Journal of Integer Sequences, Vol. 28 (2025),
Article 25.1.7

Fibonacci Numbers via Dirichlet Convolution and Lucas Numbers via Unitary Convolution

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Abstract

We apply the multiplicative arithmetic functions strategy to sequences defined by linear two-term recurrences. We associate Fibonacci numbers with Dirichlet convolution and Lucas numbers with unitary convolution. Some Fibonacci-Lucas sums are shown as applications by using the Kesava Menon quasi-distributive law (a distributive-like property of Dirichlet convolution over unitary convolution) and the Fibonacci-Lucas quotients under the Dirichlet and the unitary convolutions.

1 Introduction and preliminaries

1.1 Brief introduction and motivation

Haukkanen [9], McCarthy, and Sivaramakrishnan [17] were the first to create a bridge between the two chapters of elementary number theory: arithmetic functions and Fibonacci numbers. In two recent papers, E. D. Schwab and G. Schwab [19, 21] found new links between specially multiplicative (quadratic) arithmetic functions and generalized Fibonacci numbers. These two chapters of number theory have interacted, creating the possibility to widen the field of action for both. The basic identities of Fibonacci numbers, such as Cassini's, d'Ocagne's, Catalan's, Vajda's, and Honsberger's identities, are all particular cases of Busche-Ramanujan identities from the theory of specially multiplicative arithmetic functions (see [21, Theorem 3.1]). This relationship leads to interesting paths in the parallel approach of these two chapters of elementary number theory.

This paper focuses on some Fibonacci-Lucas-type identities using the Dirichlet convolution and the unitary convolution of arithmetic functions. The Kesava Menon quasi-distributive law of Dirichlet convolution over unitary convolution (see Equation (5) below) is a new source for Fibonacci-Lucas-type identities. For instance, the following Fibonacci-Lucas-type identities (F_i, L_i, J_i, and j_i are the i^{th} Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas numbers, respectively):

$$\sum_{i=0}^{n} L_{n-i}J_i = j_{n+1} - L_{n+1} \qquad \text{(Griffiths and Bramham [8, Section 2])} \tag{1}$$

and
$$\sum_{i=0}^{n} j_{n-i}F_i = j_{n+1} - L_{n+1} \qquad \text{(Koshy and Griffiths [15, Equation (2.7)])} \tag{2}$$

are particular cases of a general identity (see Corollary 14) which is proved below using the Kesava Menon quasi-distributive law.

Essentially, this paper is not only about an alternative technique for proving identities; but also about the close relationship between arithmetic functions and Fibonacci numbers, and their effects on each other. Fibonacci numbers are associated with Dirichlet convolution and Lucas numbers with unitary convolution of two completely multiplicative functions (see Equation (6) and Theorem 3 (i)). This simple observation motivated us to choose Fibonacci-Lucas relations for our applications.

The paper is self-contained and the next Subsections 1.2 and 1.3 include all results and notation used in Sections 2 and 3. Section 2 is devoted to Fibonacci $\mathcal{F}_{a,b}$ and Lucas $\mathcal{L}_{a,b}$ multiplicative arithmetic functions; in particular to Dirichlet and unitary quotients $\left[\frac{\mathcal{F}_{a,b}}{\mathcal{L}_{a,b}}\right]_D$ and $\left[\frac{\mathcal{L}_{a,b}}{\mathcal{F}_{a,b}}\right]_U$, respectively. Both quotients are used in Subsections 3.1 and 3.2 as a starting point to establish some Fibonacci-Lucas sums. Using the Kesava Menon quasi-distributive law, we prove Theorem 13 and implicitly Corollary 14. Perhaps Subsection 3.4 best illustrates how the strategy of approaching Fibonacci numbers works using the tools of multiplicative arithmetic functions.

1.2 Fibonacci-type sequences and their companions

Throughout this paper, the focus of attention is on the Fibonacci and the Lucas sequences in their general form (i.e., (a,b)-Fibonacci and (a,b)-Lucas sequences).

The companion sequence of the famous Fibonacci sequence $\{F_m\}_{m\geq 0}$ (A000045),

$$F_0 = 0, F_1 = 1, \qquad F_{m+2} = F_{m+1} + F_m,$$

is the Lucas sequence $\{L_m\}_{m\geq 0}$ (A000032) defined by the same recurrence relation but with different initial conditions

$$L_0 = 2, L_1 = 1, \qquad L_{m+2} = L_{m+1} + L_m.$$

Some well-known Fibonacci-type numbers modify the recurrence relation slightly, leaving the initial conditions unchanged. The Pell numbers $\{P_m\}_{m\geq 0}$ (A000129), the Jacobsthal numbers $\{J_m\}_{m\geq 0}$ (A0001045), the k-Fibonacci numbers $\{F_{k,m}\}_{m\geq 0}$ ([6, 7]), the k-Jacobsthal numbers $\{J_{k,m}\}_{m\geq 0}$ ([12]), are such numbers:

$$\begin{aligned} P_0 = 0, P_1 = 1, \qquad & P_{m+2} = 2P_{m+1} + P_m; \\ J_0 = 0, J_1 = 1, \qquad & J_{m+2} = J_{m+1} + 2J_m; \\ F_{k,0} = 0, F_{k,1} = 1, \qquad & F_{k,m+2} = kF_{k,m+1} + F_{k,m}; \\ J_{k,0} = 0, J_{k,1} = 1, \qquad & J_{k,m+2} = kJ_{k,m+1} + 2J_{k,m}. \end{aligned}$$

The companion of the Pell sequence (say Pell-Lucas sequence) $\{Q_m\}_{m\geq 0}$ ([11, A002203]), the Jacobsthal-Lucas $\{j_m\}_{m\geq 0}$ ([11, A014551]), the k-Lucas $\{L_{k,m}\}_{m\geq 0}$ ([4, 5]), and the k-Jacobsthal-Lucas $\{j_{k,m}\}_{m\geq 0}$ ([3, 25]) sequences are defined by

$$\begin{aligned} Q_0 = 2, Q_1 = 2, \qquad & Q_{m+2} = 2Q_{m+1} + Q_m; \\ j_0 = 2, j_1 = 1, \qquad & j_{m+2} = j_{m+1} + 2j_m; \\ L_{k,0} = 2, L_{k,1} = k, \qquad & L_{k,m+2} = kL_{k,m+1} + L_{k,m}; \end{aligned}$$

and

$$j_{k,0} = 2, j_{k,1} = k, \qquad j_{k,m+2} = kj_{k,m+1} + 2j_{k,m},$$

respectively.

