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A 2-Regular Sequence That Counts The Divisors of $n^2 + 1$

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Abstract

We introduce the 2-regular integer sequence A383066 $= (s(n))_{n\geq 1}$, which begins $0, 1, 1, 2, 3, 3, 2, \ldots$. We prove that the number of occurrences of an integer $m \geq 0$ in this sequence is equal to $\tau(m^2+1)$, the number of divisors of m^2+1. Using this fact, we give a generating function for $\tau(m^2 + 1)$. We also discuss other interesting properties of $s(n)$, including its relationship to the Fibonacci sequence.

1 Introduction and proof of the main result

We begin by recalling the definition of k-regular sequences, which were introduced by Allouche and Shallit [1] as a generalization of automatic sequences [2].

Definition 1. A sequence $s(n)$ is k-regular if there exists an integer E such that, for all $e_j > E$ and $0 \leq r_j \leq k^{e_j} - 1$, every subsequence of s of the form $s(k^{e_j} n + r_j)$ is expressible as an $\mathbb{Z}$-linear combination

$$\sum_i c_{ij} s(k^{f_{ij}} n + b_{ij}),$$

where $f_{ij} \leq E$, and $0 \leq b_{ij} \leq k^{f_{ij}} - 1$.

In the previous definition, the integers $\mathbb{Z}$ can be replaced by any commutative Noetherian ring R', in which case we would say that $s(n)$ is (R', k)-regular. However, for the purposes of this paper, we consider only integer sequences. We begin by giving some well-known examples of 2-regular integer sequences.

Example 2. The 2-adic valuation of a positive integer n A007814, defined by $v_2(n) := \sup\{k \in \mathbb{N}_0 : 2^k \mid n\}$ is a 2-regular sequence, since it satisfies the recursions

$$\begin{cases} v_2(2k+1) = 0 \\ v_2(2k) = v_2(k) + 1 \end{cases}$$

with initial condition $v_2(1) = 0$.

Example 3. The Cantor sequence A005823 is a 2-regular sequence which consists of integers whose ternary expansions contain no 1s. The Cantor sequence $c(n)$ satisfies the recursions

$$\begin{cases} c(2k) = 3c(k) + 2 \\ c(2k+1) = 3c(k+1) \end{cases}$$

with initial condition $c(1) = 0$.

For more examples of k-regular sequences, see Allouche and Shallit [1, pp. 186–194]. We now state the main result of this paper.

Theorem 4. *We have*

$$\sum_{m \geq 0} \tau(m^2+1) x^m = \sum_{n \geq 1} x^{s(n)}$$

where τ is the usual divisor counting function and $s(n)_{n \geq 1}$ is a 2-regular sequence defined recursively by

$$\begin{cases} s(4k) = 2s(2k) - s(k) \\ s(4k+1) = 2s(2k) + s(2k+1) \\ s(4k+2) = 2s(2k+1) + s(2k) \\ s(4k+3) = 2s(2k+1) - s(k) \end{cases}$$

with initial conditions $s(1) = 0,\ s(2) = 1,\ s(3) = 1$.

In other words, we prove that

$$\#\{n : s(n) = m\} = \tau(m^2+1)$$

for all integers $m \geq 0$.

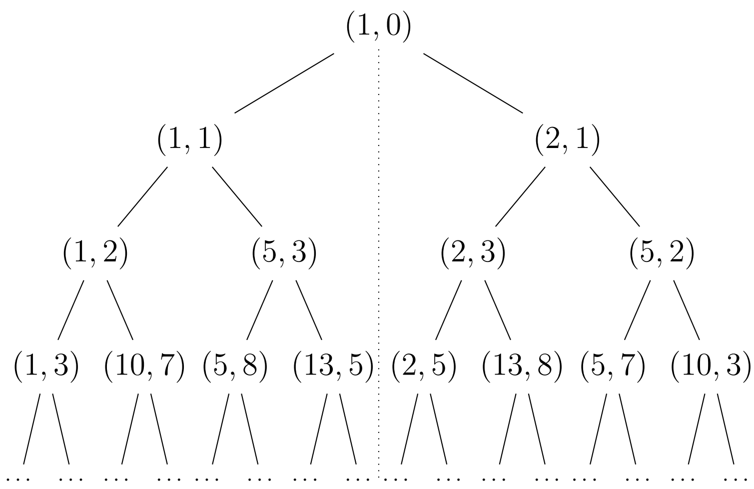

Figure 1: Integer pair tree.

