defined by

$$\begin{cases} A_0 = a, A_1 = b; \\ A_{n+1} = pA_n + A_{n-1}. \end{cases}$$

The generating function of $(A_n)_{n\geq 0}$ is

$$A(x) = \frac{a(1 - px) + bx}{1 - px - x^2}.$$

Lemma 4. *Let r, s be integers such that $\gcd(r, s) = 1$. Then the value $A(r/s)$ is an integer if and only if*

$$b \equiv -ars^{-1} \pmod{|s^2 - prs - r^2|}.$$

Proof. We have

$$A(r/s) = \frac{a(1 - p\frac{r}{s}) + b\frac{r}{s}}{1 - p\frac{r}{s} - \frac{r^2}{s^2}} = \frac{as^2 - aprs + brs}{s^2 - prs - r^2}.$$

Then $A(r/s)$ is an integer if and only if

$$as^2 - aprs + brs - a(s^2 - prs - r^2) \equiv 0 \pmod{|s^2 - prs - r^2|}.$$

Because $\gcd(r, s^2 - prs - r^2) = \gcd(s, s^2 - prs - r^2) = 1$, this is equivalent to

$$bs + ar \equiv 0 \pmod{|s^2 - prs - r^2|}$$

or

$$b \equiv -ars^{-1} \pmod{|s^2 - prs - r^2|}.$$

□

Remark 5. From Lemma 4, if $A(r/s) \in \mathbb{Z}$, then we have

$$(s^2 - prs - r^2) \mid (as^2 - aprs + brs).$$

Because $\gcd(s, s^2 - prs - r^2) = 1$, this is equivalent to

$$(s^2 - prs - r^2) \mid (as - apr + br).$$

Lemma 6. *Let r, s be integers such that $r, s \neq 0, r + ps \neq 0, \gcd(r, s) = 1$ and $A(r/s)$ is an integer. Then $A(s/(r + ps))$ and $A((s - pr)/r)$ are integers.*

Proof. We have

$$A(s/(r + ps)) = \frac{(b - ap)s(r + ps) + a(r + ps)^2}{r^2 + prs - s^2} = \frac{(r + ps)(bs + ar)}{r^2 + prs - s^2}$$

and

$$A((s - pr)/r) = \frac{(b - ap)(s - pr)r + ar^2}{r^2 + prs - s^2} = \frac{r(bs + ar - aps - bpr + ap^2 r)}{r^2 + prs - s^2}.$$

Applying Lemma 4, we conclude that both $A(s/(r+ps))$ and $A((s-pr)/r)$ are integers. □

Proof of Theorem 1. We have that $A\big(A_0^{(r,s)}/A_1^{(r,s)}\big) = A(r/s)$ is an integer. By Lemma 6 and the induction, we see that $A\big(A_n^{(r,s)}/A_{n+1}^{(r,s)}\big)$ is an integer whenever $A_{n+1}^{(r,s)} \neq 0$. Note that if there exists $n_0 > 0$ such that $A_{n_0}^{(r,s)} = 0$, we can still continue the induction from n_0 because $A(0) = a$ is an integer. Also, if there exists $n_0 < 0$ such that $A_{n_0}^{(r,s)} = 0$, then $A\big(A_{n_0-2}^{(r,s)}/A_{n_0-1}^{(r,s)}\big) = A(-p)$ is an integer. □

Proof of Theorem 2.

(i) We have
$$A\big(A_0^{(0,1)}/A_1^{(0,1)}\big) = A(0) = a \in \mathbb{Z}.$$
Then by Theorem 1, $A\big(A_n^{(0,1)}/A_{n+1}^{(0,1)}\big)$ is an integer for all integers $n \neq -1$.

(ii) We see that $A\big(A_0^{(-b,a)}/A_1^{(-b,a)}\big) = A(-b/a) = a$ is an integer. So by Theorem 1, we imply that $A\big(A_n^{(-b,a)}/A_{n+1}^{(-b,a)}\big)$ is an integer whenever $A_{n+1}^{(-b,a)} \neq 0$.

□

Remark 7.