In the mathematical literature, generalizations of Fibonacci numbers have an extensive place. For fixed real numbers a and b (can also be complex), let $\mathcal{R}(a,b)$ denote (see [13]) the set of all sequences $\{A_m\}_{m\geq 0}$ with initial values A_0 and A_1 for which all succeeding terms are determined by

$$A_{m+2} = aA_{m+1} + bA_m.$$

The set of sequences $\mathcal{R}(a,b)$ is a two-dimensional subspace of $\mathbb{R}^\infty$. Two distinguished elements of $\mathcal{R}(a,b)$ are the (a,b)-Fibonacci sequence, say $\{F_m^{a,b}\}_{m\geq 0}$, and the (a,b)-Lucas sequence, say $\{L_m^{a,b}\}_{m\geq 0}$, defined by

$$F_0^{a,b} = 0, F_1^{a,b} = 1, \qquad F_{m+2}^{a,b} = aF_{m+1}^{a,b} + bF_m^{a,b}$$

and

$$L_0^{a,b} = 2, L_1^{a,b} = a, \qquad L_{m+2}^{a,b} = aL_{m+1}^{a,b} + bL_m^{a,b},$$

respectively. The (a,b)-Fibonacci number Binet formula and the (a,b)-Lucas number Binet formula are (see [13, Equations (9) and (10)])

$$F_m^{a,b} = \frac{\alpha^m - \beta^m}{\alpha - \beta} \qquad \text{and} \qquad L_n^{a,b} = \alpha^m + \beta^m, \tag{3}$$

respectively, where

$$\alpha + \beta = a \qquad \text{and} \qquad \alpha\beta = -b.$$

The Lucas-Fibonacci basic connection is stated by the following equation (see [13, Equation (5)]):

$$L_{m+1}^{a,b} = F_{m+2}^{a,b} + bF_m^{a,b}. \tag{4}$$

The sequence $\{H_m^{a,b}(r,s)\}_{m\geq 0}$ (extensively studied by Horadam [10]) as an element of $\mathcal{R}(a,b)$ is a second-order linear recurrence sequence depending on four parameters (two initial values and two in the defining recursion itself):

$$H_0^{a,b}(r,s) = r, H_1^{a,b}(r,s) = s \qquad \text{and} \qquad H_{m+2}^{a,b}(r,s) = aH_{m+1}^{a,b}(r,s) + bH_m^{a,b}(r,s).$$

Extension of the definition of $\{H_m^{a,b}(r,s)\}_{m\geq 0}$ to negative subscript is provided by the recurrence relation

$$H_{-n}^{a,b}(r,a) = \frac{1}{b}(H_{-n+2}^{a,b} + aH_{-n+1}^{a,b}).$$

Now, $F_m^{a,b} = H_m^{a,b}(0,1)$, $L_m^{a,b} = H_m^{a,b}(2,a)$, and $F_{-1}^{a,b} = \frac{1}{b}$, $L_{-1}^{a,b} = -\frac{a}{b}$, etc.

1.3 Arithmetic functions

For information on arithmetic functions, we refer to McCarthy's [16] and Sivaramakrishnan's [23] books.

An arithmetic function is a complex-valued function defined on the set of positive integers. The Dirichlet convolution $f*g$ and the unitary convolution $f \sqcup g$ of two arithmetic functions f and g are defined by

$$(f*g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) \qquad \text{and} \qquad (f \sqcup g)(n) = \sum_{d'||n} f(d')g\left(\frac{n}{d'}\right),$$

respectively, where the summation is over the positive divisors d of n in the first case, and over the unitary divisors d' (i.e., $d' \mid n$ and $\gcd(d', \frac{n}{d'}) = 1$) of n in the second case.

An arithmetic function f is called multiplicative if $f(1) = 1$ and $f(nn') = f(n)f(n')$ whenever $\gcd(n,n') = 1$. A multiplicative arithmetic function f is said to be completely multiplicative if $f(nn') = f(n)f(n')$ holds for all positive integers n and n'. The set of multiplicative arithmetic functions $\mathcal{M}$ forms a group $(\mathcal{M}, *)$ under the Dirichlet convolution and simultaneously forms a group $(\mathcal{M}, \sqcup)$ under the unitary convolution. These two groups are isomorphic abelian groups and the multiplicative arithmetic function e defined by $e(n) = 0$ if $n = 1$ and $e(n) = 0$ otherwise is the identity element in both groups. In fact, $(\mathcal{M}, \sqcup, *)$ is a quasi-field in the sense of Kesava Menon [14] with the following quasi-distributive law:

$$f*(g \sqcup h) \sqcup f = (f*g) \sqcup (f*h), \tag{5}$$

that we call the Kesava Menon quasi-distributive law.

Note that a multiplicative arithmetic function is completely determined by its values at the prime powers.

If f and g are two multiplicative arithmetic functions, then the quotients $\left[\frac{f}{g}\right]_D$ and $\left[\frac{f}{g}\right]_U$ denote the multiplicative arithmetic functions for which $f = \left[\frac{f}{g}\right]_D * g$ and $f = \left[\frac{f}{g}\right]_U \sqcup g$, respectively. In other words,

$$\left[\frac{f}{g}\right]_D = f * g^{-1} \qquad \text{and} \qquad \left[\frac{f}{g}\right]_U = f \sqcup g^{<},$$

where g^{-1} and $g^{<}$ are the inverse elements of g in the groups $(\mathcal{M}, *)$ and $(\mathcal{M}, \sqcup)$, respectively.