Proof. Consider the binary tree of integer pairs (d, m) generated in the following way. We begin with the pair $(1, 0)$. Each pair has two children, left and right, given by the maps $L(d, m) := (d, m + d)$ and $R(d, m) := \left(\frac{(m+d)^2+1}{d}, m + \frac{m^2+1}{d}\right)$. For the first four rows of the tree, see Figure 1.

The involution map $\iota(d, m) := (\frac{m^2+1}{d}, m)$ sends each pair to its reflection with respect to the tree's central line of symmetry, represented by the dotted line in Figure 1. We note that $R(d, m) = (\iota \circ L \circ \iota)(d, m)$, which can either be checked by direct computation or by fixing an integer pair (d, m) on the tree and visually seeing that $\iota \circ L \circ \iota$ (reflection, left-child map, reflection) sends (d, m) to the same pair as the right-child map $R(d, m)$.

Lemma 5. *If a pair (d, m) appears on the integer pair tree then $d \geq 1$, $m \geq 0$, and d divides $m^2 + 1$.*

Proof. Suppose an integer pair (d, m) appears on the tree with the properties that $d \geq 1$, $m \geq 0$, and $d \mid (m^2 + 1)$. We claim that these properties also hold for the transformed pairs $L(d, m)$ and $R(d, m)$. It is easy to see that both transformed pairs $L(d, m)$ and $R(d, m)$ are still integer pairs, that their first components are ≥ 1, and that their second components are ≥ 0. To see that the first component still divides $1+$ the square of the second component, it suffices to check that this property is preserved by L and ι, since we saw that $R = \iota \circ L \circ \iota$. Indeed, the property $d \mid (m^2 + 1)$ is preserved by both L and ι, since

$$d \mid (m^2 + 1) \implies d \mid \big((m + d)^2 + 1\big) \text{ and } \frac{m^2 + 1}{d} \mid (m^2 + 1).$$

The first integer pair on the tree is $(1, 0)$, which satisfies all three properties. Therefore, all of its descendants must also satisfy all three properties. $\square$

Lemma 6. *Suppose $d \geq 1$, $m \geq 0$, and d divides m^2+1. Then the pair (d, m) appears on the integer pair tree exactly once.*

Proof. We begin by restating the lemma so that it can be proved using induction. We let $P(M)$ denote the statement "If $d \geq 1$ divides m^2+1 with $0 \leq m \leq M$, then the pair (d, m) appears on the tree exactly once."

To prove the lemma, we show that $P(M)$ is true for all integers $M \geq 0$ by induction on M. As our base case, we see that $P(0)$ is true, since the pair $(1, 0)$ appears exactly once on the tree, in the first row. This is because L and R each increase the second component of a pair by at least 1, so there are no more pairs on the tree with second component 0.

Now suppose $M > 0$. Our induction assumption is that $P(M-1)$ is true. Namely, we assume that for all $d \geq 1$ with $d \mid (m^2+1)$ and $0 \leq m \leq M-1$ we have (d, m) appearing on the tree exactly once. We show that this implies $P(M)$ is true by assuming that some $d \geq 1$ divides M^2+1 and using the induction assumption to prove that (d, M) must appear on the tree exactly once. The claim that (d, M) appears on the tree exactly once is equivalent to the statement that there exists a unique path from the root pair $(1, 0)$ to (d, M) in terms of the maps L and R. It is easy to check that $L^{-1}(d, M) = (d, M-d)$ and $R^{-1}(d, M) = \left(\frac{(M-d)^2+1}{d}, M - \frac{M^2+1}{d}\right)$. We show that exactly one of the second components of these inverse mappings $\{M-d,\ M-\frac{M^2+1}{d}\}$ is nonnegative (which we showed in the previous lemma is a necessary condition for pairs to appear on the tree). The nonnegativity of exactly one of $\{M-d,\ M-\frac{M^2+1}{d}\}$ is a result of the inequalities

