Journal of Integer Sequences, Vol. 28 (2025),
Article 25.1.1

Two New Integer Sequences Related to Crossroads and Catalan Numbers

Julien Rouyer and Alain Ninet
Laboratoire de Mathématiques de Reims (LMR) - CNRS UMR 9008
Université de Reims Champagne-Ardenne
U.F.R. Sciences Exactes et Naturelles
Moulin de la Housse - BP 1039
51687 Reims cedex 2
France
julien.rouyer@univ-reims.fr
alain.ninet@univ-reims.fr

Abstract

The marriageable singles sequence represents the number of noncrossing partitions of the finite set $\{1, \ldots, n\}$ in which some pair of singleton blocks can be joined while remaining noncrossing. The lonely singles sequence represents the number of all the other noncrossing partitions of the finite set $\{1, \ldots, n\}$ and is the difference between the Catalan numbers sequence and the marriageable singles sequence. The first 16 terms of these sequences are given, as well as some of their properties. These sequences appear when one wants to count the number of ways to cross certain road intersections simultaneously.

1 Introduction

The number of noncrossing partitions of the finite set $\{1, \ldots, n\}$ (with n a positive integer) is very well known to be the Catalan number C_n. See, for example, Stanley [2, entry 159, p. 43] and Roman [4, pp. 51–60] for a quick introduction to Catalan numbers and noncrossing partitions. Noncrossing partitions have been hugely studied, since at least Becker [7], where they are called *planar rhyme schemes* but their systematic study began with Kreweras [5] and Poupard [6]. Simion [8] presents a summary of the related results available in 2000 and some further work can be found in McCammond [9], Callan [10] and Kim [11].

The study of combinatorial properties of crossroads led us to determine the number of noncrossing partitions such that no pair of singleton blocks $\{i\}$ and $\{j\}$ (with $i \neq j$) can be merged into the pair $\{i, j\}$ while the partition remains noncrossing. These noncrossing partitions appear when one wants to determine the number of possible manners to cross simultaneously a road intersection in which entries and exits alternate, with the constraint that U-turns are prohibited. For a quick introduction to road intersection crossing management for intelligent vehicles, see Rouyer et al. [1] and Bai et al. [3].

In the remaining part of this section and in the next one, we give definitions and quick examples. We then prove some properties of both marriageable singles and lonely singles sequences. At the end, we give the first values of those sequences and formulate several conjectures concerning them.

In all the text, we let $\mathbb{N}$ denote the set of natural numbers, including 0.

Definition 1. For all $n \in \mathbb{N}$, we let $[n]$ denote the n-set $\{1, \ldots, n\}$. In particular, $[0] = \emptyset$.

Definition 2. Let n be a non-negative integer and let $\pi = \{A_1, \ldots, A_k\}$ be a partition of $[n]$ (i.e., $\cup_{i=1}^{k} A_i = [n]$ and for all $1 \leq i < j \leq k, A_i \cap A_j = \emptyset$ and for all $1 \leq i \leq k, A_i \neq \emptyset$), this partition π is said to be a

- *crossing partition* if there exist $1 \leq i \leq k$ and $1 \leq j \leq k$, $i \neq j$ and $a < b \in A_i$ and $c < d \in A_j$, such as $a < c < b < d$;
- *noncrossing partition* if, for all $1 \leq i < j \leq k$, for all $a < b \in A_i$, for all $c < d \in A_j$, we have

$$\begin{aligned} & a < b < c < d, \\ & \text{or } c < d < a < b, \\ & \text{or } a < c < d < b, \\ & \text{or } c < a < b < d. \end{aligned}$$

Definition 3. A noncrossing partition π of $[n]$ is called *marriageable singles partition* if there exists a pair of singleton blocks $\{i\}$ and $\{j\}$ in π that can be joined, the partition thus obtained remaining noncrossing.