2 Fibonacci-Lucas quotients under the Dirichlet and the unitary convolutions

For any positive integer n, let $\Omega(n)$ be the number of prime factors of n counted with multiplicity. If c is a non-zero complex number then the arithmetic function $c^{\Omega(n)}$ (in short c^{Ω}) is completely multiplicative.

Definition 1. Given two complex numbers a and b, $b \neq 0$, we define the (a, b)-Fibonacci $\mathcal{F}_{a,b}$, and the (a, b)-Lucas $\mathcal{L}_{a,b}$ multiplicative arithmetic functions as

$$\mathcal{F}_{a,b} = \alpha^{\Omega} * \beta^{\Omega} \qquad \text{and} \qquad \mathcal{L}_{a,b} = \alpha^{\Omega} \sqcup \beta^{\Omega}, \tag{6}$$

respectively, where

$$\alpha = \frac{a + \sqrt{a^2 + 4b}}{2} \qquad \text{and} \qquad \beta = \frac{a - \sqrt{a^2 + 4b}}{2}. \tag{7}$$

Definition 2. We define the $\mathcal{H}_{a,b}^{(s)}$ multiplicative arithmetic function as

$$\mathcal{H}_{a,b}^{(s)}(p^m) = H_m^{a,b}(1, s),$$

where p is a prime number and m is a non-negative integer.

In [21, Section 2], the authors highlighted the relationship between the multiplicative function $\mathcal{F}_{a,b}$ and (a, b)-Fibonacci numbers, and on the other hand in [20, Section 5] the relationship between the multiplicative function $\mathcal{L}_{a,b}$ and (a, b)-Lucas numbers.

Theorem 3. *For complex numbers a, b and integers m, p (m non-negative and p prime) the following identities hold:*

(i) ([21, 20])

$$\mathcal{F}_{a,b}(p^m) = F_{m+1}^{a,b} \qquad \textit{and} \qquad \mathcal{L}_{a,b}(p^m) = \begin{cases} 1, & \text{if } m = 0; \\ L_m^{a,b}, & \text{if } m > 0. \end{cases}$$

(ii)

$$\left[\frac{\mathcal{F}_{a,b}}{\mathcal{L}_{a,b}}\right]_D = \mathcal{F}_{0,-b} \qquad \textit{and} \qquad \left[\frac{\mathcal{L}_{a,b}}{\mathcal{F}_{a,b}}\right]_U = \mathcal{H}^{(0)}_{a,b} = \mu_a * \mathcal{F}_{a,b},$$

where μ_a is the multiplicative arithmetic function defined by

$$\mu_a(p^m) = \begin{cases} 1, & \text{if } m = 0; \\ -a, & \text{if } m = 1; \\ 0, & \text{if } m > 1. \end{cases}$$

(μ_1 is the classical Möbius function, denoted by μ in the proof of Theorem 15.)

Proof.

(i) Since $\mathcal{F}_{a,b}$ and $\mathcal{L}_{a,b}$ are multiplicative, it is clear that

$$\mathcal{F}_{a,b}(p^0) = \mathcal{L}_{a,b}(p^0) = 1 = F^{a,b}_1.$$

For $m > 0$, using the Binet formulas (see (3)) we get

$$\mathcal{F}_{a,b}(p^m) = (\alpha^\Omega * \beta^\Omega)(p^m) = \sum_{k=0}^{m} \alpha^k \beta^{m-k} = \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} = F^{a,b}_{m+1},$$

and

$$\mathcal{L}_{a,b}(p^m) = (\alpha^\Omega \sqcup \beta^\Omega)(p^m) = \alpha^m + \beta^m = L^{a,b}_m.$$

(ii) It is straightforward to check that the inverse $\mathcal{F}^{-1}_{a,b}$ of $\mathcal{F}_{a,b}$ in the group $(\mathcal{M}, *)$ is given by (see also [21])

$$\mathcal{F}^{-1}_{a,b}(p^m) = \begin{cases} 1, & \text{if } m = 0; \\ -a, & \text{if } m = 1; \\ -b, & \text{if } m = 2; \\ 0, & \text{if } m > 2. \end{cases} \tag{8}$$

Obviously, if $m = 0$ and $m = 1$, then we have

$$\mathcal{F}_{a,b} * \mathcal{F}^{-1}_{0,-b}(p^m) = \mathcal{F}_{a,b}(p^m) = \mathcal{L}_{a,b}(p^m).$$

Invoking Equation (4) it follows that if $m > 1$ then

$$(\mathcal{F}_{a,b} * \mathcal{F}^{-1}_{0,-b})(p^m) = \mathcal{F}_{a,b}(p^m) + b\mathcal{F}_{a,b}(p^{m-2}) = F^{a,b}_{m+1} + bF^{a,b}_{m-1} = L^{a,b}_m = \mathcal{L}_{a,b}(p^m).$$

We conclude that $\mathcal{F}_{a,b} = \mathcal{L}_{a,b} * \mathcal{F}_{0,-b}$.

Now, it is clear that $\mathcal{H}_{a,b}(1,0)(p^0) = H^{a,b}_0(1,0) = 1$, $\mathcal{H}_{a,b}(1,0)(p) = H^{a,b}_1(1,0) = 0$, and by induction we get

$$\mathcal{H}_{a,b}(1,0)(p^m) = H^{a,b}_m(1,0) = \begin{cases} 1, & \text{if } m = 0; \\ bF^{a,b}_{m-1}, & \text{if } m > 0. \end{cases} \tag{9}$$

while

$$H^{a,b}_{m+1}(1,0) = aH^{a,b}_m(1,0) + bH^{a,b}_{m-1}(1,0) = abF^{a,b}_{m-1} + b^2F^{a,b}_{m-2} = bF^{a,b}_m$$

if $m > 1$. Invoking Equations (9) and (4), we may conclud that

$$(\mathcal{F}_{a,b} \sqcup \mathcal{H}_{a,b}(1,0))(p^m) = F^{a,b}_{m+1} + bF^{a,b}_{m-1} = L^{a,b}_m \qquad (m > 0),$$

and therefore $\mathcal{L}_{a,b} = \mathcal{F}_{a,b} \sqcup \mathcal{H}_{a,b}(1,0)$.