$$\inf\left\{d, \frac{m^2+1}{d}\right\} \leq m < \sup\left\{d, \frac{m^2+1}{d}\right\},$$

which hold for all positive integers d and m with $d \mid (m^2+1)$. These inequalities can be proved by considering the cases $d \leq m$ and $d > m$ and using the fact that $m^2 < m^2+1 < (m+1)^2$ for all $m > 0$. Equality occurs in the left inequality when $(d, m) \in \{(1, 1), (2, 1)\}$. Thus, exactly one of the pairs $\{L^{-1}(d, M), R^{-1}(d, M)\}$ has a nonnegative second component. Furthermore, this component is strictly less than M. The properties $d \mid (m^2+1)$ and $d \geq 1$ are clearly preserved by L^{-1} and R^{-1}. Therefore, by our induction assumption and Lemma 5, exactly one of the pairs $\{L^{-1}(d, M), R^{-1}(d, M)\}$ appears on the tree and there exists a unique path from $(1, 0)$ to this pair in terms of L and R. In other words, the pair (d, M) has exactly one parent appearing on the tree, which is guaranteed by our induction assumption to have a unique path back to $(1, 0)$ in terms of L and R.

Finally, this proves that (d, M) has a unique path to $(1, 0)$ in terms of L and R and therefore appears on the tree exactly once, which shows that $P(M)$ is true. $\square$

We now write only the second pair components as they appear on the integer pair tree. Let us temporarily define $s(n)$ as the sequence one gets by reading the integers on the second component tree left-to-right or right-to-left, starting from the top. Figure 2 shows the first four rows of the second component tree. For example, $s(1) = 0,\ s(2) = 1,\ s(3) = 1$, etc. We

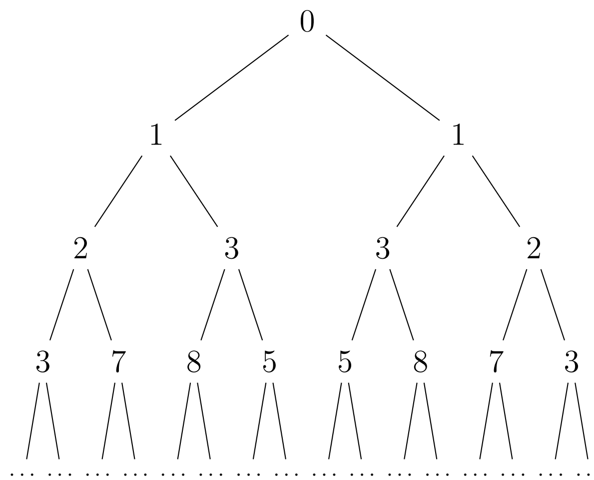

Figure 2: Second component tree.

will show that this agrees with the original definition we gave in Theorem 4. Using the new definition, and in light of Lemmas 5 and 6, this proves that the number of occurrences of an integer $m \geq 0$ on the second component tree is equal to $\#\{(d,m) : d \geq 1,\ d \mid (m^2+1)\} = \tau(m^2+1)$. To see that $s(n)$ satisfies the recursions we gave in Theorem 4, we keep track of the second components as they are changed by the maps L and R. For example, since $L(d,m) = (d, m+d)$ and $R(d,m) = \left(\frac{(m+d)^2+1}{d}, m + \frac{m^2+1}{d}\right)$, we write m_L for $m+d$ and m_R for $m + \frac{m^2+1}{d}$. Figure 3 shows three generations of second pair components. Figure 4 shows how to write components in the third generation as linear combinations of components from the previous two generations.

We now rewrite the parent-child relationships in terms of the sequence $s(n)$. Reading a two-child binary tree left-to-right, we see that for a parent with index k, its left child's index is $2k$ while its right child's index is $2k+1$. Figure 5 relates three generations of components to the corresponding indices of $s(n)$.

