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Associated Stirling Eulerian Numbers and Their q-Analogues

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Abstract

In this paper, we study the excedance distribution over permutations while considering the parameters of cycle length and the number of cycles. We refer to the number of such permutations as the *associated Stirling Eulerian number*. Moreover, if we consider the permutations in which the first s integers are in different cycles, we denote their count as the *associated s-Stirling Eulerian number*. We provide a formula defining these numbers, along with their generating functions. We establish q-analogues and offer extensions of the results.

1 Introduction

The Stirling numbers of the first kind, denoted $S_{n,k}$, count the permutations with k cycles over n objects. These are classical numbers. Appell [1], Carlitz [5, 6], Comtet [7], Foata [8], Riordan [13], Tricomi [15] have defined the *associated Stirling numbers of the first kind*, denoted $d(n,k)$, which count the numbers of permutations over n objects having exactly k cycles, where each cycle has a length 2 or greater. These numbers are defined as follows:

$$d(n+1,k) = n\big(d(n,k) + d(n-1,k-1)\big), \tag{1}$$

with $d(n,1) = (n-1)!$ and $d(n,k) = 0$ for $n \leq 2k-1$.

The generating function $D(q,u) = \sum_k \sum_n d(n,k) q^k \frac{u^n}{n!}$ has the closed form

$$D(q,u) = (\exp(-u)/(1-u))^q. \tag{2}$$

Belbachir and Bousbaa [2], as well as Broder [4] have conducted in-depth studies of these numbers.

Definition 1. An r-*cycle* is a cycle in a permutation of length r. A *circular permutation* over $[n]$ is a permutation consisting of an n-cycle. An r-*cycle permutation* is a permutation in which each cycle has length r.

An r^--*cycle* (respectively, r^+-*cycle*) *permutation* is a permutation where each cycle length is at most r (respectively, greater than r). An *involution* is a 2^--cycle permutation. Following Broder's definitions [4], we define the number of r^--cycle (respectively, r^+-cycle) permutations as the r^--*associated* (respectively, r^+-*associated*) *Stirling number of the first kind*.

An (r^-, k)-*cycle* (respectively, (r^+, k)-*cycle*) *permutation* is an r^--cycle (respectively, r^+-cycle) permutation consisting of k cycles. If each cycle has length r, we refer to it as an $((r), k)$-*cycle permutation*. Furthermore, if the first s integers lie in distinct cycles, we define an (r^-, k, s)-*cycle* (respectively, (r^+, k, s)-*cycle*) *permutation* as an (r^-, k)-cycle (respectively, (r^+, k)-cycle) permutation. The r^--*associated* s-*Stirling number of the first kind* (respectively, r^+-*associated* s-*Stirling number of the first kind*) counts the number of such permutations. If the length of each cycle is equal to r, we call it an $((r), k, s)$-*cycle permutation*.

An $(r^-, \star, s)$-cycle permutation (respectively, $(r^+, \star, s)$-cycle permutation) is an r^--cycle (respectively, r^+-cycle) permutation where the first s integers are in distinct cycles.

We can generalize this by considering an $(r_1) \cdots (r_\ell)$-*cycle permutation* consisting of cycles of lengths $1 < r_1 < \cdots < r_\ell$, and $r_1 \cdots r_s$-*cycle permutations*, where the first s integers are in distinct cycles of lengths $r_i > 1$ for $i = 1, \ldots, s$.

An $((r_1) \cdots (r_\ell), k)$-*cycle permutation* is an $(r_1) \cdots (r_\ell)$-cycle permutation with k cycles, and an $((r_1) \cdots (r_\ell), k, s)$-*cycle permutation* has the first s integers in distinct cycles. An $r_1 \cdots r_s(r_{s+1}) \cdots (r_\ell)$-*cycle permutation* is a permutation where the first s integers belong to different cycles, each of length $r_i > 1$ for $i = 1, \ldots, s$, and the remaining cycles have lengths $1 < r_{s+1} < \cdots < r_\ell$.

An $r_1^\star \cdots r_\ell^\star$-*cycle permutation* is a permutation that does not contain any cycles of length r_i for $1 \leq i \leq \ell$.

Definition 2. An integer i is a *fixed point* of a permutation σ if $\sigma(i) = i$.

A *derangement* is a permutation without fixed points. We can also define a derangement as a 1^+-cycle permutation.