Finally, since $(\mu_a * \mathcal{F}_{a,b})(p^0) = 1$ and

$$(\mu_a * \mathcal{F}_{a,b})(p^m) = \sum_{i=0}^{m} \mu_a(p^i)\mathcal{F}_{a,b}(p^{m-i}) = F^{a,b}_{m+1} - aF^{a,b}_m = bF^{a,b}_{m-1}$$

if $m > 0$, the proof is complete. □

Also taking into account the quasi-distributive law (5), we arrive at the next well-known Fibonacci-Lucas identity.

Corollary 4. *For (a,b)-Fibonacci and (a,b)-Lucas numbers, and $\Delta = a^2 + 4b \neq 0$, the following identity holds:*

$$\Delta F^{a,b}_{m+1} = L^{a,b}_{m+2} + bL^{a,b}_m. \qquad (10)$$

Proof. Since $\mathcal{F}^{-1}_{0,-b} * \mathcal{L}_{a,b} = \mathcal{F}^{-1}_{0,-b} * (\mathcal{F}_{a,b} \sqcup \mathcal{H}^0_{a,b})$, and $\mathcal{F}^{-1}_{0,-b}(p^m) = 0$ if $m > 2$, by the quasi-distributive law (5) we have

$$(\mathcal{F}^{-1}_{0,-b} * \mathcal{L}_{a,b})(p^{m+2}) = (\mathcal{F}^{-1}_{0,-b} * \mathcal{F}_{a,b})(p^{m+2}) + (\mathcal{F}^{-1}_{0,-b} * \mathcal{H}^0_{a,b})(p^{m+2}) \quad \text{if } m > 0.$$

Thus,

$$\begin{aligned} L^{a,b}_{m+2} + bL^{a,b}_m &= F^{a,b}_{m+3} + bF^{a,b}_{m+1} + bF^{a,b}_{m+1} + b^2F^{a,b}_{m-1} \\ &= aF^{a,b}_{m+2} + 4bF^{a,b}_{m+1} - abF^{a,b}_m \\ &= a^2F^{a,b}_{m+1} + 4bF^{a,b}_{m+1} = \Delta F^{a,b}_{m+1}. \end{aligned}$$

The proof is complete. □

In the special case of k-Fibonacci numbers we have

$$(k^2 + 4)F_{k,m+1} = L_{k,m+2} + L_{k,m}. \qquad (11)$$

3 Applications to Fibonacci-Lucas relations

3.1 Sums involving consecutive even and odd subscript (a,b)-Lucas numbers

As an initial application, in Theorem 5 (a) we use the convolution tool of arithmetic functions to prove two sums that involve the first n consecutive odd and even subscript (a,b)-Lucas numbers. Then we extend these sums in part (b).

Theorem 5. *For any non-negative integers m and n, and $b \neq 0$ we have the following:*

(a) *(i)* $\sum_{i=0}^{n}(-b)^{n-i}L_{2i+1}^{a,b} = F_{2n+2}^{a,b}$;

(ii) $\sum_{i=0}^{n}(-b)^{n-i}L_{2i}^{a,b} = F_{2n+1}^{a,b} + (-b)^n$;

(b) *(i)*

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2(i+m)+1}^{a,b} &= F_{2(n+m)+2}^{a,b} + H_{2m+1}^{a,b}((-b)^n, 0)\\ &= F_{2(n+m)+2}^{a,b} + (-1)^n b^{n+1} F_{2m}^{a,b}\\ &= F_{n+m+1}^{a,b}L_{n+m+1}^{a,b} + (-1)^n b^{n+1} F_{m}^{a,b}L_{m}^{a,b};\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2(i+m)}^{a,b} &= F_{2(n+m)+1}^{a,b} + H_{2m}^{a,b}((-b)^n, 0)\\ &= F_{2(n+m)+1}^{a,b} + (-1)^n b^{n+1} F_{2m-1}^{a,b}\\ &= F_{n+m+1}^{a,b}L_{n+m}^{a,b} + (-1)^n b^{n+1} F_{m}^{a,b}L_{m-1}^{a,b}.\end{aligned}$$

Proof.

(a) We use the first quotient $\left[\frac{\mathcal{F}_{a,b}}{\mathcal{L}_{a,b}}\right]_D = \mathcal{F}_{0,-b}$, that is $\mathcal{F}_{a,b} = \mathcal{L}_{a,b} * \mathcal{F}_{0,-b}$. It is straightforward to check that $\mathcal{F}_{0,-b}$ is given by

$$\mathcal{F}_{0,-b}(p^m) = \begin{cases}(-b)^{\frac{m}{2}}, & \text{if } m \text{ is an even integer;}\\ 0, & \text{if } m \text{ is an odd integer.}\end{cases}$$

(i) Let $\ell = 2n+1$. Then

$$F_{2n+2}^{a,b} = \mathcal{F}_{a,b}(p^\ell) = \sum_{i=0}^{\ell}\mathcal{L}_{a,b}(p^i)\mathcal{F}_{0,-b}(p^{\ell-i}) = \sum_{i=0}^{n}(-b)^{n-i}L_{2i+1}^{a,b}.$$

(ii) If $\ell = 2n$ then

$$\begin{aligned}F_{2n+1}^{a,b} + (-b)^n &= \mathcal{F}_{a,b}(p^\ell) + (-b)^n\\ &= \sum_{i=0}^{\ell}\mathcal{L}_{a,b}(p^i)\mathcal{F}_{0,-b}(p^{\ell-i}) + (-b)^n\\ &= \mathcal{F}_{0,-b}(p^\ell) + \sum_{i=1}^{n}L_{2i}^{a,b}\mathcal{F}_{0,-b}(p^{2n-2i}) + (-b)^n\\ &= \sum_{i=0}^{n}(-b)^{n-i}L_{2i}^{a,b}.\end{aligned}$$