More precisely, let $\pi = \{ A_1, \ldots, A_k \}$ be a noncrossing partition of $[n]$ with at least two singleton blocks $A_1 = \{i\}$ and $A_2 = \{j\}$. Then π is a marriageable singles partition iff $\pi' = \{ \{i, j\}, A_3, \ldots, A_k \}$ is a noncrossing partition of $[n]$.

Conversely, a noncrossing partition of $[n]$ is called *lonely singles partition* if it is not a marriageable singles partition.

Let M_n and L_n be the number of marriageable singles and lonely singles partitions of $[n]$ respectively. The *marriageable singles* and *lonely singles sequences* are $(M_n)_{n \geq 0}$ and $(L_n)_{n \geq 0}$ respectively.

Remark 4. A partition that contains at most one singleton block is clearly a lonely singles partition. In an equivalent way, a marriageable singles partition contains at least two singleton blocks.

Example 5. The set [4] has 5 marriageable singles partitions, which are

$$\{\,\{1\},\{2\},\{3\},\{4\}\,\},\qquad \{\,\{1\},\{2\},\{3,4\}\,\},\qquad \{\,\{1\},\{2,3\},\{4\}\,\},$$
$$\{\,\{1,4\},\{2\},\{3\}\,\},\qquad \{\,\{1,2\},\{3\},\{4\}\,\}.$$

For example, $\pi = \{\,\{1,2\},\{3\},\{4\}\,\}$ is a marriageable singles partition because the singleton blocks $\{3\}$ and $\{4\}$ can be joined to give the noncrossing partition $\pi' = \{\,\{1,2\},\{3,4\}\,\}$.

The set [4] has 9 lonely singles partitions, which are

$$\{\,\{1,2,3,4\}\,\},\qquad \{\,\{1,2,3\},\{4\}\,\},\qquad \{\,\{1,2,4\},\{3\}\,\},$$
$$\{\,\{1,3,4\},\{2\}\,\},\qquad \{\,\{1\},\{2,3,4\}\,\},\qquad \{\,\{1,2\},\{3,4\}\,\},$$
$$\{\,\{1,4\},\{2,3\}\,\},\qquad \{\,\{1,3\},\{2\},\{4\}\,\},\qquad \{\,\{1\},\{2,4\},\{3\}\,\}.$$

The first seven have no pair of singleton blocks and are clearly not marriageable singles partitions. The last two have only one pair of singleton blocks, but after merging it, they both give $\{\,\{1,3\},\{2,4\}\,\}$ which is a crossing partition.

Lemma 6. *Let C_n (for $n \geq 0$) denote the number of noncrossing partitions of $[n]$. Then we have $C_n = L_n + M_n$.*

Proof. As the set of noncrossing partitions is the disjoint union of the sets of lonely singles and marriageable singles partitions, the result follows immediately. □

Remark 7. It is well known that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. The sequence $(C_n)_{n\geq 0}$ is referenced as A000108 in the On-Line Encyclopedia of Integer Sequences [12].

Remark 8. The unique partition of $[0] = \emptyset$ is the empty partition $\emptyset$. This partition is noncrossing (the first Catalan number is $C_0 = 1$) and it is a lonely singles partition.

2 Standard road intersection

Noncrossing complete matchings of $2n$ points lying on a line and noncrossing complete set of chords are two of the many standard combinatorial objects counted by the Catalan numbers. Noncrossing complete matchings and noncrossing complete set of chords are in natural bijection with the noncrossing partitions, as explained by Stanley [2, entries 59 and 61, p. 28].

Here we introduce some notions about road intersections corresponding to noncrossing sets of chords.

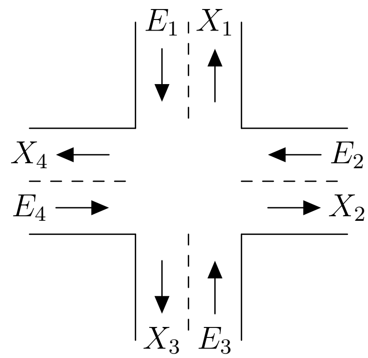

Figure 1: A Standard Road Intersection of size $n = 4$.