Definition 3 (Rakotondrajao [12])**.** An *s-fixed-points-permutation* is a permutation whose set of fixed points is a subset of the set $[s]$, and the first s integers lie in different cycles.

For example, the permutation $(1\ 4)(2)(3)(5\ 7)(6)$ is a $(0^+, 5, 3)$-cycle permutation and not a 3-fixed-points-permutation.

Definition 4. An integer i is an *excedance* for a permutation σ if $\sigma(i) > i$.

The number of permutations of n objects with exactly m excedances is the classical *Eulerian number* $A_{n,m}$. They satisfy the following relation:

$$A_{n,m} = (n-m)A_{n-1,m-1} + (m+1)A_{n-1,m}, \tag{3}$$

with $A_{0,0} = 1$ and $A_{0,1} = 0$. The egf of the Eulerian polynomial

$$A_n(x) = \sum_{\sigma \in S_n} x^{e(\sigma)} = \sum_m A_{n,m} x^m \tag{4}$$

has the closed form

$$A(x,u) = \sum_n A_n(x) \frac{u^n}{n!} = \frac{x-1}{x - \exp\left((x-1)u\right)}. \tag{5}$$

Many works on Eulerian numbers can be easily found in the literature, including those by Foata and Schützenberger [8], Mantaci and Rakotondrajao [11], Riordan [13], Stanley [14]. The function $\mathcal{P}(x,u)$ defined by

$$\mathcal{P}(x,u) = \frac{\ln(A(x,u)) + u(x-1)}{x}, \tag{6}$$

is the primitive of $A(x,u)$ and is computed combinatorially in Sec. 3.

Note that a permutation is a 0^+-cycle permutation, and a derangement a 1^+-cycle permutation. The length of an r-cycle permutation is a multiple of r and the length of an $((r),k)$-cycle permutation is equal to kr. We are interested in the study of associated Stirling Eulerian numbers. More precisely, we consider the parameter of the number of cycles with the constraint on the length of cycles in the study of excedance distribution over permutations. We also study the same distribution by adding the placement of the first s integers. Recurrence relations, formulas, exponential generating functions, and their q-analogues are provided. We extend Brenti's results [3, Sec. 7] on his study of a q-analogue of the Eulerian polynomials, as well as general identities. A general identity extends the result on derangement numbers established by Ksavrelof and Zeng [9], along with new identities. We provide generalizations of our results by specifying the lengths of cycles. Note that the associated Stirling numbers of the first kind $d(n,k)$ in Equation (1) enumerate the $(1^+,k)$-cycle permutations, while $D(q,x)$ in Equation (2) is a q-analogue of the exponential generating function (egf) of the derangement numbers. We denote by $e(\sigma)$ the number of excedances of the permutation σ and by $c(\sigma)$ the number of cycles of σ. The parameter q associated with the number of cycles defines the q-analogue in our study. We use the expression *Stirling number* to refer to the Stirling number of the first kind and introduce the notation in each section throughout this paper.

2 Circular permutations

$\mathcal{C}_n$	the set of circular permutations over $[n]$	
$C_{n,m}$	$\lvert\{\sigma \in \mathcal{C}_n : e(\sigma) = m\}\rvert$	Th. 5 Eq. (7)
$C_n(x)$	$\sum_m C_{n,m} x^m = \sum_{\sigma\in\mathcal{C}_n} x^{e(\sigma)}$	Th. 5 Eq. (8)
$C(x,u)$	$\sum_n \sum_{\sigma\in\mathcal{C}_n} x^{e(\sigma)} \frac{u^n}{n!} = \sum_n C_n(x) \frac{u^n}{n!}$	Th. 6 Eq. (9)
$\mathcal{C}_{r^-}$	the set of r^--cycle circular permutations	
$C^{r^-}(x,u)$	$\sum_{n=0}^{r} \sum_{\sigma\in\mathcal{C}_n} x^{e(\sigma)} \frac{u^n}{n!} = \sum_{n=0}^{r} C_n(x) \frac{u^n}{n!}$	Th. 7 Eq. (10)
		Cor. 12 Eq. (15)
$\mathcal{C}_{r^+}$	the set of r^+-cycle circular permutations	
$C^{(r)}(x,u)$	$\sum_{\sigma\in\mathcal{C}_r} x^{e(\sigma)} \frac{u^r}{r!} = C_r(x) \frac{u^r}{r!}$	Th. 8 Eq. (11)
$C^{r^+}(x,u)$	$\sum_{n\geq r+1} \sum_{\sigma\in\mathcal{C}_n} x^{e(\sigma)} \frac{u^n}{n!} = \sum_{n\geq r+1} C_n(x) \frac{u^n}{n!}$	Th. 9 Eq. (12)
		Cor. 33 Eq. (33)

Table 1: Notation.