(i) By induction, we can prove that
$$A_n = (-1)^{n-1} A_{1-n}^{(-b,a)}$$
for $n \geq 0$. Then
$$\frac{-A_{n+1}}{A_n} = \frac{(-1)^{n+1} A_{-n}^{(-b,a)}}{(-1)^{n-1} A_{-n+1}^{(-b,a)}} = \frac{A_{-n}^{(-b,a)}}{A_{-n+1}^{(-b,a)}}.$$
This implies $A\big(-A_{n+1}/A_n\big)$ are integers whenever $A_n \neq 0$.

(ii) From Theorem 1, if we find out $x_0 = r/s$ such that $A(x)$ is an integer, we can construct a family of elements of $\mu(A)$ including x_0.

The next part of this section presents an algorithm to find all families of values x for which $A(x)$ is an integer.

Following the approach of Knapp [3], we assume that $A(x) = k$ is an integer, which is equivalent to
$$\frac{a(1-px)+bx}{1-px-x^2} = k. \tag{1}$$
If $k = 0$, we can imply that $x = a/(ap-b)$ is in the family generated by the sequence $(A_n^{(-b,a)})_{n\in\mathbb{Z}}$, which was shown in Theorem 2. So we can assume $k \neq 0$. Then the equation (1) is equivalent to
$$kx^2 + (kp - ap + b)x + a - k = 0.$$
Solving for x, we see that
$$x = \frac{ap - b - kp \pm \sqrt{(kp+b-ap)^2 - 4k(a-k)}}{2k}.$$
Define $M = \pm\sqrt{(kp+b-ap)^2 - 4k(a-k)}$ and $x = r/s$, we have
$$k = A(r/s) = \frac{as^2 - aprs + brs}{s^2 - prs - r^2}$$

and

$$M = 2kx - ap + kp + b = \frac{br^2 + bs^2 - apr^2 + 2ars}{s^2 - prs - r^2}.$$

From now on, we write M in the form $M(r, s)$. Note that if there exists an integer n such that $A_n^{(r,s)} = 0$, then the family of values x generated by $(A_n^{(r,s)})$ is identical to the one generated by $(A_n^{(0,1)})$, which is mentioned in Theorem 2. So we can assume that the conditions $r \neq 0, s \neq 0, r + ps \neq 0$ automatically hold.

Lemma 8. *Assume that $r \neq 0, s \neq 0, r + ps \neq 0$. Then*

(i) $M(s - pr, r) + (p^2 + 2)M(r, s) + M(s, r + ps) = 0$.

(ii) $M(r, s)^2 + (p^2 + 2)M(r, s)M(s, r + ps) + M(s, r + ps)^2 = abp^3 + (a^2 - b^2)p^2$.

Proof.

(i) We have

$$\begin{aligned} M(s - pr, r) &= \frac{b(s - pr)^2 + br^2 - ap(s - pr)^2 + 2ar(s - pr)}{r^2 + prs - s^2} \\ &= \frac{br^2 + bs^2 + 2ars - 2bprs + bp^2r^2 - ap^3r^2 - aps^2 - 2apr^2 + 2ap^2rs}{r^2 + prs - s^2} \end{aligned}$$

$$\begin{aligned} M(s, r + ps) &= \frac{bs^2 + b(r + ps)^2 - aps^2 + 2as(r + ps)}{r^2 + prs - s^2} \\ &= \frac{bs^2 + br^2 + 2bprs + bp^2s^2 - aps^2 + 2ars + aps^2}{r^2 + prs - s^2} \end{aligned}$$

$$(p^2 + 2)M(r, s) = (p^2 + 2)\frac{br^2 + bs^2 - apr^2 + 2ars}{s^2 - prs - r^2}$$