Definition 9. A road intersection with n entries and n exits that alternate is called a *standard road intersection of size* n.

Let $E_1, \ldots, E_n$ denote the entries and $X_1, \ldots, X_n$ denote the exits of a standard road intersection of size n. These entries and exits are numbered clockwise (Figure 1 gives such a representation of a standard road intersection of size $n = 4$).

We use a bipartite graph to graphically represent a way to simultaneously cross a standard road intersection (see Figures 2 and 3 for two examples with $n = 4$). In these graphs, black vertices represent entries and white vertices represent exits. Each edge represents the crossing of the intersection by a vehicle going from an entry to an exit.

Each entry can be connected to each exit, unless restrictions are indicated.

Definition 10. For a given standard road intersection, an edge starting from one entry E_i and ending at one exit X_j is a *lane*. We let E_iX_j denote such a lane.

Definition 11. For a given standard road intersection, a lane of the type E_iX_i, i.e., an edge starting from one entry E_i and ending at exit X_i is a *U-turn*.

Definition 12. A standard road intersection of size n where U-turns are forbidden is a *restricted standard road intersection*.

Definition 13. For a given standard road intersection, a *Maximal Set of Lanes*, abbreviated *MSL*, is a set of noncrossing lanes (i.e., each pair of lanes have no common point) such that any additional lane would cross at least one of them (i.e., would have a common point with at least one of them).

Remark 14. An MSL corresponds to a set of n nonintersecting chords (or a noncrossing complete matching on $2n$ vertices); see Stanley [2, entries 59 and 61, p. 28]. Three examples of MSL are given by Figures 2, 3 and 6.

Definition 15. For a given standard road intersection, an MSL is said to be *absolute* when it does not contain two U-turns E_iX_i and E_jX_j (with $i \neq j$) that can be changed into two lanes E_iX_j and E_jX_i to give another MSL.

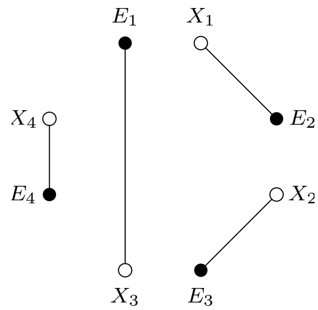

Figure 2: A bipartite graph associated with the intersection represented in Figure 1, with an example of absolute MSL corresponding to the noncrossing partition $\{\{1,2,3\},\{4\}\}$.

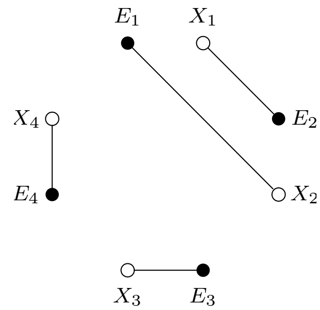

Figure 3: A bipartite graph associated with the intersection represented in Figure 1, with an example of a nonabsolute MSL corresponding to the noncrossing partition $\{\{1,2\},\{3\},\{4\}\}$.

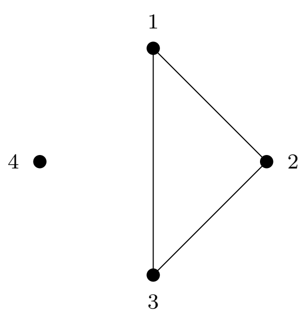

Figure 4: Simplified graph of Figure 2, showing explicitly the noncrossing lonely singles partition $\{\{1,2,3\},\{4\}\}$.

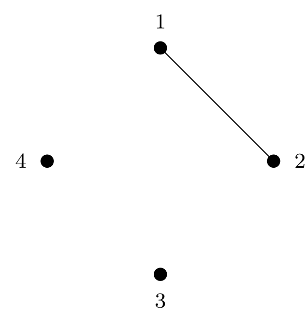

Figure 5: Simplified graph of Figure 3, showing explicitly the noncrossing marriageable singles partition $\{\{1,2\},\{3\},\{4\}\}$.