(b) We prove the first part of $b(i)$ and $b(ii)$ by induction on m. For $m=0$ see (a). For $m=1$ we get

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2(i+1)+1}^{a,b} &= \sum_{i=0}^{n}(-b)^{n-i}(aL_{2(i+1)}^{a,b}+bL_{2i+1}^{a,b})\\ &= aF_{2n+3}^{a,b}+aH_{2}^{a,b}((-b)^{n},0)+bF_{2n+2}^{a,b}\\ &= F_{2n+4}^{a,b}+H_{3}^{a,b}((-b)^{n},0).\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2(i+1)}^{a,b} &= \sum_{i=0}^{n}(-b)^{n-i}(aL_{2i+1}^{a,b}+bL_{2i}^{a,b})\\ &= aF_{2n+2}^{a,b}+bF_{2n+1}^{a,b}+(-1)^{n}b^{n+1}\\ &= F_{2n+3}^{a,b}+H_{2}^{a,b}((-b)^{n},0),\end{aligned}$$

Thus we can proceed to the second step of induction ($m>1$). We have

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2(i+m)+1}^{a,b} &= \sum_{i=0}^{n}(-b)^{n-i}(aL_{2(i+m)}^{a,b}+bL_{2(i+m)-1}^{a,b})\\ &= aF_{2(n+m)+1}^{a,b}+aH_{2m}^{a,b}((-b)^{n},0)+bF_{2(n+m)}^{a,b}+bH_{2m-1}^{a,b}((-b)^{n},0)\\ &= F_{2(n+m)+2}^{a,b}+H_{2m+1}^{a,b}((-b)^{n},0).\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2(i+m)}^{a,b} &= \sum_{i=0}^{n}(-b)^{n-i}(aL_{2(i+m)-1}^{a,b}+bL_{2(i+m)-2}^{a,b})\\ &= aF_{2(n+m)}^{a,b}+aH_{2m-1}^{a,b}((-b)^{n},0)+bF_{2(n+m)-1}^{a,b}+bH_{2m-2}^{a,b}((-b)^{n},0)\\ &= F_{2(n+m)+1}^{a,b}+H_{2m}^{a,b}((-b)^{n},0),\end{aligned}$$

The proof of the first part of (b) (i) and (b) (ii) is complete. By applying the following lemma, the second part of our proof is also solved.

Lemma 6. *For all non-negative integer m, the following identity holds:*

$$H_{m}^{a,b}((-b)^{n},0)=(-1)^{n}b^{n+1}F_{m-1}^{a,b}.$$

The proof of this lemma is by induction on m and is omitted.

Now, before proceeding to the proof of the last part we make the following remark.

Remark 7. Bitim and Topal [1, Corollary 2] established the following Fibonacci-Lucas relation:

$$\sum_{i=0}^{n}(-b)^i L_{2i}^{a,b} = (-b)^{-n} F_{n+1}^{a,b} L_n^{a,b}.$$

This relation together with Part (a) (ii) of Theorem 5, gives the following formula for odd subscript (a, b)-Fibonacci numbers:

$$F_{2n+1}^{a,b} = F_{n+1}^{a,b} L_n^{a,b} + (-1)^{n+1} b^n. \tag{12}$$

The identity (12) is missing from the fairly comprehensive list of connected formulas (2.1)-(2.13) inserted in [1], but the twin formula (with even subscript),

$$F_{2n}^{a,b} = F_n^{a,b} L_n^{a,b}, \tag{13}$$

appears in [1, (2.3)].

Now, using these two Fibonacci-Lucas identities (12) and (13), the last part of the theorem becomes an immediate consequence. □

Remark 8. Perhaps the simplest proof of Theorem 5(a) is the one below, using (4) and then rearranging the parentheses.

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i} L_{2i+1}^{a,b} &= \sum_{i=0}^{n}(-b)^{n-i}(F_{2i+2}^{a,b} + bF_{2i}^{a,b}) \\ &= (-b)^n bF_0^{a,b} + \sum_{i=0}^{n-1}((-b)^{n-i} F_{2i+2}^{a,b} + (-b)^{n-i-1} bF_{2i+2}^{a,b}) + (-b)^0 F_{2n+2}^{a,b} = F_{2n+2}^{a,b}.\end{aligned}$$

This procedure proves the Jacobsthal version of Theorem 5(a) in [22]. The perspective offered by the initial proof in Theorem 5(a) is broader and highlights the interdependence of the two chapters of number theory: multiplicative arithmetic functions and Fibonacci numbers (in this regard, see also [21, Section 2]).

Corollary 9. *The k-Fibonacci version of Theorem 5(b)(ii) is the following:*

$$\sum_{i=0}^{n}(-1)^i L_{k,2(i+m)} = (-1)^n F_{k,2(n+m)+1} + F_{k,2m-1}.$$

This identity is fully consistent with the following formula given in [18, Theorem 3.4]:

$$\sum_{i=0}^{n}(-1)^i L_{k,2(i+m)} = \frac{1}{k^2+4}(L_{k,2m-2} - L_{k,2(n+m)} + L_{k,2m} - L_{k,2(n+m+1)}),$$

only if n is odd, since $L_{k,u} + L_{k,u+2} = (k^2+4)F_{k,u+1}$ (see Equation (11)). If the non-negative integer n is even then the two minus signs before $L_{k,2(n+m)}$ and $L_{k,2(n+m+1)}$ should have been plus.

3.2 Regarding even and odd subscript (a,b)-Fibonacci numbers

In this subsection, the (a,b)-Lucas numbers from Theorem 5 (a) are replaced by the (a,b)-Fibonacci numbers; and the new Fibonacci-Lucas relations are proven directly using Equation (10). Taking into consideration Corollary 4, we can observe that the origin of these relations is the Fibonacci-Lucas quotients (ii) of Theorem 3 and the quasi-distributive law (5). Then we consider particular cases referring to ordinary Fibonacci, Pell and Jacobsthal numbers.