By addition, we have

$$M(s - pr, r) + (p^2 + 2)M(r, s) + M(s, r + ps) = 0.$$

(ii) We have

$$\begin{aligned} M(r, s)^2 &= \frac{(br^2 + bs^2 - apr^2 + 2ars)^2}{(s^2 - prs - r^2)^2} \\ &= \frac{b^2r^4 + b^2s^4 + a^2p^2r^4 + 4a^2r^2s^2 + 2b^2r^2s^2 - 2abpr^4 + 4abr^3s}{(s^2 - prs - r^2)^2} \\ &+ \frac{-2abpr^2s^2 + 4abrs^3 - 4a^2pr^3s}{(s^2 - prs - r^2)^2}; \end{aligned}$$

$$
\begin{aligned}
M(s,r+ps)^2 &= \frac{(bs^2+b(r+ps)^2-aps^2+2as(r+ps))^2}{(s^2-prs-r^2)^2}\\
&= \frac{b^2r^4+b^2s^4+6b^2p^2r^2s^2+b^2p^4s^4+a^2p^2s^4+2b^2p^2s^4+4a^2r^2s^2}{(s^2-prs-r^2)^2}\\
&+ \frac{2b^2r^2s^2+4a^2prs^3+4b^2prs^3+2abps^4+4abrs^3}{(s^2-prs-r^2)^2}\\
&+ \frac{4abr^3s+4b^2pr^3s+10abpr^2s^2+4b^2p^3rs^3+8abp^2rs^3+2abp^3s^4}{(s^2-prs-r^2)^2};
\end{aligned}
$$

$$
\begin{aligned}
&(p^2+2)M(r,s)M(s,r+ps)\\
&= -\frac{(p^2+2)(br^2+bs^2-apr^2+2ars)(bs^2+b(r+ps)^2-aps^2+2as(r+ps))}{(s^2-prs-r^2)^2}\\
&= -\frac{b^2p^2r^4+2a^2p^2r^2s^2+4b^2p^2r^2s^2-abp^3r^4+2b^2r^4+2b^2s^4+8a^2r^2s^2+4b^2r^2s^2}{(s^2-prs-r^2)^2}\\
&- \frac{-2abpr^4+3b^2p^2s^4+2abp^3r^2s^2+8abp^2rs^3+8abpr^2s^2+8abrs^3-2abp^4r^3s}{(s^2-prs-r^2)^2}\\
&- \frac{4a^2prs^3+4b^2prs^3+b^2p^4s^4-abp^5r^2s^2+2abp^4rs^3+2abps^4+abp^3s^4+2a^2p^3rs^3}{(s^2-prs-r^2)^2}-\\
&\frac{2b^2p^3rs^3+8abr^3s-2a^2p^3r^3s+2b^2p^3r^3s-a^2p^4r^2s^2+b^2p^4r^2s^2-4a^2pr^3s+4b^2pr^3s}{(s^2-prs-r^2)^2}.
\end{aligned}
$$

Let T be the left-hand side of the needed-proving assertion. By addition, we obtain

$$
T = \frac{(abp^3+(a^2-b^2)p^2)(s^2-prs-r^2)^2}{(s^2-prs-r^2)^2} = abp^3+(a^2-b^2)p^2.
$$

□

Lemma 9. *Assume that $abp+a^2-b^2<0$ and that there exists a rational value x such that $A(x)$ is an integer. Construct a family of elements of $\mu(A)$ by Theorem 1. For each x in the family, calculate the corresponding value M. Then, there exists a value $M=M_0$ yielded by some x in the family such that $|M_0| \le \sqrt{-abp-a^2+b^2}$.*

Proof. Let $M_0 = M(r,s)$ be a value of M such that $|M_0|$ is minimal. Define

$$
M_1 = M(s, r+ps) \text{ and } M_{-1} = M(s-pr, r).
$$

From Lemma 8, we deduce that both M_1 and M_{-1} have opposite signs to M_0. Without loss of generality, suppose that $M_0>0$ and $M_1, M_{-1}<0$. Since $|M_0|$ is minimal, we have $M_1, M_{-1} \le -M_0$. By Lemma 8, we have

$$
-M_{-1}-(p^2+2)M_0 = M_1 \le -M_0.
$$

This implies $M_{-1} \geq -(p^2+1)M_0$ or $M_{-1} = tM_0$, where $t \in [-p^2-1, -1] \cap \mathbb{Q}$. On the other hand, we see that

$$M_{-1}^2 + (p^2+2)M_{-1}M_0 + M_0^2 = abp^3 + (a^2-b^2)p^2.$$

This implies

$$M_0^2 = \frac{abp^3 + (a^2-b^2)p^2}{t^2 + (p^2+2)t + 1}.$$

Define $f(t) = (abp^3 + (a^2-b^2)p^2)/(t^2+(p^2+2)t+1)$ with $t \in [-p^2-1, -1]$. Some calculations give us