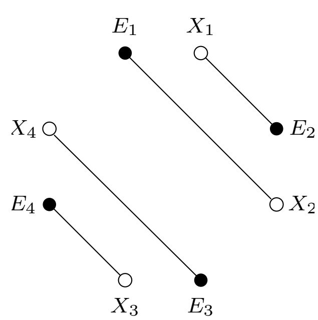

Figure 6: Modification of Figure 3 to obtain an absolute MSL corresponding to the noncrossing partition $\{\{1,2\},\{3,4\}\}$.

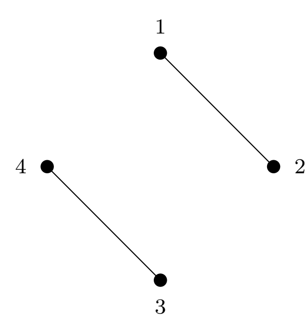

Figure 7: Simplified graph of Figure 6, showing explicitly the noncrossing lonely singles partition $\{\{1,2\},\{3,4\}\}$.

Lemma 16. *For a given standard road intersection of size n, the set of MSL is in one-to-one correspondence with the set of noncrossing partitions of $[n]$. The number of MSL of a standard road intersection of size n is equal to the Catalan number C_n.*

Proof. As a MSL can be seen as a set of n nonintersecting chords joining $2n$ points, this result is very well known. See Stanley [2, entry 59, p. 28]. □

The following three corollaries are just reinterpretations of the lonely singles and marriageable singles definitions in the language of road intersections.

Corollary 17. *The number of nonabsolute MSL of a standard road intersection of size n is M_n.*

Corollary 18. *The number of absolute MSL of a standard road intersection of size n is L_n.*

Corollary 19. *The number of MSL for a restricted standard road intersection of size n is L_n.*

3 Properties

Proposition 20. *Both sequences $(L_n)_{n\geq 0}$ and $(M_n)_{n\geq 0}$ are increasing: $L_n < L_{n+1}$ for all $n \geq 2$ and $M_n < M_{n+1}$ for all $n \geq 3$.*

Proof. We build a simple injective map f_n from the set LS_n of the lonely singles partitions of $[n]$ to the set LS_{n+1} of the lonely singles partitions of $[n+1]$ by merging the singleton $\{n+1\}$ to the unique element A_1 of a partition π that contains the number 1 (one could equally prefer to use the number n instead of the number 1: the idea is to attach the number $n+1$ to one of its two direct neighbours 1 or n):

$$\begin{gathered} \mathrm{LS}_n \to \mathrm{LS}_{n+1} \\ f_n\colon \pi = \{\, A_1, \ldots, A_k \,\} \mapsto \{\, A_1 \cup \{n+1\}, \ldots, A_k \,\}, \end{gathered}$$

where $1 \in A_1$ and $A_1 \cup \cdots \cup A_k = [n]$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. For example,

$$\begin{gathered} f_3\,(\{\, \{1,2\}, \{3\} \,\}) = \{\, \{1,2,4\}, \{3\} \,\}, \\ f_4\,(\{\, \{1\}, \{2,4\}, \{3\} \,\}) = \{\, \{1,5\}, \{2,4\}, \{3\} \,\}. \end{gathered}$$

The map f_n is clearly injective and every lonely singles partition π gives a lonely singles partition $f_n(\pi)$. As L_n is the cardinality of LS_n and L_{n+1} is the cardinality of LS_{n+1}, we obtain $L_n \leq L_{n+1}$.