Theorem 10. *For all non-negative integers n, and $\Delta = a^2 + 4b \neq 0$, we have*

(a) *(i)* $\sum_{i=0}^{n}(-b)^{n-i}F_{2i}^{a,b} = \frac{1}{\Delta}(L_{2n+1}^{a,b} + (-1)^{n+1}ab^n)$;

(ii) $\sum_{i=0}^{n}(-b)^{n-i}F_{2i+1}^{a,b} = \frac{1}{\Delta}(L_{2n+2}^{a,b} + (-1)^{n}2b^{n+1})$;

(b) *(i)* $\sum_{i=0}^{n}(-1)^{i}F_{2i} = \frac{1}{5}((-1)^nL_{2n+1} - 1)$; $\sum_{i=0}^{n}(-1)^{i}L_{2i} = (-1)^nF_{2n+1} + 1$;

(ii) $\sum_{i=0}^{n}(-1)^{i}F_{2i+1} = \frac{1}{5}((-1)^nL_{2n+2} + 2)$; $\sum_{i=0}^{n}(-1)^{i}L_{2i+1} = (-1)^nF_{2n+2}$;

(c) *(i)* $\sum_{i=0}^{n}(-1)^{i}P_{2i} = \frac{1}{8}((-1)^nQ_{2n+1} - 2)$; $\sum_{i=0}^{n}(-1)^{i}Q_{2i} = (-1)^nP_{2n+1} + 1$;

(ii) $\sum_{i=0}^{n}(-1)^{i}P_{2i+1} = \frac{1}{8}((-1)^nQ_{2n+2} + 2)$; $\sum_{i=0}^{n}(-1)^{i}Q_{2i+1} = (-1)^nP_{2n+2}$;

(d) *(i)* $\sum_{i=0}^{n}(-1)^{i}2^{n-i}J_{2i} = \frac{1}{9}((-1)^nj_{2n+1} - 2^n)$;
$\sum_{i=0}^{n}(-1)^{i}2^{n-i}j_{2i} = (-1)^nJ_{2n+1} + 2^n$;

(ii) $\sum_{i=0}^{n}(-1)^{i}2^{n-i}J_{2i+1} = \frac{1}{9}((-1)^nj_{2n+2} + 2^{n+2})$;
$\sum_{i=0}^{n}(-1)^{i}2^{n-i}j_{2i+1} = (-1)^nJ_{2n+2}$;
(for the alternating Jacobsthal-Lucas sums see [22]).

Proof.
(a) Using the formula $\Delta F_n^{a,b} = L_{n+1}^{a,b} + bL_{n-1}^{a,b}$ (see Equation (10)), where $\Delta = a^2 + 4b$, we get

(i)

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2i+1}^{a,b} &= \Delta\sum_{i=0}^{n}(-b)^{n-i}F_{2i}^{a,b} - b\sum_{i=0}^{n}(-b)^{n-i}L_{2i-1}^{a,b}\\ &= \Delta\sum_{i=0}^{n}(-b)^{n-i}F_{2i}^{a,b} + (-b)^{n+1}(-\frac{a}{b}) + \sum_{i=0}^{n}(-b)^{n-i}L_{2i+1}^{a,b} - L_{2n+1}^{a,b}.\end{aligned}$$

Hence

$$\Delta\sum_{i=0}^{n}(-b)^{n-i}F_{2i}^{a,b} = L_{2n+1}^{a,b} + (-1)^{n+1}ab^n,$$

which is the desired identity.

(ii)

$$\begin{aligned}\sum_{i=0}^{n}(-b)^{n-i}L_{2i}^{a,b} &= \frac{\Delta}{b}\sum_{i=0}^{n}(-b)^{n-i}F_{2i+1}^{a,b} - \frac{1}{b}\sum_{i=0}^{n}(-b)^{n-i}L_{2i+2}^{a,b} \\ &= \frac{\Delta}{b}\sum_{i=0}^{n}(-b)^{n-i}F_{2i+1}^{a,b} + \sum_{i=0}^{n}(-b)^{n-i}L_{2i}^{a,b} + (-1)^{n+1}2b^n - \frac{1}{b}L_{2n+2}^{a,b} \\ &= \frac{\Delta}{b}\sum_{i=0}^{n}(-b)^{n-i}F_{2i+1}^{a,b} - (-b)^n L_0^{a,b} + \sum_{i=0}^{n}(-b)^{n-i}L_{2i}^{a,b} - \frac{1}{b}L_{2n+2}^{a,b}.\end{aligned}$$

Hence

$$\frac{\Delta}{b}\sum_{i=0}^{n}(-b)^{n-i}F_{2i+1}^{a,b} = \frac{1}{b}L_{2n+2}^{a,b} + (-b)^n 2.$$

The proof of (a) is now complete.

The formulas (b), (c), and (d) are special cases of Theorems 10(a) and 5(a) for the Fibonacci (Lucas), Pell (Pell-Lucas), and Jacobsthal (Jacobsthal-Lucas) numbers, respectively. □

The established formulas are of Fibonacci-Lucas type in the sense that both numbers (Fibonacci and Lucas) are present in the formula. Of course, we can find more friendly formulas by giving up this restriction . For instance, by eliminating the (a,b)-Lucas number $L_{2n+1}^{a,b}$ from Theorem 10 (a)(i), we get the next corollary.