Theorem 5. *For all integers $n \geq 2$ and $1 \leq m \leq n$, we have*

$$C_{n,m} = A_{n-1,m-1} \text{ with } C_{1,0} = 1 \tag{7}$$

and

$$C_n(x) = xA_{n-1}(x). \tag{8}$$

Proof. Let $n \geq 2$ and $m \geq 1$. Let $\pi \in \mathcal{C}_n$ be a permutation such that $e(\pi) = m$. It is common to put the smallest letter at the beginning of the cycle. The integer 1 is always an excedance for a circular permutation. The remaining permutation after this integer, which is an ordinary permutation over the set $\{2, \ldots, n\}$, has $m - 1$ excedances. Multiplying by x^m and summing over m in Equation (7), we get

$$C_n(x) = xA_{n-1}(x).$$

□

Theorem 6. *The exponential generating function of the excedance distribution over circular permutations has the closed form*

$$C(x,u) = \sum_{n\geq 0} \sum_{\pi\in\mathcal{C}_n} x^{e(\pi)} \frac{u^n}{n!} = \ln A(x,u). \tag{9}$$

Proof. Since $C_n(x) = xA_{n-1}(x)$, we get

$$\begin{aligned}\sum_{n\geq 2} C_n(x)\frac{u^n}{n!} &= \sum_{n\geq 2} xA_{n-1}(x)\frac{u^n}{n!},\\ &= x(\mathcal{P}(x,u)) - xu.\end{aligned}$$

Then, we obtain

$$u + \sum_{n\geq 2} C_n(x)\frac{u^n}{n!} = u + x\frac{\ln(A(x,u)) + u(x-1)}{x} - xu.$$

Therefore, we have

$$C(x,u) = \ln(A(x,u)).$$

□

Theorem 7. *For a fixed integer $r \geq 1$, the exponential generating function of the excedance distribution over circular r^--cycle permutations has the closed form*

$$C^{r^-}(x,u) = \sum_{n=1}^{r} C_n(x)\frac{u^n}{n!} = u + x\sum_{n=1}^{r-1} A_n(x)\frac{u^{n+1}}{(n+1)!}. \tag{10}$$

Proof. From Equation (8), we have

$$\begin{aligned}\sum_{n=2}^{r} C_n(x)\frac{u^n}{n!} &= x\sum_{n=2}^{r} A_{n-1}(x)\frac{u^n}{n!}.\\ \sum_{n=1}^{r} C_n(x)\frac{u^n}{n!} &= u + x\sum_{n=1}^{r-1} A_n(x)\frac{u^{n+1}}{(n+1)!}.\end{aligned}$$

□

Theorem 8. *For a fixed integer $r \geq 2$, the exponential generating function of the excedance distribution over circular r-cycle permutations has the closed form*

$$C^{(r)}(x,u) = xA_{r-1}(x)\frac{u^r}{r!}. \tag{11}$$

Theorem 9. *For a fixed integer $r \geq 2$, the exponential generating function of the excedance distribution over circular r^+-cycle permutations has the closed form*

$$C^{r^+}(x,u) = \sum_{n\geq r+1} C_n(x)\frac{u^n}{n!} = \ln\left(A(x,u)\exp\left(-u - x\sum_{i=1}^{r-1} A_i(x)\frac{u^{i+1}}{(i+1)!}\right)\right). \tag{12}$$

Proof. We have

$$
\begin{aligned}
C^{r^+}(x,u) &= C(x,u) - C^{r^-}(x,u),\\
&= \ln A(x,u) - u - x\sum_{i=1}^{r-1} A_i(x)\frac{u^{i+1}}{(i+1)!},\\
&= \ln\left(A(x,u)\exp\left(-u - x\sum_{i=1}^{r-1} A_i(x)\frac{u^{i+1}}{(i+1)!}\right)\right).
\end{aligned}
$$