In a similar way, we build a simple injective map g_n from the set MS_n of the marriageable singles partitions of $[n]$ to the set MS_{n+1} of the marriageable singles partitions of $[n+1]$ by adding the singleton block $\{n+1\}$ to any partition π:

$$\begin{aligned} \mathrm{MS}_n &\to \mathrm{MS}_{n+1} \\ g_n \colon \pi &\mapsto \pi \cup \{\, \{n+1\} \,\}. \end{aligned}$$

For example,

$$g_4\,(\{\,\{1\},\{2,3\},\{4\}\,\}) = \{\,\{1\},\{2,3\},\{4\},\{5\}\,\}.$$

The map g_n is clearly injective, and a pair of marriageable singletons $\{i\}$ and $\{j\}$ of $\pi \in \mathrm{MS}_n$ remains marriageable singletons as elements of $g_n(\pi) \in \mathrm{MS}_{n+1}$. As M_n is the cardinality of MS_n and M_{n+1} is the cardinality of MS_{n+1}, we obtain $M_n \leq M_{n+1}$.

More precisely, we have $L_n < L_{n+1}$ for all $n \geq 2$ and $M_n < M_{n+1}$ for all $n \geq 3$: it is easy to build a lonely singles and a marriageable singles partitions of $[n+1]$ that are not in the images of the maps f_n and g_n, e.g., respectively $\{\,[n],\{n+1\}\,\}$ and $\{\,\{1\},\dots,\{n-1\},\{n,n+1\}\,\}$. □

Corollary 21. *We have*

$$\lim_{n\to+\infty} M_n = \lim_{n\to+\infty} L_n = +\infty.$$

Proof. This is an immediate consequence of Proposition 20. □

Proposition 22. *For all $n \geq 0$, we have $C_n + 3M_n \leq M_{n+2}$.*

Proof. We build four simple injective maps h_n, i_n, j_n and k_n with disjoint images included in the set MS_{n+2} of the marriageable singles partitions of $[n+2]$.

- The map h_n is defined on the set NC_n of all noncrossing partitions of $[n]$ by adding both the singleton blocks $\{n+1\}$ and $\{n+2\}$ to a noncrossing partition π:

$$\begin{aligned} \mathrm{NC}_n &\to \mathrm{MS}_{n+2} \\ h_n \colon \pi &\mapsto \pi \cup \{\, \{n+1\},\{n+2\} \,\}. \end{aligned}$$

  The maps i_n, j_n and k_n are defined on the set MS_n of the marriageable singles partitions of $[n]$.

- The map i_n adds the pair $\{n+1, n+2\}$ to a marriageable singles partition π:

$$\begin{aligned} \mathrm{MS}_n &\to \mathrm{MS}_{n+2} \\ i_n \colon \pi &\mapsto \pi \cup \{\, \{n+1, n+2\} \,\}. \end{aligned}$$

- The map j_n merges the singleton block $\{n+2\}$ with the unique element A_1 of a marriageable singles partition π that contains the number 1 and adds the singleton block $\{n+1\}$ to π:

$$\begin{aligned} &\mathrm{MS}_n \to \mathrm{MS}_{n+2} \\ j_n \colon \pi = \{\, A_1, \ldots, A_k \,\} &\mapsto \{\, A_1 \cup \{n+2\}, A_2, \ldots, A_k, \{n+1\} \,\}, \end{aligned}$$

  where $1 \in A_1$, $A_1 \cup \cdots \cup A_k = [n]$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

- The map k_n merges the singleton block $\{n+1\}$ with the unique element A_1 of a marriageable singles partition π that contains the number n and adds the singleton block $\{n+2\}$ to π:

$$\begin{aligned} &\mathrm{MS}_n \to \mathrm{MS}_{n+2} \\ k_n : \pi = \{\, A_1, \ldots, A_k \,\} &\mapsto \{\, A_1 \cup \{n+1\}, A_2, \ldots, A_k, \{n+2\} \,\}, \end{aligned}$$

  where $n \in A_1$, $A_1 \cup \cdots \cup A_k = [n]$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

For example,

$$\begin{aligned} h_4\,(\{\,\{1,2,3\},\{4\}\,\}) &= \{\,\{1,2,3\},\{4\},\{5\},\{6\}\,\}, \\ i_4\,(\{\,\{1,2\},\{3\},\{4\}\,\}) &= \{\,\{1,2\},\{3\},\{4\},\{5,6\}\,\}, \\ j_4\,(\{\,\{1,2\},\{3\},\{4\}\,\}) &= \{\,\{1,2,6\},\{3\},\{4\},\{5\}\,\}, \\ k_4\,(\{\,\{1,2\},\{3\},\{4\}\,\}) &= \{\,\{1,2\},\{3\},\{4,5\},\{6\}\,\}. \end{aligned}$$