Corollary 11. *For (a,b)-Fibonacci numbers the following identity holds:*

$$\sum_{i=0}^{n}(-b)^{n-i}F_{2i}^{a,b} = F_n^{a,b}F_{n+1}^{a,b}.$$

Proof. Using Equations (4), (10), (12), and (13) we get

$$\begin{aligned}\Delta F_n^{a,b}F_{n+1}^{a,b} &= (L_{n+1}^{a,b} + bL_{n-1}^{a,b})F_{n+1}^{a,b} \\ &= F_{2n+2}^{a,b} + (L_{n+1}^{a,b} - aL_n^{a,b})F_{n+1}^{a,b} \\ &= F_{2n+2}^{a,b} + F_{2n+2}^{a,b} - aL_n^{a,b}F_{n+1}^{a,b} \\ &= F_{2n+2}^{a,b} + F_{2n+2}^{a,b} - a(F_{2n+1}^{a,b} + (-b)^n) \\ &= F_{2n+2}^{a,b} + bF_{2n}^{a,b} + (-1)^{n+1}ab^n \\ &= L_{2n+1}^{a,b} + (-1)^{n+1}ab^n.\end{aligned}$$

The proof is complete by Theorem 10 (a)(i). □

3.3 Convolution sums involving (a,b)-Fibonacci and (a,b)-Lucas numbers

As a new application of our approach, we start with a proof of a theorem using the Dirichlet inverse of the multiplicative arithmetic function $\mathcal{F}_{a,b}$. Then Kesava Menon quasi-distributive law leads to the Fibonacci-Lucas convolution sum. A specific case reveals the identities (1) and (2) mentioned in Section 1.

Theorem 12. *Let c be a non-zero complex number such that $c^2 - ac - b \neq 0$ and let n be a positive integer. We have*

$$\sum_{i=1}^{n} c^{n-i} F_i^{a,b} = \frac{c^{n+1} - cF_{n+1}^{a,b} - bF_n^{a,b}}{c^2 - ac - b} \tag{14}$$

Proof. If $f = \mathcal{F}_{a,b} * c^{\Omega}$ then $f(p^{n-1}) = \sum_{i=0}^{n-1} \mathcal{F}_{a,b}(p^i) c^{\Omega(p^{n-i})} = \sum_{i=1}^{n} c^{n-i} F_i^{a,b}$, and $c^{\Omega} = \mathcal{F}_{a,b}^{-1} * f$.

Now, Equation (14) holds for $n = 1$. We assume that $n > 1$. Invoking Equation (8) we have

$$\begin{aligned} c^n = c^{\Omega(p^n)} = (\mathcal{F}_{a,b}^{-1} * f)(p^n) &= \sum_{i=1}^{n+1} c^{n+1-i} F_i^{a,b} - a \sum_{i=1}^{n} c^{n-i} F_i^{a,b} - b \sum_{i=1}^{n-1} c^{n-1-i} F_i^{a,b} \\ &= c \sum_{i=1}^{n} c^{n-i} F_i^{a,b} + F_{n+1}^{a,b} - a \sum_{i=1}^{n} c^{n-i} F_i^{a,b} - \frac{b}{c} \sum_{i=1}^{n} c^{n-i} F_i^{a,b} + \frac{b}{c} F_n^{a,b}, \end{aligned}$$

and the proof is finished. □

In what follows, a fundamental role is played by the Kesava Menon quasi-distributive law (5).

Theorem 13. *We have*

$$\sum_{i=1}^{n} L_{n-i}^{a,b} F_i^{c,d} = \frac{\alpha^{n+1} - \alpha F_{n+1}^{c,d} - dF_n^{c,d}}{\alpha^2 - c\alpha - d} + \frac{\beta^{n+1} - \beta F_{n+1}^{c,d} - dF_n^{c,d}}{\beta^2 - c\beta - d}, \tag{15}$$

if any of the roots α, β of the quadratic equation $x^2 - ax - b = 0$ (see (7)) are not zeros of $x^2 - cx - d$.

Proof. Using the quasi-distributive law (5) we get

$$(\mathcal{F}_{c,d} * \mathcal{L}_{a,b}) \sqcup \mathcal{F}_{c,d} = (\mathcal{F}_{c,d} * \alpha^{\Omega}) \sqcup (\mathcal{F}_{c,d} * \beta^{\Omega}).$$

Since $(\mathcal{F}_{c,d} * \mathcal{L}_{a,b})(p^{n-1}) = \sum_{i=1}^{n} L_{n-i}^{a,b} F_i^{c,d} - F_n^{c,d}$ we have

$$[(\mathcal{F}_{c,d} * \mathcal{L}_{a,b}) \sqcup \mathcal{F}_{c,d}](p^{n-1}) = \sum_{i=1}^{n} L_{n-i}^{a,b} F_i^{c,d}.$$

Invoking Equation (14), it follows that

$$[(\mathcal{F}_{c,d} * \alpha^{\Omega}) \sqcup (\mathcal{F}_{c,d} * \beta^{\Omega})](p^{n-1}) = \sum_{i=1}^{n} \alpha^{n-i} F_i^{c,d} + \sum_{i=1}^{n} \beta^{n-i} F_i^{c,d}$$
$$= \frac{\alpha^{n+1} - \alpha F_{n+1}^{c,d} - dF_n^{c,d}}{\alpha^2 - c\alpha - d} + \frac{\beta^{n+1} - \beta F_{n+1}^{c,d} - dF_n^{c,d}}{\beta^2 - c\beta - d},$$

and the proof is finished. □

In the case of $a = c$ and $b \neq d$, Equation (15) is significantly simplified. We thus arrive (via the Kesava Menon quasi-distributive law) to the Equations (1) and (2) mentioned at the beginning of the paper.