Using the linear dependencies we found, we finally recover the recursions from Theorem 4, namely

$$\begin{cases} s(4k) = 2s(2k) - s(k) \\ s(4k+1) = 2s(2k) + s(2k+1) \\ s(4k+2) = 2s(2k+1) + s(2k) \\ s(4k+3) = 2s(2k+1) - s(k). \end{cases}$$

$\square$

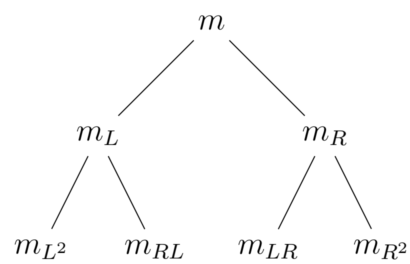

Figure 3: Three generations of second pair components.

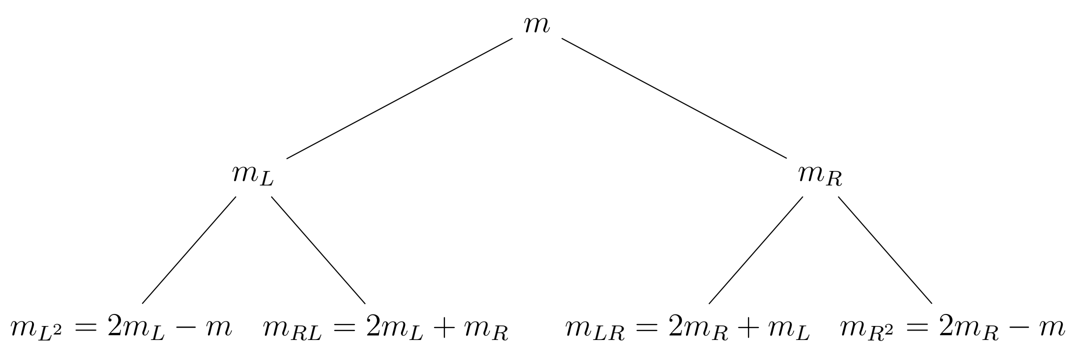

Figure 4: Linear dependencies between second pair components.

2 Other properties of the sequence

We discuss some other interesting properties of the sequence $s(n)$. From the recursions for $s(n)$, it is easy to show that the row sums of the second component tree satisfy the linear recurrence $r_n = 5r_{n-1} - 2r_{n-2}$ with initial conditions $r_0 = 0$, $r_1 = 2$. Here, r_n denotes the sum of integers on row $n \geq 0$ of the second component tree, or, equivalently, $r_n := \sum_{2^n \leq t < 2^{n+1}} s(t)$. By diagonalizing the 2×2 integer matrix corresponding to this recurrence, we write down an exact formula for the average value of an integer on row $n \geq 0$ of the tree.

Proposition 7.

$$\frac{1}{2^n} \sum_{2^n \leq t < 2^{n+1}} s(t) = \frac{(5+\sqrt{17})^n - (5-\sqrt{17})^n}{2^{2n-1}\sqrt{17}}$$

Proposition 8. *The integer n^2+1 is a prime number if and only if*

$$\{m : s(m) = n\} = \{2^n, 2^{n+1} - 1\}.$$

Proof. Note that $s(2^n) = s(2^{n+1} - 1) = n$ for all $n \geq 0$. These correspond to the leftmost and rightmost integers on row n of the second component tree. In other words, $\{2^n, 2^{n+1} - 1\} \subseteq \{m : s(m) = n\}$. Now, if $n^2 + 1$ is a prime number, then $\tau(n^2 + 1) = 2$ and it follows from Theorem 4 that $\{m : s(m) = n\} = \{2^n, 2^{n+1} - 1\}$. Conversely, if $n^2 + 1$ is composite,