□

3 r^--cycle permutations

$(n_1,\ldots,n_k)!$	$\dfrac{n_1+\cdots+n_k}{n_1!\cdots n_k!}$	
$\mathcal{S}_n^{r^-}$	the set of r^--cycle permutations	
$A_{n,m}^{r^-}$	$\lvert\{\sigma\in\mathcal{S}_n^{r^-} : e(\sigma)=m\}\rvert$ *r^--associated Stirling Eulerian number*	Th. 10 Eq. (13)
$A_n^{r^-}(x)$	$\sum_m A_{n,m}^{r^-}x^m = \sum_{\sigma\in\mathcal{S}_n^{r^-}} x^{e(\sigma)}$	
$A^{r^-}(x,u)$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^-}} x^{e(\sigma)}\dfrac{u^n}{n!} = \sum_n A_n^{r^-}(x)\dfrac{u^n}{n!}$	Th. 11 Eq. (14)
$\mathcal{P}(x,u)$	the primitive of $A(x,u)$ with respect to u	Cor. 13 Eq. (16)
$\mathcal{S}_n^{r^-,k}$	the set of (r^-,k)-cycle permutations	
$A_{n,m,k}^{r^-}$	$\lvert\{\sigma\in\mathcal{S}_n^{r^-,k} : e(\sigma)=m\}\rvert$	
$A_{n,k}^{r^-}(x)$	$\sum_m A_{n,m,k}^{r^-}x^m = \sum_{\sigma\in\mathcal{S}_n^{r^-,k}} x^{e(\sigma)}$	
$A_k^{r^-}(x,u)$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^-,k}} x^{e(\sigma)}\dfrac{u^n}{n!} = \sum_n A_{n,k}^{r^-}(x)\dfrac{u^n}{n!}$	Th. 14 Eq. (17)
$A^{r^-}(q,x,u)$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^-,k}} x^{e(\sigma)}q^{c(\sigma)}\dfrac{u^n}{n!}$ q-analogue of $A^{r^-}(x,u)$	Th. 15 Eq. (18) Cor. 16 Eq. (19)
$\mathcal{S}_n^{r^-,\star,s}$	the set of $(r^-,\star,s)$-cycle permutations	
$[A_{n,m}^{r^-}]_s$	$\lvert\{\sigma\in\mathcal{S}_n^{r^-,\star,s} : e(\sigma)=m\}\rvert$	
$[A_n^{r^-}(x)]_s$	$\sum_m [A_{n,m}^{r^-}]_s x^m = \sum_{\sigma\in\mathcal{S}_n^{r^-,\star,s}} x^{e(\sigma)}$	
$[A^{r^-}(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^-,\star,s}} x^{e(\sigma)}\dfrac{u^n}{n!} = \sum_n [A_n^{r^-}(x)]_s\dfrac{u^n}{n!}$	Th. 17 Eq. (20)
$[A_{n,m,k}^{r^-}]_s$	$\lvert\{\sigma\in\mathcal{S}_n^{r^-,k,s} : e(\sigma)=m\}\rvert$	
$[A_{n,k}^{r^-}(x)]_s$	$\sum_m [A_{n,m,k}^{r^-}]_s x^m = \sum_{\sigma\in\mathcal{S}_n^{r^-,k,s}} x^{e(\sigma)}$	

$[A_k^{r^-}(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^-,k,s}} x^{e(\sigma)}\frac{u^n}{n!} = \sum_n [A_{n,k}^{r^-}(x)]_s \frac{u^n}{n!}$		Th. 18 Eq. (21)
$\left[A^{r^-}(q,x,u)\right]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^-,\star,s}} x^{e(\sigma)} q^{c(\sigma)}\frac{u^n}{n!}$	q-analogue of $[A^{r^-}(x,u)]_s$	Th. 19 Eq. (22)

Table 2: Notation.

Theorem 10. *For a fixed integer $r \geq 2$, for all integers n and m such that $0 \leq m \leq n$, we have*

$$A_{n+1,m}^{r^-} = \sum_{j=1}^{r-1} \binom{n}{j} \sum_{i\geq 0} C_{j+1,i} A_{n-j,m-i}^{r^-} + A_{n,m}^{r^-}. \tag{13}$$

Proof. Consider the integer $n+1$ in an r^--cycle permutation of $\mathcal{S}_{n+1}^{r^-}$.