$$f'(t) = \frac{-(abp^3 + (a^2-b^2)p^2)(2t+p^2+2)}{(t^2+(p^2+2)t+1)^2}.$$

The only root of $f'(t) = 0$ (which is the minimum point of $f(t)$) is $t = (-p^2-2)/2$. Since $f(-p^2-1) = f(-1) = -abp + b^2 - a^2$, we conclude that $M_0^2 \leq -abp + b^2 - a^2$. This implies

$$|M_0| \leq \sqrt{-abp + b^2 - a^2}.$$

$\square$

Lemma 10. *Assume that $abp + (a^2 - b^2) \geq 0$ and there exists a rational value x such that $A(x)$ is an integer. Construct a family of elements of $\mu(A)$ by Theorem 1. For each x in the family, calculate the corresponding value M. Then there exists a value $M = M_0$ coming from some x in the family such that $|M_0| \leq \sqrt{(abp^3 + (a^2-b^2)p^2)/(p^2+4)}$.*

Proof. If $M_0 = 0$ then the statement is obvious. So suppose that $M_0 \neq 0$.

If both M_1 and M_{-1} have the opposite sign from M_0, using the same argument as in Lemma 9, we imply that

$$M_0^2 = \frac{abp^3 + (a^2-b^2)p^2}{t^2 + (p^2+2)t + 1},$$

which is impossible since the right-hand side is negative for all $t \in [-p^2-1, -1]$.

Thus either M_1 or M_{-1} has the same sign as M_0. Assume that M_1 and M_0 have the same sign. Then Lemma 8 gives us

$$abp^3 + (a^2-b^2)p^2 = M_0^2 + (p^2+2)M_0M_1 + M_1^2 = |M_0|^2 + (p^2+2)|M_0||M_1| + |M_1|^2 \geq (p^2+4)|M_0|^2.$$

This gives us

$$|M_0|^2 \leq \frac{abp^3 + (a^2-b^2)p^2}{p^2+4}.$$

This implies

$$|M_0| \leq \sqrt{\frac{abp^3 + (a^2-b^2)p^2}{p^2+4}}.$$

$\square$

Proof of Theorem 3. From the above theorems, we can construct an algorithm to find all families of values x which make $A(x)$ an integer. First, by Theorem 2, we know some trivial families which always make $A(x)$ an integer. Then we can find other families by the following steps:

(i) Find all integer values M such that

$$\begin{cases} |M| \le \sqrt{(abp^3 + (a^2-b^2)p^2)/(p^2+4)}, & \text{if} \quad abp + a^2 - b^2 \ge 0; \\ |M| \le \sqrt{-abp - a^2 + b^2}, & \text{if} \quad abp + a^2 - b^2 < 0, \end{cases}$$

and the equation

$$M^2 + (p^2+2)MN + N^2 = abp^3 + (a^2-b^2)p^2 \tag{2}$$

has an integer solution N.

(ii) For each value M, use the relation

$$M^2 = (kp + b - ap)^2 - 4k(a-k) \tag{3}$$

to find the corresponding integer values of k.

(iii) For each $k \neq 0$, use the relation

$$x = \frac{ap - kp - b + M}{2k} \tag{4}$$

where $M = \pm\sqrt{(kp - ap + b)^2 - 4k(a-k)}$ to find the value x.

For $k = 0$, $A(x) = 0$ leads to $a(1-px)+bx = 0$. Thus, we can find that $x = a/(ap-b)$, which generates the second family in Theorem 2.

(iv) For each satisfied $x = r/s$, define the sequence $(A_n^{(r,s)})$ the same as Theorem 1. Then all the values $A_n^{(r,s)}/A_{n+1}^{(r,s)}$ becomes a family of elements of $\mu(A)$.