We have the immediate following properties:

- $h_n(\pi)$ is a marriageable singles partition of $[n+2]$ for all noncrossing partition π of $[n]$,
- $i_n(\pi), j_n(\pi)$ and $k_n(\pi)$ are marriageable singles partitions of $[n+2]$ for all marriageable singles partition π of $[n]$ (the marriageable pairs can change depending on the map selected),
- h_n, i_n, j_n and k_n are injective maps; thus,
  - NC_n and $h_n(\mathrm{NC}_n)$ have the same cardinality C_n,
  - MS_n and $i_n(\mathrm{MS}_n)$ and $j_n(\mathrm{MS}_n)$ and $k_n(\mathrm{MS}_n)$ have the same cardinality M_n,
- the sets $h_n(\mathrm{NC}_n)$, $i_n(\mathrm{MS}_n)$, $j_n(\mathrm{MS}_n)$ and $k_n(\mathrm{MS}_n)$ are disjoint; thus,

$$h_n(\mathrm{NC}_n) \sqcup i_n(\mathrm{MS}_n) \sqcup j_n(\mathrm{MS}_n) \sqcup k_n(\mathrm{MS}_n) \subset \mathrm{MS}_{n+2},$$

- as M_{n+2} is the cardinality of MS_{n+2}, we obtain $C_n + 3M_n \leq M_{n+2}$.

□

Definition 23. For all nonnegative integers n, m and k, let $\mathrm{NC}_{n,m,k}$ be the number of noncrossing partitions of $[n]$ in m classes with k singleton blocks.

Proposition 24. *For all non-negative integers n and m with $(n,m) \neq (0,0)$, the number of noncrossing partitions of $[n]$ in m classes with no singleton block is*

$$\mathrm{NC}_{n,m,0} = \frac{1}{n-m+1}\binom{n}{m}\binom{n-m-1}{m-1}.$$

When $(n,m) \neq (1,1)$, the number of noncrossing partitions of $[n]$ in m classes with exactly one singleton block is

$$\mathrm{NC}_{n,m,1} = \binom{n}{m-1}\binom{n-m-1}{m-2}.$$

Proof. Poupard [6] proved that, when $n \geq 1$ and $m \geq 1$, the number of noncrossing partitions of $[n]$ in m classes with no singleton block is $\mathrm{NC}_{n,m,0} = \frac{1}{n-m+1}\binom{n}{m}\binom{n-m-1}{m-1}$, and $n \geq 2m$. When $n = 0$ and $m \geq 1$ or $n \geq 1$ and $m = 0$ or $m \geq \lfloor \frac{n}{2} \rfloor + 1$, this equality holds and gives a number of such partitions equal to 0.

When $n \geq 1$ and $m \geq 1$, the set of noncrossing partitions of $[n]$ in m classes with exactly one singleton block is clearly in one-to-one correspondence with the set of couples (π, i), where π is a noncrossing partition of $[n-1]$ in $m-1$ classes with no singleton block and $1 \leq i \leq n$. Then, when $(n,m) \neq (1,1)$, we obtain that the number of noncrossing partitions of $[n]$ in m classes with exactly one singleton block is $\mathrm{NC}_{n,m,1} = n \times \mathrm{NC}_{n-1,m-1,0} = \frac{n}{n-m+1}\binom{n-1}{m-1}\binom{n-m-1}{m-2} = \binom{n}{m-1}\binom{n-m-1}{m-2}$. □

Remark 25. Here are a few details on specific cases:

- the empty partition is the unique noncrossing partition of the empty set $[0]$ in 0 class with no singleton block. Thus, $\mathrm{NC}_{0,0,0} = 1$,
- the set $[1]$ has a unique noncrossing partition. This partition has $m = 1$ class and it is a singleton block. Thus, $\mathrm{NC}_{1,1,1} = 1$ and $\mathrm{NC}_{1,m,k} = 0$ when $(m,k) \neq (1,1)$ and, in particular, $\mathrm{NC}_{1,1,0} = 0$,
- for all positive integer n, $[n]$ has no noncrossing partition in $m = 0$ class. Thus, for all non-negative integer k, $\mathrm{NC}_{n,0,k} = 0$ and in particular $\mathrm{NC}_{n,0,0} = 0$.