1. If $n+1$ is a fixed point, the remaining elements form a permutation of $[n]$ with m excedances, where the lengths of the cycles are at most r. Therefore, there are $A_{n,m}^{r^-}$ possibilities.

2. Otherwise, we have $\binom{n}{j} C_{j+1,i}$ ways to form a cycle of length $j+1$ with i excedances that contains the integer $n+1$, for $1 \leq j \leq n$. The remaining elements of $[n]$ form a permutation with $m-i$ excedances, whose cycle lengths are at most r. Thus, we get $\sum_{j=1}^{r-1}\binom{n}{j}\sum_{i\geq 0} C_{j+1,i} A_{n-j,m-i}^{r^-}$ possibilities.

□

Theorem 11. *For a fixed integer $r \geq 2$, the exponential generating function of the r^--associated Stirling Eulerian numbers has the closed form*

$$A^{r^-}(x,u) = \exp\left(u + x\sum_{j=1}^{r-1} A_j(x)\frac{u^{j+1}}{(j+1)!}\right). \tag{14}$$

Proof. Since

$$A_{n+1,m}^{r^-} = \sum_{j=1}^{r-1} \binom{n}{j} \sum_{i\geq 0} C_{j+1,i} A_{n-j,m-i}^{r^-} + A_{n,m}^{r^-},$$

we have

$$A_{n+1}^{r^-}(x) = \sum_{j=1}^{r-1} \binom{n}{j} C_{j+1}(x) A_{n-j}^{r^-}(x) + A_n^{r^-}(x).$$

We get

$$\begin{aligned}\frac{\partial}{\partial u}A^{r^-}(x,u) &= \sum_{n\geq 0}\sum_{j=1}^{r-1}\binom{n}{j}C_{j+1}(x)A^{r^-}_{n-j}(x)\frac{u^n}{n!} + A^{r^-}(x,u),\\ &= \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^j}{j!}A^{r^-}_{n-j}(x)\frac{u^{n-j}}{(n-j)!} + A^{r^-}(x,u),\\ &= \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^j}{j!}A^{r^-}(x,u) + A^{r^-}(x,u).\end{aligned}$$

Hence,

$$\ln\left(A^{r^-}(x,u)\right) = \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^{j+1}}{(j+1)!} + u.$$

Finally, we have

$$A^{r^-}(x,u) = \exp\Big(u + \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^{j+1}}{(j+1)!}\Big) = \exp\Big(u + x\sum_{j=1}^{r-1}A_j(x)\frac{u^{j+1}}{(j+1)!}\Big).$$

□

Corollary 12. *For all integers $r \geq 2$, we have*

$$C^{r^-}(x,u) = \ln A^{r^-}(x,u). \tag{15}$$

Corollary 13. *The primitive of the exponential generating function of the Eulerian numbers has the closed form*

$$\mathcal{P}(x,u) = \frac{\ln(A(x,u)) + u(x-1)}{x}. \tag{16}$$

Proof. A permutation without restrictions on the cycle length can be considered an ordinary permutation. From Equation (14), we deduce

$$\begin{aligned}\exp(u + x\sum_{n\geq 1}A_n(x)\frac{u^{n+1}}{(n+1)!}) &= A(x,u),\\ u + x\sum_{n\geq 1}A_n(x)\frac{u^{n+1}}{(n+1)!} &= \ln A(x,u),\\ u + x(\mathcal{P}(x,u) - u) &= \ln A(x,u).\end{aligned}$$

Thus, we have

$$\mathcal{P}(x,u) = \frac{\ln A(x,u) + u(x-1)}{x}.$$

□

Theorem 14. *For fixed integers $r \geq 2$ and k, the exponential generating function of the excedance distribution over (r^-, k)-cycle permutations has the closed form*

$$A_k^{r^-}(x,u) = \frac{(\ln A^{r^-}(x,u))^k}{k!} = \frac{\left(u + x\sum_{n=1}^{r-1} A_n(x)\frac{u^{n+1}}{(n+1)!}\right)^k}{k!}. \tag{17}$$

Proof. A permutation of $\mathcal{S}_n^{r^-,k}$ is a product of k disjoint circular permutations of length smaller or equal than r, hence

$$A_{n,m,k}^{r^-} = \frac{1}{k!}\sum_{\substack{i_1,\ldots,i_k \leq r\\ i_1+\cdots+i_k=n}} \sum_{j_1+\cdots+j_k=m} (i_1,\ldots,i_k)!\; C_{i_1,j_1}\cdots C_{i_k,j_k}.$$

