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Generalizing Recent Results of Kathiravan, Majumdar, Sangale, and Srinivas on (ℓ, k)-regular Partitions

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Abstract

In two recent papers, Kathiravan, Srinivas, and Sangale and Kathiravan, Sangale, and Majumdar proved several congruences satisfied by (ℓ, k)-regular partitions; that is, partitions whose parts are not divisible by integers ℓ or k. In this short note, we generalize some of their results, and prove two infinite family of congruences modulo 2 and one infinite family of congruences modulo 12. Our proofs involve only elementary techniques.

1 Introduction

An integer partition of n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq k-1$ and $\sum_{i=1}^{k} \lambda_i = n$. The λ_i's are called parts of the partition λ. For instance, there are 7 partitions of 5, and they are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), \text{ and } (1,1,1,1,1).$$

We let $p(n)$ denote the number of partitions of n, and we have the following generating function, due to Euler:

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{\prod_{i\geq 1}(1-q^i)} = \frac{1}{(q;q)_\infty} = \frac{1}{f_1},$$

where we have used the shorthand notation

$$(a;q)_\infty := \prod_{i\geq 0}(1-aq^i) \quad \text{and} \quad f_k := (q^k;q^k)_\infty.$$

Partitions have been very well-studied since the time of Euler, and prominent mathematicians such as Euler, Jacobi, and Ramanujan are associated with the study of partitions. For a general overview of the area, we refer to the book of Andrews [2]. One area of study in partitions is that of generalized partition functions. For instance, if we do not allow parts divisible by ℓ, then the resulting partitions are called ℓ-regular partitions (see [11, A000009, A000726, A001935, A035959, A219601]). We let $b_\ell(n)$ denote the number of ℓ-regular partitions of n, and it is known that

$$\sum_{n\geq 0} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

This class of partitions is very well-studied; see for instance the work of Andrews, Hirschhorn, and Sellers [1], Cui and Gu [5], Wang [12], Du and Tang [6], and the references therein.

Generalizing this idea further, Kathiravan, Srinivas, and Sangale [9] introduced the concept of (ℓ, k)-regular partitions, which are partitions such that no part is divisible by ℓ or k, where $(\ell, k) = 1$. They let $b_{\ell,k}(n)$ denote the number of such partitions of n and gave the generating function

$$\sum_{n\geq 0} b_{\ell,k}(n)q^n = \frac{f_\ell f_k}{f_1 f_{\ell k}}. \tag{1}$$

Kathiravan, Srinivas, and Sangale [9] proved congruences modulo 2 for $(3,8)$ and $(4,7)$-regular partitions and modulo $8, 9$ and 12 for $(4,9)$-regular partitions. Kathiravan, Sangale, and Majumdar [8] proved congruences modulo 2 for $(2,7)$, $(4,11)$, and $(5,8)$-regular partitions and modulo 4 for $(4,5)$-regular partitions. Naika, Hemanthkumar, and Sumanth Bharadwaj [10] proved congruences modulo 2 for $(3,5)$-regular partitions.

Our goal in this note is to generalize the recent congruence results of Kathiravan, Majumdar, Sangale, and Srinivas on (ℓ, k)-regular partitions. For example, Kathiravan, Sangale, and Majumdar [8, Theorem 3.2] proved the following congruences: for all $n \geq 0$, we have

$$b_{4,11}(22n + i) \equiv 0 \pmod 2, \quad i \in \{2, 8, 16, 18, 20\}.$$

The result below strengthens their result.

Theorem 1. *Let $p \geq 5$ be a prime, and r is an integer such that $8r + 1$ is a quadratic nonresidue modulo p. Then we have, for all $n \geq 0$,*

$$b_{4,p}(pn + r) \equiv 0 \pmod 2.$$

A similar result also holds for the functions $b_{2,p}(n)$.

Theorem 2. *Let $p \geq 5$ be a prime, and r is an integer such that $24r + 1$ is a quadratic nonresidue modulo p. Then we have, for all $n \geq 0$,*

$$b_{2,p}(pn + r) \equiv 0 \pmod 2.$$

Theorems 1 and 2 are proved in Section 2 using elementary techniques. In Section 3, we provide a proof of the following infinite family of congruences (related to [11, A187020]) which significantly generalizes a result in [9].