By Theorem 1, we know that the family can be constructed. Also by Lemma 9 and Lemma 10, we clearly see that all families of elements of $\mu(A)$ can be found by this way, except the families mentioned in Theorem 2. ☐

Example 11. Given $(a, b, p) = (9, 1, 2)$, we find all families of x. First, we see that Theorem 2 leads to the families

$$\ldots, \frac{5}{-2}, \frac{-2}{1}, \frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \ldots \tag{5}$$

and

$$\ldots, \frac{-23}{11}, \frac{11}{-1}, \frac{-1}{9}, \frac{9}{17}, \frac{17}{43}, \ldots. \tag{6}$$

To find other families, the equation (2) becomes

$$M^2 + 6MN + N^2 = 392,$$

and we need to find all integers M such that $|M| \le \sqrt{392/8} = 7$ and the above equation has an integer solution N. The test gives us $M \in \{-7, -1, 1, 7\}$. For $M = -7$, the relation (3) becomes

$$49 = (2k - 17)^2 - 36k + 4k^2$$

This leads to k-values of 3 and 10. For $k = 3$, by (4) we obtain that $x = 2/3$, which leads to the family

$$\ldots, \frac{4}{-1}, \frac{-1}{2}, \frac{2}{3}, \frac{3}{8}, \frac{8}{19}, \ldots. \tag{7}$$

Then $k = 10$ implies $x = -1/2$, which is already in the above family. Similarly, the value $M = -1$ leads to $k = 9$ and $k = 4$. For $k = 9$, we have $x = -1/9$, which is already in (6). For $k = 4$, we can see that $x = 1$ and the satisfied family

$$\ldots, \frac{3}{-1}, \frac{-1}{1}, \frac{1}{1}, \frac{1}{3}, \frac{3}{7}, \ldots. \tag{8}$$

For $M = 1$, we obtain $k = 9$ or $k = 4$. The former case leads to $x = 0$, which is already in (5). The latter leads to $x = 5/4$ and the family

$$\ldots, \frac{17}{-6}, \frac{-6}{5}, \frac{5}{4}, \frac{4}{13}, \frac{13}{30}, \ldots. \tag{9}$$

When $M = 7$ and $k = 10$, it turns out that $x = 1/5$, which yields the family

$$\ldots, \frac{-5}{3}, \frac{3}{1}, \frac{1}{5}, \frac{5}{11}, \frac{11}{27}, \ldots \tag{10}$$

and $k = 3$ leads to $x = 3$, which is already in (10). Thus, we have six families of elements of $\mu(A)$.

Example 12. Given $(a, b, p) = (1, 1, 2)$, we can prove that A_n is the n-th Pell-Lucas number (A001333). From Theorem 2, we have two families already

$$\ldots, \frac{5}{-2}, \frac{-2}{1}, \frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \ldots \tag{11}$$

and

$$\ldots, \frac{-7}{3}, \frac{3}{-1}, \frac{-1}{1}, \frac{1}{1}, \frac{1}{3}, \ldots. \tag{12}$$

We can see that $abp^3 + (a^2 - b^2)p^2 = 8 > 0$. Then the equation from Theorem 3 becomes

$$M^2 + 6MN + N^2 = 8$$

and we have to find all M satisfying $|M| \leq 1$ and the above equation has an integer solution N. If $M = 0$, the relation (3) gives no satisfied value k. When $M = 1$, the relation leads to $k = 0$ or $k = 1$. If $k = 1$, we find that $x = 0$, which leads to the family (11). The case $k = 0$ leads to the family (12). When $M = -1$, by the same arguments, we conclude that (11) and (12) are the only satisfied families.

In the following part, we consider the sequence $(U_n)_{n\geq 0}$ defined by

$$\begin{cases} U_0 = 0, U_1 = 1, \\ U_{n+1} = pU_n + U_{n-1}. \end{cases}$$

Now, by applying Theorem 3, we explain how our results can obtain a special case of Theorem 1.1 of Bulawa and Lee [1] and, also a special case of Theorem 1.7 of Knapp et al. [4] .