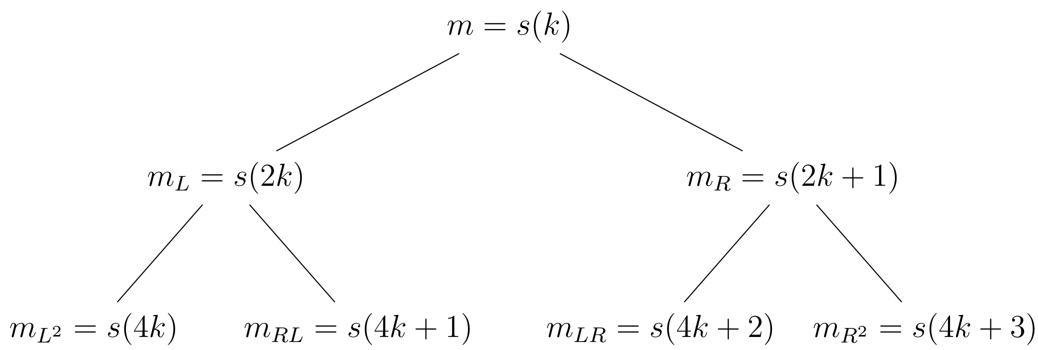

Figure 5: Dependencies re-indexed in terms of $s(n)$.

then $\tau(n^2+1) > 2$ and so Theorem 4 implies there is some $m \notin \{2^n, 2^{n+1}-1\}$ such that $s(m) = n$. $\square$

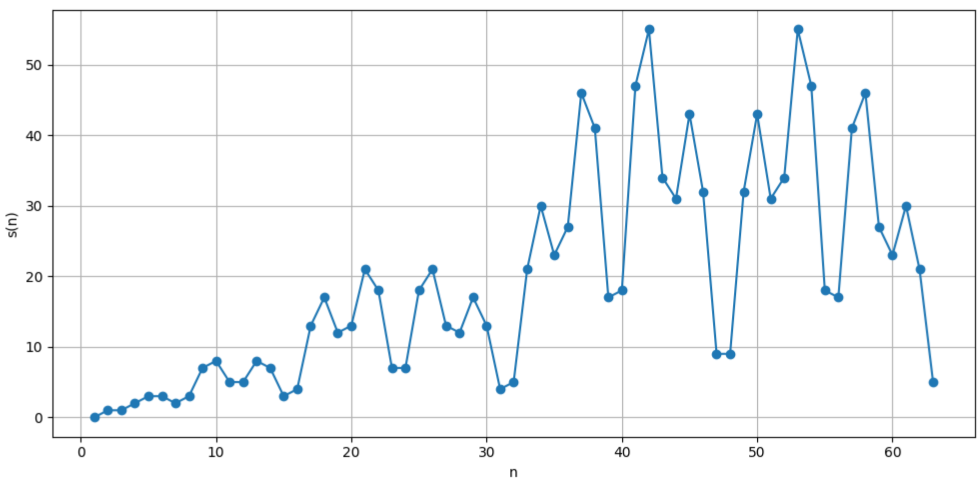

Figure 6: Line plot of the sequence $s(n)$ for $n \in [1, 63]$

The Fibonacci sequence also makes an appearance in the second component tree. The Fibonacci sequence is defined by the recursion $F_{n+1} = F_n + F_{n-1}$ with initial conditions $F_1 = 0, F_2 = 1$.

Proposition 9. *Consider the sequence defined by*

$$a(n) = \begin{cases} 1, & \text{if } n = 1; \\ 2a(n-1), & \text{if } n = 4k; \\ a(n-1)+1, & \text{if } n = 4k+1; \\ 2a(n-1)+1, & \text{if } n = 4k+2; \\ a(n-1)-1, & \text{if } n = 4k+3. \end{cases}$$

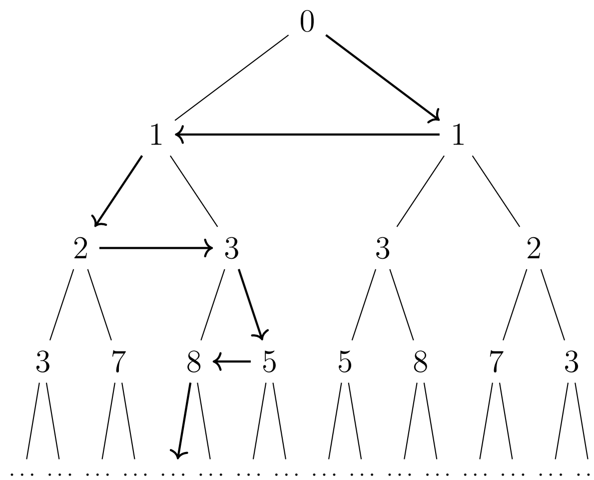

Figure 7: Path inside the second component tree that runs over the Fibonacci sequence.