We get

$$A_{n,k}^{r^-}(x) = \frac{1}{k!}\sum_{\substack{i_1,\ldots,i_k \leq r\\ i_1+\cdots+i_k=n}} (i_1,\ldots,i_k)!\; C_{i_1}(x)\cdots C_{i_k}(x),$$

and

$$\begin{aligned} A_k^{r^-}(x,u) &= \frac{1}{k!}(\sum_{n=1}^{r} C_n(x)\frac{u^n}{n!})^k = \frac{(C^{r^-}(x,u))^k}{k!},\\ &= \frac{(\ln A^{r^-}(x,u))^k}{k!}, \text{ from Equation (15)},\\ &= \frac{1}{k!}\left(u + x\sum_{n=1}^{r-1} A_n(x)\frac{u^{n+1}}{(n+1)!}\right)^k, \text{ from Equation (14)}. \end{aligned}$$

□

Theorem 15. *For a fixed integer $r \geq 2$, a q-analogue exponential generating function of the r^--associated Stirling Eulerian numbers has the closed form*

$$A^{r^-}(q,x,u) = \exp\left(q\left(u + x\sum_{j=1}^{r-1} A_j(x)\frac{u^{j+1}}{(j+1)!}\right)\right). \tag{18}$$

Proof. From equation (17), we get

$$\begin{aligned} \sum_{k\geq 0} A_k^{r^-}(x,u)q^k &= \sum_{k\geq 0}\frac{\left(u + x\sum_{j=1}^{r-1} A_j(x)\frac{u^{j+1}}{(j+1)!}\right)^k}{k!}q^k,\\ &= \exp\left(q\left(u + x\sum_{j=1}^{r-1} A_j(x)\frac{u^{j+1}}{(j+1)!}\right)\right). \end{aligned}$$

□

Corollary 16. *For all integer $r \geq 2$, we have*

$$A^{r^-}(q, x, u) = \Big(A^{r^-}(x, u)\Big)^q. \tag{19}$$

Theorem 17. *For a fixed integer $r \geq 2$, the partial differential exponential generating function of the r^--associated s-Stirling Eulerian numbers has the closed form*

$$\frac{\partial^s [A^{r^-}(x, u)]_s}{\partial u^s} = \left(1 + x \sum_{i=1}^{r-1} A_i(x) \frac{u^i}{i!}\right)^s A^{r^-}(x, u). \tag{20}$$

Proof. Considering the first $s-1$ integers and their cycles, we have

$$[A^{r^-}_{n+s,m}]_s = \sum_{\substack{i_1+\cdots+i_{s-1}=n \\ 0\leq i_1,\ldots,i_{s-1}\leq r-1}} (i_1, \ldots, i_{s-1})! \sum_{j_1+\cdots+j_s=m} C_{i_1+1,j_1} \cdots C_{i_{s-1}+1,j_{s-1}} A^{r^-}_{n+1-\sum_{\ell=1}^{s-1} i_\ell, j_s}.$$

We get

$$[A^{r^-}_{n+s}(x)]_s = \sum_{\substack{i_1+\cdots+i_{s-1}=n \\ 0\leq i_1,\ldots,i_{s-1}\leq r-1}} (i_1, \ldots, i_{s-1})!\ C_{i_1+1}(x) \cdots C_{i_{s-1}+1}(x) A^{r^-}_{n+1-\sum_{\ell=1}^{s-1} i_\ell}(x),$$

and

$$\begin{aligned}
\frac{\partial^s [A^{r^-}(x, u)]_s}{\partial u^s} &= \Big(\sum_{i=0}^{r-1} C_{i+1}(x) \frac{u^i}{i!}\Big)^{s-1} \frac{\partial A^{r^-}(x, u)}{\partial u}, \\
&= \Big(1 + \sum_{i=1}^{r-1} C_{i+1}(x) \frac{u^i}{i!}\Big)^{s-1} \Big(1 + x \sum_{i=1}^{r-1} A_i(x) \frac{u^i}{i!}\Big) A^{r^-}(x, u), \\
&= \Big(1 + x \sum_{i=1}^{r-1} A_i(x) \frac{u^i}{i!}\Big)^s A^{r^-}(x, u).
\end{aligned}$$