Theorem 3. *For all $n \geq 0$ and $\alpha \geq 3$, we have*

$$b_{4,9}(2^\alpha n + 6 \cdot 2^{\alpha-3} + 1) \equiv 0 \pmod{12}. \tag{2}$$

Remark 4. Kathiravan, Srinivas, and Sangale [9, Theorem 1.3] proved the $\alpha = 3$ case of (2).

All of the proofs which appear below are elementary in nature, relying on well–known q–series identities and generating function manipulations.

2 Proof of Theorems 1 and 2

In order to prove Theorems 1 and 2, we simply need two well–known identities that date back to Euler and Jacobi.

Lemma 5 (Euler)**.** *We have*

$$f_1 = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2}.$$

Proof. See [7, Section 1.6]. □

Lemma 6 (Jacobi)**.** *We have*

$$f_1^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2}.$$

Proof. See [7, Section 1.7]. □

With the above in hand, we can now prove Theorems 1 and 2 relatively quickly.

Proof of Theorem 1. From (1) and Lemma 6, we have

$$\begin{aligned}\sum_{n\geq 0} b_{4,p}(n) q^n &= \frac{f_4 f_p}{f_1 f_{4p}} \\ &\equiv \frac{f_1^3}{f_p^3} \pmod 2 \\ &\equiv \frac{1}{f_p^3} \sum_{k=0}^{\infty} q^{k(k+1)/2} \pmod 2.\end{aligned}$$

Since f_p^3 is a function of q^p, and since we are interested in arguments of the form $pn + r$ where r is not divisible by p, we can focus our attention on the f_1^3. In particular, we need to ask whether

$$pn + r = \frac{k(k+1)}{2}$$

for some nonnegative integer k. This is equivalent to asking whether $8(pn+r)+1 = (2k+1)^2$ by completing the square. This would mean that $8r+1 \equiv (2k+1)^2 \pmod p$. However, this cannot be the case since we defined r such that $8r+1$ is a quadratic nonresidue modulo p. This completes the proof. □

We next prove Theorem 2 in very similar fashion.

Proof of Theorem 2. From (1) and Lemma 5, we know

$$\begin{aligned}\sum_{n\geq 0} b_{2,p}(n) q^n &= \frac{f_2 f_p}{f_1 f_{2p}} \\ &\equiv \frac{f_1}{f_p} \pmod 2 \\ &\equiv \frac{1}{f_p} \sum_{k=-\infty}^{\infty} q^{(3k^2-k)/2} \pmod 2.\end{aligned}$$

Since f_p is a function of q^p, and since we are interested in arguments of the form $pn + r$ where r is not divisible by p, we can focus our attention on the f_1. In particular, we need to ask whether

$$pn + r = \frac{(3k^2 - k)}{2}$$

for some nonnegative integer k. This is equivalent to asking whether $24(pn+r)+1 = (6k+1)^2$ by completing the square. This would mean that $24r + 1 \equiv (6k + 1)^2 \pmod{p}$. However, this cannot be the case since we defined r such that $24r+1$ is a quadratic nonresidue modulo p. □

3 An elementary proof of Theorem 3

Our goal in this section is to prove Theorem 3 as noted above. In order to complete the proof, we need a few more q–series dissection results that appear in the literature which we list here.

Lemma 7. *We have*

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}.$$

Proof. See [13, Lemma 3.5]. □

Lemma 8. *We have*

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}.$$

Proof. See [4, Eq. (49)]. (Also see [11, A273845].) □

Lemma 9. *We have*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}.$$

Proof. See [14, Eq. (3.29)]. (Also see [11, A328547].) □

Lemma 10. *We have*

$$\begin{aligned}
\frac{1}{f_1^4} &= \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}},\\
\frac{1}{f_1^8} &= \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_4^4 f_8^8}{f_2^{20}},\\
\frac{1}{f_1^{12}} &= \frac{f_4^{42}}{f_2^{42} f_8^{12}} + 12q\frac{f_4^{30}}{f_2^{38} f_8^4} + 48q^2\frac{f_4^{18} f_8^4}{f_2^{34}} + 64q^3\frac{f_4^6 f_8^{12}}{f_2^{30}}.
\end{aligned}$$

Proof. See [3, Lemma 2.2]. □

Lemma 11. *We have*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}.$$

Proof. See [4, Lemma 2.2]. (Also see [11, A318026].) □

Our proof relies on various generating function dissections and manipulations (utilizing the lemmas above), along with a very straightforward application of mathematical induction.