Proposition 26. *For all $n \geq 2$,*

$$L_n \geq \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n-m+1}\binom{n}{m}\binom{n-m-1}{m-1} + \sum_{m=2}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{m-1}\binom{n-m-1}{m-2}.$$

Proof. As every noncrossing partition with at most one singleton block is a lonely singles partition, we have $L_n \geq \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \mathrm{NC}_{n,m,0} + \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \mathrm{NC}_{n,m,1}$ and the result follows, using Proposition 24. □

Proposition 27. *For all $n \geq 3$,*

$$M_n \geq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\sum_{m=0}^{\lfloor \frac{n+i-j-1}{2} \rfloor} \frac{1}{n+i-j-m} \binom{n+i-j-1}{m} \binom{n+i-j-m-2}{m-1} \right.$$
$$\left. \times \sum_{m=0}^{\lfloor \frac{j-i-1}{2} \rfloor} \frac{1}{j-i-m} \binom{j-i-1}{m} \binom{j-i-m-2}{m-1} \right).$$

Proof. Let π be a marriageable singles partition of $[n]$ with exactly two singleton blocks $\{i\}$ and $\{j\}$ (with $i < j$). The set of such partitions π is clearly in one-to-one correspondence with the set of triples $(\{i,j\}, \pi_1, \pi_2)$ where $1 \leq i < j \leq n$ and π_1 and π_2 are noncrossing partitions of the sets $[n+i-j-1]$ and $[j-i-1]$ respectively, with no singleton blocks. Thus,

$$M_n \geq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\sum_{m=0}^{\lfloor \frac{n+i-j-1}{2} \rfloor} \mathrm{NC}_{n+i-j-1,m,0} \sum_{m=0}^{\lfloor \frac{j-i-1}{2} \rfloor} \mathrm{NC}_{j-i-1,m,0} \right).$$

□

Remark 28. The previous inequality can be improved, considering three or more singleton blocks that can all be joined.

4 Known values and conjectures

Very few values of the sequences L_n and M_n are known at the moment. They are given in Table 1. L_n and M_n are, respectively, sequences A363448 and A363449 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [12].

The values of M_n were obtained with the algorithms available in the accompanying files, provided in pdf and ipynb formats. The complexity of these algorithms is exponential: they need to be improved and to be run with a more powerful computer to obtain values of L_n and M_n for $n \geq 15$. Calculations were performed under Python 3.7.3 using Jupyter Notebook 5.7.6 with a 3.19 GHz i7-8700 processor and 32 GB RAM.

n	L_n	L_n/L_{n-1}	M_n	M_n/M_{n-1}	C_n	M_n/L_n	M_n/C_n	Computing time for M_n
0	1		0		1	0	0	0s
1	1	1	0		1	0	0	0s
2	1	1	1		2	1	0.5	0s
3	4	4	1	1	5	0.25	0.2	0s
4	9	2.25	5	5	14	0.56	0.36	263μs
5	26	2.89	16	3.2	42	0.62	0.38	657μs
6	77	2.96	55	3.44	132	0.71	0.42	1.8ms
7	232	3.01	197	3.58	429	0.85	0.46	7.5ms
8	725	3.13	705	3.58	1430	0.97	0.49	24ms
9	2299	3.17	2563	3.64	4862	1.11	0.53	90ms
10	7415	3.22	9381	3.67	16796	1.27	0.56	507ms
11	24223	2.99	34563	3.90	58786	1.66	0.62	3.7s
12	79983	3.29	128029	3.69	208012	1.86	0.65	39.4s
13	266553	3.23	476347	3.75	742900	2.16	0.68	15m, 36s
14	895333	3.25	1779107	3.76	2674440	2.50	0.71	6h, 2m, 40s
15	3028093	3.31	6666752	3.75	9694845	2.83	0.74	6days, 7h, 13m