□

Theorem 18. *For fixed integers r, k and s, the partial differential exponential generating function of the excedance distribution over $(r^-, k, s)-$cycle permutations has the closed form*

$$\frac{\partial^s [A^{r^-}_k(x, u)]_s}{\partial u^s} = \left(1 + x \sum_{i=1}^{r-1} A_i(x) \frac{u^i}{i!}\right)^s \frac{\big(\ln A^{r-}(x, u)\big)^{k-s}}{(k-s)!}. \tag{21}$$

Proof. Considering the first s integers and their cycles, we have

$$[A^{r^-}_{n+s,m,k}]_s = \sum_{\substack{i_1+\cdots+i_s=n \\ 0\leq i_1,\ldots,i_s\leq r-1}} (i_1, \ldots, i_s)! \sum_{m_1+\cdots+m_s+m_t=m} C_{i_1+1,m_1} \cdots C_{i_s+1,m_s} A^{r^-}_{n-\sum_{l=1}^{s} i_l, m_t, k-s}.$$

We get

$$[A^{r^-}_{n+s,k}(x)]_s = \sum_{\substack{i_1+\cdots+i_s=n\\ 0\le i_1,\ldots,i_s\le r-1}} (i_1,\ldots,i_s)!\; C_{i_1+1}(x)\cdots C_{i_s+1}(x) A^{r^-}_{n-\sum_{l=1}^{s} i_l,k-s}(x),$$

and

$$\begin{aligned}\frac{\partial^s [A^{r^-}_k(x,u)]_s}{\partial u^s} &= \left(\sum_{i=0}^{r-1} C_{i+1}(x)\frac{u^i}{i!}\right)^s A^{r^-}_{k-s}(x,u) = (1+\sum_{i=1}^{r-1} C_{i+1}(x)\frac{u^i}{i!})^s \frac{(\ln A^{r^-}(x,u))^{k-s}}{(k-s)!},\\ &= \left(1+x\sum_{i=1}^{r-1} A_i(x)\frac{u^i}{i!}\right)^s \frac{(\ln A^{r^-}(x,u))^{k-s}}{(k-s)!}.\end{aligned}$$

□

Theorem 19. *For fixed integers r and s, a partial differential q-analogue exponential generating function of the excedance distribution over $(r^-,\star,s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A^{r^-}(q,x,u)]_s}{\partial u^s} = q^s \left(1+x\sum_{i=1}^{r-1} A_i(x)\frac{u^i}{i!}\right)^s \left(A^{r^-}(x,u)\right)^q. \tag{22}$$

Proof. We immediately have the result by multiplying by q^k the right member of Equation (21) and summing over k. □

4 r-cycle permutations

$\mathcal{S}_n^{(r)}$	the set of r-cycle permutations	
$A^{(r)}_{n,m}$	$\lvert\{\sigma\in\mathcal{S}_n^{(r)} : e(\sigma)=m\}\rvert$	Th. 20 Eq. (23)
$A^{(r)}_n(x)$	$\sum_m A^{(r)}_{n,m}x^m = \sum_{\sigma\in\mathcal{S}_n^{(r)}} x^{e(\sigma)}$	
$A^{(r)}(x,u)$	$\sum_n A^{(r)}_n(x)\frac{u^n}{n!}$ the egf	Th. 21 Eq. (24)
$\mathcal{S}_n^{(r),k}$	the set of $((r),k)$-cycle permutations	
$A^{(r)}_{n,m,k}$	$\lvert\{\sigma\in\mathcal{S}_n^{(r),k} : e(\sigma)=m\}\rvert$	
$A^{(r)}_{n,k}(x)$	$\sum_m A^{(r)}_{n,m,k}x^m = \sum_{\sigma\in\mathcal{S}_n^{(r),k}} x^{e(\sigma)}$	
$A^{(r)}_k(x,u)$	$\sum_n\sum_{\sigma\in\mathcal{S}_n^{(r),k}} x^{e(\sigma)}\frac{u^n}{n!} = \sum_n A^{(r)}_{n,k}(x)\frac{u^n}{n!}$	Th. 22 Eq. (25)
$A^{(r)}(q,x,u)$	$\sum_n\sum_{\sigma\in\mathcal{S}_n^{(r),k}} x^{e(\sigma)}q^{c(\sigma)}\frac{u^n}{n!}$ q-analogue of $A^{(r)}(x,u)$	Th. 23 Eq. (26)
$\mathcal{S}_n^{(r),\star,s}$	the set of $((r),\star,s)$-cycle permutations	