We begin by completing a 2–dissection to obtain the generating function for $b_{4,9}(2n+1)$. (Note that all of the arguments in the statement of Theorem 3 are odd, so we do not need the generating function for $b_{4,9}(2n)$, although we could easily obtain it as well.)

Theorem 12. *We have*

$$\sum_{n\geq 0} b_{4,9}(2n+1)q^n = \frac{f_2^3 f_3}{f_1^3 f_6}.$$

Proof. Using Lemma 7, we can rewrite the generating function for $b_{4,9}(n)$ in the following way:

$$\begin{aligned}\sum_{n\geq 0} b_{4,9}(n)q^n &= \frac{f_4 f_9}{f_1 f_{36}} \\ &= \frac{f_4}{f_{36}}\left(\frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}\right)\end{aligned}$$

which allows us to see immediately that

$$\sum_{n\geq 0} b_{4,9}(2n+1)q^{2n+1} = q\frac{f_4}{f_{36}}\cdot\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}$$

or

$$\sum_{n\geq 0} b_{4,9}(2n+1)q^n = \frac{f_2}{f_{18}}\cdot\frac{f_2^2 f_3 f_{18}}{f_1^3 f_6} = \frac{f_2^3 f_3}{f_1^3 f_6}.$$

□

From here, we can take care of the modulo 3 portion of Theorem 3 almost immediately. Namely, note that

$$\begin{aligned}\sum_{n\geq 0} b_{4,9}(2n+1)q^n &= \frac{f_2^3 f_3}{f_1^3 f_6} \\ &\equiv \frac{f_6 f_3}{f_3 f_6} \pmod 3 \\ &= 1.\end{aligned}$$

This means that, for all $n \geq 1$,

$$b_{4,9}(2n+1) \equiv 0 \pmod 3. \tag{3}$$

(Note that the 1 that appears at the end of the string of congruences above arises because $b_{4,9}(1) = 1$.) Hence, the mod 3 portion of all of the congruences associated with Theorem 3 is now proven.

Thus, we need to focus our attention now on the mod 4 aspects of the congruences in Theorem 3. In order to complete the proof of Theorem 3, we need a few specific results mod 4. We now prove those results, which serve as the building blocks of our elementary proof of Theorem 3.

Theorem 13. *For all $n \geq 0$, we have $b_{4,9}(16n+1) \equiv b_{4,9}(4n+1) \pmod 4$.*

Proof. Using Theorem 12 and Lemma 8, we know that

$$\sum_{n\geq 0} b_{4,9}(2n+1)q^n = \frac{f_2^3 f_3}{f_1^3 f_6}$$
$$= \frac{f_2^3}{f_6}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right)$$

which means

$$\sum_{n\geq 0} b_{4,9}(2(2n)+1)q^{2n} = \frac{f_2^3}{f_6}\cdot\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2}$$

or

$$\sum_{n\geq 0} b_{4,9}(4n+1)q^n = \frac{f_1^3}{f_3}\cdot\frac{f_2^6 f_3^3}{f_1^9 f_6^2} = \frac{f_2^6 f_3^2}{f_1^6 f_6^2}. \tag{4}$$

We now 2–dissect this expression again in order to find a result for the generating function for $b_{4,9}(8n+1)$ in the following way:

$$\sum_{n\geq 0} b_{4,9}(4n+1)q^n = \frac{f_2^6 f_3^2}{f_1^6 f_6^2}$$
$$= \frac{f_2^6}{f_6^2}\cdot\frac{f_3^2}{f_1^2}\cdot\frac{1}{f_1^4}.$$