Table 1: Values of L_n, M_n, C_n, L_n/L_{n-1}, M_n/M_{n-1}, M_n/L_n and M_n/C_n (given with 2 digits), with computing time of M_n, for all $n \leq 15$.

Conjecture 29. We conjecture the five following propositions:

$$\text{for all } n \in \mathbb{N}, n \geq 9 \Rightarrow M_n > L_n, \tag{1}$$

$$\lim_{n\to+\infty} \frac{M_n}{L_n} = +\infty, \tag{2}$$

$$\lim_{n\to+\infty} \frac{M_n}{C_n} = 1, \tag{3}$$

$$\lim_{n\to+\infty} \frac{L_n}{C_n} = 0, \tag{4}$$

$$\lim_{n\to+\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to+\infty} \frac{L_{n+1}}{L_n} = 4. \tag{5}$$

Remark 30. Obviously, conjecture (2) implies conjecture (1). Table 1 shows that $\frac{M_n}{L_n}$ grows slowly and conjecture 2 may stand. More clearly, Table 1 shows that $\frac{M_n}{C_n}$ seems to grow quite quickly and lets think that conjecture 3 stands. As $C_n = M_n + L_n$, conjectures (2), (3) and (4) are clearly equivalent. It is well known and easy to prove that $\lim_{n\to+\infty} \frac{C_{n+1}}{C_n} = 4$. Table 1 shows that the similar limits given in conjecture (5) look very realistic as well.

5 Acknowledgements

We would like to thank Frédéric Blanchard and Rupert Wei Tze Yu for their constant help. Our deepest gratitude goes to the anonymous referee who gave us directions to improve this paper.

References

[1] J. Rouyer, A. Ninet, H. Fouchal and A. Keziou, A road intersection control in urban intelligent transportation systems, in *ICC 2022: Proc. IEEE International Conference on Communications*, Seoul, South Korea, May 2022, pp. 3562–3567.

[2] S. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

[3] S. Bai and X. Bai, A general framework for intersection traffic control with backpressure routing, *IEEE Access* **9** (2021), 102125–102136.

[4] S. Roman, *An Introduction to Catalan Numbers*, Birkhauser, 2015.

[5] G. Kreweras, Sur les partitions non croisées d'un cycle, *Discrete Math.* **1** (1972), 333–350.

[6] Y. Poupard, Étude et dénombrement parallèles des partitions non-croisées d'un cycle et des découpages d'un polygone convexe, *Discrete Math.* **2** (1972), 279–288.

[7] H. W. Becker, Planar rhyme schemes, *Bull. Amer. Math. Soc.* **58** (1952), 39.

[8] R. Simion, Noncrossing partitions, *Discrete Math.* **217** (2000), 367–409.

[9] J. McCammond, Noncrossing partitions in surprising locations, 2006. Available at https://arxiv.org/abs/math/0601687.

[10] D. Callan, Sets, lists and noncrossing partitions, *J. Integer Sequences* **11** (2008), Article 08.1.3.

[11] J. S. Kim, New interpretations for noncrossing partitions of classical types, *J. Combin. Theory Ser. A* **118** (2011), 1168–1189.

[12] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2024. Available at https://oeis.org.

2020 *Mathematics Subject Classification*: Primary 05A18.
Keywords: noncrossing partition.

(Concerned with sequences A000108, A363448, and A363449.)

Received November 11 2023; revised versions received November 12 2023; November 13 2023; February 12 2024; October 1 2024. Published in *Journal of Integer Sequences*, January 3 2025.

Return to Journal of Integer Sequences home page.