Corollary 13 ([4])**.** *Let $U(x)$ be the generating function of $(U_n)_{n\geq 0}$. Then*

$$\mu(U) = \left\{ \frac{A_n^{(0,1)}}{A_{n+1}^{(0,1)}} \,\middle|\, n \neq -1 \right\}.$$

Proof. We can easily see that $(U_n)_{n\geq 0}$ is $(A_n)_{n\geq 0}$ when $(a, b) = (0, 1)$. Therefore, applying Theorem 2, we have $A_n^{(0,1)}/A_{n+1}^{(0,1)} \in \mu(U)$ for all $n \neq -1$. Applying Theorem 3, we have

$$abp + (a^2 - b^2) = -1 < 0.$$

Then we must find M such that $|M| \leq 1$, and the equation $M^2 + (p^2 + 2)MN + N^2 = -p^2$ has an integer solution N. If $M = 0$, we find out $N^2 = -p^2$, which is impossible for $p \neq 0$. In case $M = \pm 1$, let k_0 and x_0 be the values k and x corresponding to M.

From the definition of M, we have

$$(k_0 p + 1)^2 + 4k_0^2 = 1,$$

which has $k_0 = 0$ be the only integer solution. It means that the family generated by $(A_n^{(r,s)})$ is identical to the one generated by $(A_n^{(0,1)})$, which was mentioned by Theorem 2. They are also all the elements of $\mu(U)$. $\square$

Corollary 14 ([1])**.** *Let $U(x)$ be the generating function of $(U_n)_{n\geq 0}$. Then*

(i) *$U(U_n/U_{n+1})$ is an integer for all $n \geq 0$.*

(ii) *Assume that $U(x) = k$ is an integer for some rational number x. Then there exists a non-negative integer n such that $U(U_n/U_{n+1}) = k$.*

Proof.

(i) Follows easily from Theorem 2.

(ii) Recall that for $(U_n)_{n\geq 0}$, we have

$$U(x) = \frac{x}{1 - px - x^2}.$$

By induction, we can prove that $A_{-n}^{(0,1)} = (-1)^{n+1} A_n^{(0,1)}$ for $n \in \mathbb{Z}$. Then

$$\begin{aligned}
U\big(A_{-n}^{(0,1)}/A_{-n+1}^{(0,1)}\big) &= \frac{A_{-n}^{(0,1)} A_{-n+1}^{(0,1)}}{(A_{-n+1}^{(0,1)})^2 - pA_{-n+1}^{(0,1)} A_{-n}^{(0,1)} - (A_{-n}^{(0,1)})^2} \\
&= \frac{(-1)A_n^{(0,1)} A_{n-1}^{(0,1)}}{(A_{n-1}^{(0,1)})^2 + pA_{n-1}^{(0,1)} A_n^{(0,1)} - (A_n^{(0,1)})^2} \\
&= U\big(A_{n-1}^{(0,1)}/A_n^{(0,1)}\big).
\end{aligned}$$

By Corollary 13, $A_n^{(0,1)}/A_{n+1}^{(0,1)}$ are the precise rational values for which $U(x)$ an integer. Therefore, if $U(x) = k$ is an integer, we can always find $n \geq 0$ such that

$$U\big(A_n^{(0,1)}/A_{n+1}^{(0,1)}\big) = U\big(U_n/U_{n+1}\big) = k.$$

□

In the last part, we consider the case $(a, b) = (1, 1)$ and prove the following result:

Corollary 15. *In case $(a, b) = (1, 1)$, there are no other families of $\mu(A)$, apart from those given in Theorem 2.*

Proof. Since we suppose that $p \neq 0$ in the beginning of the article, we consider two cases of p: $p > 0$ and $p < 0$.

First case: $p > 0$. Applying Theorem 3, we need to find M such that $M^2 < p$ and the equation

$$N^2 + (p^2 + 2)MN + (M^2 - p^3) = 0 \tag{13}$$

has an integer solution N. The discriminant of this equation is $p^2(M^2p^2 + 4p + 4M^2)$.

It is clear that $M^2p^2 + 4p + 4M^2 \leq M^2p^2 + 8p$, so if $|M| \geq 4$, we have

$$(p|M|)^2 < M^2p^2 + 4p + 4M^2 < (p|M| + 1)^2$$

which implies that the discriminant cannot be a perfect square. Therefore, there can only possibly be integer solutions for N if $|M| \leq 3$. Note that if $k = 0$, then $x = 1/(p-1)$, which belongs to the second family in Theorem 2. Thus, in the following part, we suppose that $k \neq 0$.