Using Lemma 9 and Lemma 10, we see that

$$\sum_{n\geq 0} b_{4,9}(8n+1)q^{2n} \equiv \frac{f_2^6}{f_6^2}\cdot\frac{f_4^{18} f_6 f_{12}^2}{f_2^{19} f_8^5 f_{24}} \pmod 4$$

or

$$\sum_{n\geq 0} b_{4,9}(8n+1)q^n \equiv \frac{f_1^6}{f_3^2}\cdot\frac{f_2^{18} f_3 f_6^2}{f_1^{19} f_4^5 f_{12}} \pmod 4 \tag{5}$$
$$= \frac{f_2^{18} f_6^2}{f_1^{13} f_3 f_4^5 f_{12}}.$$

We now perform one additional 2–dissection in order to obtain an expression, modulo 4, for the generating function for $b_{4,9}(16n+1)$. In order to do so, we rewrite the above as

$$\sum_{n\geq 0} b_{4,9}(8n+1)q^n \equiv \frac{f_2^{18} f_6^2}{f_4^5 f_{12}}\cdot\frac{1}{f_1 f_3}\cdot\frac{1}{f_1^{12}} \pmod 4.$$

Lemma 10 and Lemma 11 can now be utilized to show that

$$\sum_{n\geq 0} b_{4,9}(8(2n)+1)q^{2n} \equiv \frac{f_2^{18}f_6^2}{f_4^5 f_{12}} \cdot \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} \cdot \frac{f_4^{42}}{f_2^{42} f_8^{12}} \pmod 4$$
$$= \frac{f_4^{36} f_{12}^4}{f_2^{26} f_6^2 f_8^{10} f_{24}^2}$$

which means

$$\sum_{n\geq 0} b_{4,9}(16n+1)q^n \equiv \frac{f_2^{36} f_6^4}{f_1^{26} f_3^2 f_4^{10} f_{12}^2} \pmod 4. \tag{6}$$

The goal now is to show that the expressions in (4) and (6) are congruent to one another modulo 4. The only tool necessary to prove this is the fact that $f_1^4 \equiv f_2^2 \pmod 4$ which easily follows from the Binomial Theorem. We have

$$\begin{aligned}
\sum_{n\geq 0} b_{4,9}(16n+1)q^n &\equiv \frac{f_2^{36} f_6^4}{f_1^{26} f_3^2 f_4^{10} f_{12}^2} \pmod 4 \\
&\equiv \frac{f_2^{36} f_{12}^2}{f_1^{26} f_3^2 f_4^{10} f_{12}^2} \pmod 4 \\
&\equiv \frac{f_2^{36}}{f_1^6 f_2^{10} f_3^2 f_2^{20}} \pmod 4 \\
&\equiv \frac{f_2^6}{f_1^6 f_3^2} \pmod 4 \\
&= \frac{f_2^6 f_3^2}{f_1^6 f_3^4} \\
&\equiv \frac{f_2^6 f_3^2}{f_1^6 f_6^2} \pmod 4 \\
&\equiv \sum_{n\geq 0} b_{4,9}(4n+1)q^n \pmod 4.
\end{aligned}$$

□

Theorem 13 will serve as the "engine" for a proof of Theorem 3 by mathematical induction. In order to complete that proof, we need two base cases. We now prove those two individual congruences modulo 4.

Theorem 14. *For all $n \geq 0$, we have $b_{4,9}(16n+13) \equiv 0 \pmod 4$.*

Proof. We note from (4) that

$$\sum_{n\geq 0} b_{4,9}(4n+1)q^n = \frac{f_2^6}{f_6^2}\left(\frac{f_3}{f_1^3}\right)^2.$$

Applying Lemma 8, we obtain

$$\sum_{n\geq 0} b_{4,9}(4n+1)q^n = \frac{f_2^6}{f_6^2}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right)^2$$
$$= \frac{f_2^6}{f_6^2}\left(\frac{f_4^{12} f_6^6}{f_2^{18} f_{12}^4} + 9q^2\frac{f_4^4 f_6^2 f_{12}^4}{f_2^{14}} + 6q\frac{f_4^8 f_6^4}{f_2^{16}}\right).$$