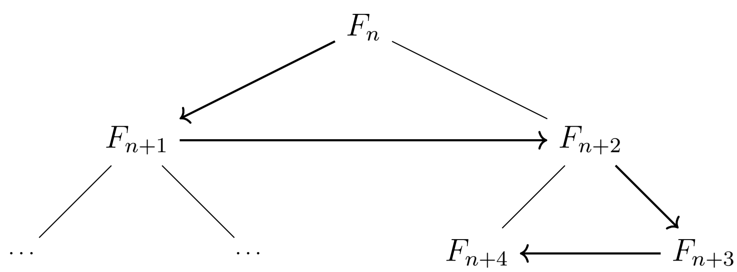

Figure 8: Fibonacci path in the case where $m_L = F_{n+1}$ and $m_R = F_{n+2}$.

Then $s(a(n)) = F_n$, namely the nth term of the Fibonacci sequence.

Proof. Note that the recursions for $a(n)$ are chosen so that the sequence $s(a(n))$ sweeps out the path on the second component tree shown in Figure 7. Starting from the top of the tree, move to the closest neighbor in a given direction, cycling through these four directions: south-east, west, south-west, east.

The recursions for $s(n)$ tell us that if we start with $m = F_n$ and $\{m_L, m_R\} = \{F_{n+1}, F_{n+2}\}$, the children of F_{n+2} will be $2F_{n+2} + F_{n+1}$ and $2F_{n+2} - F_n$. Using the Fibonacci recursion we have that $2F_{n+2} - F_n = F_{n+2} + F_{n+1} = F_{n+3}$ and $2F_{n+2} + F_{n+1} = F_{n+2} + F_{n+3} = F_{n+4}$. Therefore, if $m_L = F_{n+1}$ and $m_R = F_{n+2}$ we trace out the path given in Figure 8. If $m_L = F_{n+2}$ and $m_R = F_{n+1}$ we trace the path given in Figure 9.

Since $F_n = s(a(n))$ for $1 \leq n \leq 3$, we conclude that $s(a(n)) = F_n$ for all $n \geq 1$. □

Corollary 10. *The integer $F_n^2 + 1$ is composite for $n > 4$.*

Proof. This follows from the fact that there exists $m \notin \cup_{n \geq 0}\{2^n, 2^{n+1} - 1\}$ with $s(m) = F_n$ for all $n > 4$, as can be seen in Figure 7. □

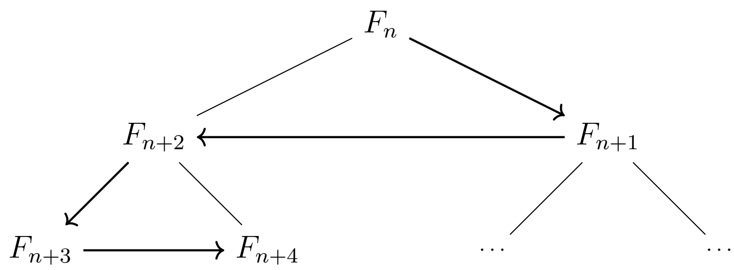

Figure 9: Fibonacci path in the case where $m_L = F_{n+2}$ and $m_R = F_{n+1}$.

Another way to see this fact is to use Cassini's Identity [5], namely $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ in the case where $n = 2k$, as well as the related identity $F_{2k-1}F_{2k+3} - F_{2k+1}^2 = 1$ in the case where $n = 2k+1$. Both identities can be proved using induction.

Corollary 11. *The largest integer on row $n \geq 1$ of the second component tree is F_{2n}.*

Proof. It can be seen from the recursions for the second component tree that the largest second component on a particular row n of the integer pair tree is given by either one of the zigzag paths $RLRLRL\cdots$ or $LRLRLR\cdots$. This, together with Proposition 9, proves the corollary. $\square$

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(Concerned with sequence A383066.)

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