For $M = 0$, from (3) we have $(kp - p + 1)^2 = 4k(1 - k)$. Since $4k(1 - k) < 0$ for all integers $k \neq 0, 1$, we only have $k = 1$. This is impossible.

For $M = \pm 1$, from (3) we have $(kp - p + 1)^2 = 4k(1-k) + 1$. Since $4k(1-k) + 1 < 0$ for all integers $k \neq 0, 1$, we have $k = 1$, which leads to $x = (M-1)/2$ by (4). Since $M = \pm 1$, we have $x = 0$ (which belongs to the first family in Theorem 2) or $x = -1$ (which belongs to the second family in Theorem 2).

For $M = \pm 2$, from (3) we have $(kp - p + 1)^2 = 4k(1-k) + 4$. Since $4k(1-k) + 4 < 0$ for all integers $k \neq 0, 1$, we have $k = 1$. This is impossible.

For $M = \pm 3$, from (3) we have $(kp - p + 1)^2 = 4k(1-k) + 9$. Since $4k(1-k) + 9 < 0$ for all integers $k \neq -1, 0, 1, 2$, we have $k \in \{-1, 1, 2\}$.

If $k = -1$, we have $(-2p + 1)^2 = 1$, which implies $p = 1$ (since $p \neq 0$)). By (4), we obtain $x \in \{-2, 1\}$. The value $x = 1$ belongs to the first family in Theorem 2, while $x = -2$ belongs to the second family in Theorem 2.

If $k = 1$, we imply $1 = 9$, which is impossible.

If $k = 2$, we have $p = 0$ or $p = -2$, which contradicts our case $p > 0$.

Second case: $p < 0$. Let $q = -p$, applying Theorem 3, we need to find M such that $M^2 \leq q$ and the equation

$$N^2 + (q^2 + 2)MN + M^2 + q^3 = 0 \tag{14}$$

has integer solutions N.

Suppose that there exists a value of M satisfying $M^2 = q$. Then the equation (14) has an integer solution $N = -M$. The equation (3) gives us

$$q = (q + 1 - kq)^2 - 4k(1-k) = (cq + 1)^2 - 4(1-c)c,$$

where $c = 1 - k$. This leads to the equation

$$c^2q^2 + (2c - 1)q + (2c - 1)^2 = 0. \tag{15}$$

If $c = 0$, we have $p = -1, M^2 = 1$, and $k = 1$, which leads to $x = 0$ or $x = -1$ by (4). Both of these values belong to families given in Theorem 2. If $c \neq 0$, in (15) we have $\Delta_q = (2c-1)^2(1 - 4c^2) < 0$ for all integers $c \neq 0$, which leads to no solution of q or M.

Therefore, all of the other values M that we need to find satisfy $M^2 < q$ and the equation (14) has integer solutions N. Then the discriminant of (14) is $q^2(M^2q^2 + 4M^2 - 4q)$. We can clearly see that if (14) has integer solutions, $M^2q^2 + 4M^2 - 4q$ must be a perfect square. We prove that $|M| \leq 1$. Otherwise, suppose that $|M| \geq 2$. Let $M^2q^2 + 4M^2 - 4q = k^2$ with $k \geq 0$. Then $(|M|q - k)(|M|q + k) = 4(q - M^2)$. Thus, both $|M|q - k$ and $|M|q + k$ are even numbers. Set $|M|q - k = 2s$ and $|M|q + k = 2t$. Then

$$st = q - M^2 \text{ and } s + t = |M|q,$$

which implies $s(|M|q - s) = q - M^2 < q$. Besides, since $M^2q^2 - k^2 = 4(q - M^2) > 0$, we have $s = (|M|q - k)/2 > 0$, thus $s \geq 1$. We also see that $s = |M|q - t \leq |M|q - 1$ since $t \geq 1$. Therefore, we have $(s-1)(|M|q - s - 1) \geq 0$, which implies $s(|M|q - s) \geq |M|q - 1$. So $|M|q - 1 < q$. By $|M| \geq 2$, we have $2q - 1 < q$, which is impossible for $q \geq 1$.

For $|M| \leq 1$, by the same arguments as those in the case $p > 0$, we can prove that there are no other families of $\mu(A)$, except for those in Theorem 2. $\square$

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