Corollary 14. *If $b \neq d$, then*

(a)

$$\sum_{i=1}^{n} L_{n-i}^{a,b} F_i^{a,d} = \frac{L_{n+1}^{a,b} - L_{n+1}^{a,d}}{b-d} = \sum_{i=1}^{n} L_{n-i}^{a,d} F_i^{a,b}; \tag{16}$$

(b) *(i)* ([8, Section 2], [2, Theorem 1.1] *and* [15, (2.6), (2.7)]*—see also (1) and (2))*

$$\sum_{i=1}^{n} L_{n-i} J_i = j_{n+1} - L_{n+1} = \sum_{i=1}^{n} j_{n-i} F_i;$$

(ii)

$$2\sum_{i=1}^{n} P_i = \frac{1}{2}(Q_{n+1} - 2) = \sum_{i=1}^{n} iQ_{n-i};$$

(iii)

$$\sum_{i=1}^{n} L_{2(n-i)}(2^i - 1) = L_{2n+2} - 2^{n+1} - 1 = \sum_{i=1}^{n} (2^{n-i} + 1) F_{2i}.$$

Proof. (a) Using Equation (15), we get

$$\sum_{i=1}^{n} L_{n-i}^{a,b} F_i^{a,d} = \frac{\alpha^{n+1} - \alpha F_{n+1}^{a,d} - dF_n^{a,d}}{\alpha^2 - a\alpha - d} + \frac{\beta^{n+1} - \beta F_{n+1}^{a,d} - dF_n^{a,d}}{\beta^2 - a\beta - d}$$
$$= \frac{\alpha^{n+1} + \beta^{n+1} - aF_{n+1}^{a,d} - 2dF_n^{a,d}}{b-d} = \frac{L_{n+1}^{a,b} - L_{n+1}^{a,d}}{b-d}$$
$$= \frac{L_{n+1}^{a,d} - L_{n+1}^{a,b}}{d-b} = \sum_{i=1}^{n} L_{n-i}^{a,d} F_i^{a,b}.$$

(b) It is straightforward to see that (i), (ii), and (iii) follow immediately from (a).

(i) The ordinary Fibonacci and Lucas sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$, respectively, are recovered when $a = b = 1$, and the Jacobsthal and Jacobsthal-Lucas sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$, respectively, are recovered when $a = 1$ and $b = 2$.

(ii) The sequence of non-negative integers and the constant sequence $2, 2, 2, \ldots$ are the $(2, -1)$-Fibonacci and the $(2, -1)$-Lucas sequences, respectively. The Pell and Pell-Lucas sequences are the $(2, 1)$-Fibonacci and the $(2, 1)$-Lucas sequences, respectively.

(iii). The sequences $(2^n - 1)$ and $(2^n + 1)$ are the Mersenne sequence (which is the $(3, -2)$-Fibonacci sequence) and the Mersenne-Lucas sequence (the $(3, -2)$-Lucas sequence), respectively. The even-numbered Fibonacci and Lucas sequences are the $(3, -1)$-Fibonacci and the $(3, -1)$-Lucas sequences, respectively. □

3.4 The Kesava Menon quasi-distributive law again

As a final application to our approach, we prove the (a, b)-Fibonacci convolution sum using again the quasi-distributive law (5).

Theorem 15. ([24, *Theorem* 4]) *For (a, b)-Fibonacci and (a, b)-Lucas numbers, and $\Delta = a^2 + 4b \neq 0$, we have*

$$\Delta \sum_{i=0}^{n} F_i^{a,b} F_{n-i}^{a,b} = (n + 1)L_n^{a,b} - 2F_{n+1}^{a,b}. \tag{17}$$

Proof. Let $\widetilde{\mathcal{F}}_{a,b}$ be the multiplicative arithmetic function defined by

$$\widetilde{\mathcal{F}}_{a,b} = (\alpha^\Omega * \beta^\Omega \mu) \sqcup (\beta^\Omega * \alpha^\Omega \mu).$$

Note that if f is a completely multiplicative arithmetic function then the usual product $f\mu$ is the inverse element of f in the commutative group $(\mathcal{M}, *)$. Thus, using the quasi-distributive law (5) we get

$$\mathcal{F}_{a,b} * \widetilde{\mathcal{F}}_{a,b} \sqcup \mathcal{F}_{a,b} = (\alpha^\Omega * \alpha^\Omega) \sqcup (\beta^\Omega * \beta^\Omega). \tag{18}$$

We observe that if p is a prime number then $\widetilde{\mathcal{F}}_{a,b}(p) = 0$, and if $n > 1$ then

$$\widetilde{\mathcal{F}}_{a,b}(p^n) = \alpha^n - \alpha^{n-1}\beta + \beta^n - \beta^{n-1}\alpha = \alpha^n + \beta^n - \alpha\beta(\alpha^{n-2} + \beta^{n-2}) = L_n^{a,b} + bL_{n-2}^{a,b}.$$

So, by Equation (10), we have

$$\widetilde{\mathcal{F}}_{a,b}(p^n) = \begin{cases} 1, & \text{if } n = 0; \\ \Delta F_{n-1}^{a,b}, & \text{if } n > 0. \end{cases}$$

Now, since

$$\begin{aligned}[(\mathcal{F}_{a,b} * \widetilde{\mathcal{F}}_{a,b}) \sqcup \mathcal{F}_{a,b}](p^n) &= \sum_{i=0}^{n} \mathcal{F}_{a,b}(p^{n-i})\widetilde{\mathcal{F}}_{a,b}(p^i) + \mathcal{F}_{a,b}(p^n) \\ &= F_{n+1}^{a,b} + \Delta \sum_{i=0}^{n} F_i^{a,b} F_{n-i}^{a,b} + F_{n+1}^{a,b},\end{aligned}$$

and

$$[(\alpha^\Omega * \alpha^\Omega) \sqcup (\beta^\Omega * \beta^\Omega)](p^n) = (n+1)\alpha^n + (n+1)\beta^n = (n+1)L_n^{a,b},$$

the proof is complete by (18). □

4 Acknowledgment

The author would like to thank the referee for carefully reading the paper and for providing many useful comments and suggestions.

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2020 *Mathematics Subject Classification*: Primary 11B39; Secondary 11A25.
Keywords: (a, b)-Fibonacci sequence, (a, b)-Lucas sequence, Fibonacci-Lucas relation, multiplicative arithmetic function, Dirichlet convolution, unitary convolution.

(Concerned with sequences A000032, A000045, A000104, A000129, A001045, A002203, and A014551.)

Received July 31 2024; revised versions received August 19 2024; January 16 2025; January 22 2025; February 13 2025. Published in *Journal of Integer Sequences*, February 13 2025.

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