Extracting the terms involving the odd powers of q, dividing by q, and replacing q^2 by q throughout, we obtain

$$\sum_{n\geq 0} b_{4,9}(8n+5)q^n = 6\frac{f_2^8 f_3^2}{f_1^{10}} = 6f_2^8 \cdot \left(\frac{f_3^2}{f_1^2}\right) \cdot \left(\frac{1}{f_1^8}\right). \tag{7}$$

Applying Lemma 9 and the second equation of Lemma 10 to (7), we see that

$$\sum_{n\geq 0} b_{4,9}(8n+5)q^n \equiv 2f_2^8\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right)\frac{f_4^{28}}{f_2^{28} f_8^8} \pmod 4.$$

Now, from the above equation, extracting the terms involving odd powers of q, dividing both sides by q and then replacing q^2 by q, we obtain

$$\sum_{n\geq 0} b_{4,9}(16n+13)q^n \equiv 0 \pmod 4.$$

This implies the theorem. □

Theorem 15. *For all $n \geq 0$, we have $b_{4,9}(32n+25) \equiv 0 \pmod 4$.*

Proof. From (5), we have

$$\sum_{n\geq 0} b_{4,9}(8n+1)q^n \equiv \frac{f_2^{18} f_6^2}{f_4^5 f_{12}} \cdot \frac{1}{f_1 f_3} \cdot \frac{1}{f_1^{12}} \pmod 4.$$

Using Lemma 11 and the third equation of Lemma 10 in the above, we obtain

$$\sum_{n\geq 0} b_{4,9}(8n+1)q^n \equiv \frac{f_2^{18} f_6^2}{f_4^5 f_{12}}\left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right)\frac{f_4^{42}}{f_2^{42} f_8^{12}} \pmod 4.$$

Extracting the terms involving the odd powers of q, dividing throughout by q and then replacing q^2 by q in the above equation, we arrive at

$$\sum_{n\geq 0} b_{4,9}(16n+9)q^n \equiv \frac{f_2^{42}}{f_1^{28} f_6^2 f_4^{14}} = \frac{f_2^{42}}{f_6^2 f_4^{14}} \cdot \left(\frac{1}{f_1^{12}}\right)^2 \cdot \frac{1}{f_1^4} \pmod 4.$$

It is easy to see that, applying Lemma 10 in the above equation, we obtain

$$\sum_{n\geq 0} b_{4,9}(16n+9)q^n \equiv \frac{f_2^{42}}{f_6^2 f_4^{14}} \left(\frac{f_4^{42}}{f_2^{42} f_8^{12}}\right)^2 \frac{f_4^{14}}{f_2^{14} f_8^4} \pmod 4.$$

Since the right-hand-side of the above equation involves only even powers of q, we readily obtain

$$\sum_{n\geq 0} b_{4,9}(32n+25)q^n \equiv 0 \pmod 4.$$

Our result now follows. □

Proof of Theorem 3. As a reminder, we are trying to prove that, for all $n \geq 0$ and $\alpha \geq 3$, we have

$$b_{4,9}(2^\alpha n + 6\cdot 2^{\alpha-3} + 1) \equiv 0 \pmod{12}.$$

As was noted above, the $\alpha = 3$ case of the above was proven by Kathiravan, Srinivas, and Sangale [9, Theorem 1.3]. Our goal is to now prove the remaining cases via induction. Also, thanks to (3), it is enough to prove the congruence modulo 4.

First, note that the $\alpha = 4$ case corresponds to the arithmetic progression $2^4 n + 6\cdot 2^1 + 1 = 16n + 13$, and Theorem 14 provides the result in this case. Next, the $\alpha = 5$ case corresponds to the arithmetic progression $2^5 n + 6\cdot 2^2 + 1 = 32n + 25$, and Theorem 15 handles this case. We use these two specific cases as base cases for arguments by induction.

We now prove our result in two stages, once for even α and once for odd α. First, let $\alpha = 2\beta$ for some integer β. We assume that, for some $\beta \geq 2$ and all $n \geq 0$,

$$b_{4,9}(2^{2\beta} n + 6\cdot 2^{2\beta-3} + 1) \equiv 0 \pmod 4.$$

We then want to prove that, for all $n \geq 0$,

$$b_{4,9}(2^{2\beta+2} n + 6\cdot 2^{2\beta-1} + 1) \equiv 0 \pmod 4.$$

Note that

$$\begin{aligned} b_{4,9}(2^{2\beta+2} n + 6\cdot 2^{2\beta-1} + 1) &= b_{4,9}(16(2^{2\beta-2} n + 6\cdot 2^{2\beta-5}) + 1) \\ &\equiv b_{4,9}(4(2^{2\beta-2} n + 6\cdot 2^{2\beta-5}) + 1) \pmod 4 \text{ from Theorem 13} \\ &= b_{4,9}(2^{2\beta} n + 6\cdot 2^{2\beta-3}) + 1) \\ &\equiv 0 \pmod 4 \end{aligned}$$

by the induction hypothesis. This completes the proof for even α.

Now, let $\alpha = 2\beta + 1$ for some integer β. We assume that, for some $\beta \geq 1$ and all $n \geq 0$,

$$b_{4,9}(2^{2\beta+1} n + 6\cdot 2^{2\beta-2} + 1) \equiv 0 \pmod 4.$$

We then want to prove that, for all $n \geq 0$,

$$b_{4,9}(2^{2\beta+3} + 6 \cdot 2^{2\beta} + 1) \equiv 0 \pmod 4.$$

This is readily seen, as before

$$\begin{aligned} b_{4,9}(2^{2\beta+3} + 6 \cdot 2^{2\beta} + 1) &= b_{4,9}(16(2^{2\beta-1}n + 6 \cdot 2^{2\beta-4}) + 1) \\ &\equiv b_{4,9}(4(2^{2\beta-1}n + 6 \cdot 2^{2\beta-4}) + 1) \pmod 4 \text{ from Theorem 13} \\ &= b_{4,9}(2^{2\beta+1}n + 6 \cdot 2^{2\beta-2}) + 1) \\ &\equiv 0 \pmod 4. \end{aligned}$$

□

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References

[1] G. E. Andrews, M. D. Hirschhorn, and J. A. Sellers, Arithmetic properties of partitions with even parts distinct, *Ramanujan J.* **23** (2010), 169–181.

[2] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1998.

[3] E. H. M. Brietzke, Robson da Silva, and J. A. Sellers, Congruences related to an eighth order mock theta function of Gordon and McIntosh, *J. Math. Anal. Appl.* **479** (2019), 62–89.

[4] N. D. Baruah and K. K. Ojah, Partitions with designated summands in which all parts are odd, *Integers* **15** (2015), Paper No. A9.

[5] S.-P. Cui and N. S. S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. in Appl. Math.* **51** (2013), 507–523.

[6] J. Q. D. Du and D. Tang, Congruence properties modulo powers of 2 for 4-regular partitions, *Electron. J. Combin.* **31** (2024), Paper No. 3.32.

[7] M. D. Hirschhorn, *The Power of q*, Developments in Mathematics, 49, Springer, 2017.

[8] T. Kathiravan, U. K. Sangale, and D. Majumdar, Infinite families of congruences modulo 2 for (ℓ, k)-regular partitions, *Hardy-Ramanujan J.* **46** (2023), 51–62.

[9] T. Kathiravan, K. Srinivas, and U. K. Sangale, Some congruences for (ℓ, k)-regular partitions, *J. Ramanujan Math. Soc.* **39** (2024), 225–237.

[10] M. S. Mahadeva Naika, B. Hemanthkumar, and H. S. Sumanth Bharadwaj, Congruences modulo 2 for certain partition functions, *Bull. Aust. Math. Soc.* **93** (2016), 400–409.

[11] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, 2024.

[12] L. Wang, Arithmetic properties of 7-regular partitions, *Ramanujan J.* **47** (2018), 99–115.

[13] E. X. W. Xia and X. M. Yao, Some modular relations for the Göllnitz-Gordon functions by an even-odd method, *J. Math. Anal. Appl.* **387** (2012), 126–138.

[14] E. X. W. Xia and O. X. M. Yao, Analogues of Ramanujan's partition identities, *Ramanujan J.* **31** (2013), 373–396.

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