$[A^{(r)}_{n,m}]_s$	$\lvert\{\sigma \in \mathcal{S}^{(r),\star,s}_n : e(\sigma) = m\}\rvert$	
$[A^{(r)}_n(x)]_s$	$\sum_m [A^{(r)}_{n,m}]_s x^m = \sum_{\sigma\in\mathcal{S}^{(r),\star,s}_n} x^{e(\sigma)}$	
$[A^{(r)}(x,u)]_s$	$\sum_n [A^{(r)}_n(x)]_s \dfrac{u^n}{n!}$ its egf	Th. 27 Eq. (27)
$[A^{(r)}_k(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}^{(r),\star,s}_n} x^{e(\sigma)} \dfrac{u^n}{n!}$	Th. 28 Eq. (28)
$[A^{(r)}(q,x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}^{(r),k,s}_n} x^{e(\sigma)} q^{c(\sigma)} \dfrac{u^n}{n!}$ q-analogue of $[A^{(r)}(x,u)]_s$	Th. 29 Eq. (29)

Table 3: Notation.

Theorem 20. *For a fixed integer $r \geq 2$, for all integers $n \geq m \geq 0$, we have the following formula*

$$A^{(r)}_{n+1,m} = \binom{n}{r-1} \sum_{k\geq 1} C_{r,k} A^{(r)}_{n-(r-1),m-k}. \tag{23}$$

Proof. First, consider the integer $n+1$ and construct the cycle that contains it in a permutation of $\mathcal{S}^r_{n+1}$. To do this, we choose $r-1$ integers from $[n]$ and form a circular permutation with $n+1$ that has k excedances, where $1 \leq k \leq m-1$. There are $\binom{n-1}{k} C_{r,k}$ possibilities for this step. Next, to obtain a permutation of $[n+1]$ with m excedances, where each cycle length is equal to r, we consider the remaining $n-(r-1)$ integers and form a permutation of $[n-(r-1)]$ with $m-k$ excedances, ensuring that each cycle length is also equal to r. This construction gives us the desired result. □

Theorem 21. *For a fixed integer $r \geq 2$, the exponential generating function of the numbers $A^{(r)}_{n,m}$ has the closed form*

$$A^{(r)}(x,u) = \exp\Big(x A_{r-1}(x) \frac{u^r}{r!}\Big). \tag{24}$$

Proof. From Equation (23), we get

$$A^{(r)}_{n+1}(x) = \binom{n}{r-1} C_r(x) A^{(r)}_{n-r+1}(x).$$

Let us consider the following variable change $r-1 = r'$, we have

$$A^{(r)}_{n+1}(x) = \binom{n}{r'} C_{r'+1}(x) A^{(r)}_{n-r'}(x).$$

We deduce

$$\begin{aligned}\frac{\partial}{\partial u} A^{(r)}(x,u) &= \sum_{n\geq 0} \binom{n}{r'} C_{r'+1}(x) A^{(r)}_{n-r'}(x) \frac{u^n}{n!} = \sum_{n\geq 0} C_{r'+1}(x) \frac{u^{r'}}{r'!} A^{(r)}_{n-r'}(x) \frac{u^{n-r'}}{(n-r')!},\\ &= C_{r'+1}(x) \frac{u^{r'}}{r'!} A^{(r)}(x,u).\end{aligned}$$

Thus,

$$\frac{(A^r(x,u))'_u}{A^{(r)}(x,u)} = C_{r'+1}(x)\frac{u^{r'}}{r'!}.$$

Integrating with respect to u, we have

$$A^{(r)}(x,u) = \exp\Big(C_{r'+1}(x)\frac{u^{r'+1}}{(r'+1)!}\Big) = \exp\Big(xA_{r'}(x)\frac{u^{r'+1}}{(r'+1)!}\Big).$$

Thus,

$$A^{(r)}(x,u) = \exp\Big(xA_{r-1}(x)\frac{u^r}{r!}\Big).$$

□

Theorem 22. *For fixed integers $r \geq 2$ and k, the exponential generating function of the excedance distribution over $((r),k)-$cycle permutations has the closed form*

$$A_k^{(r)}(x,u) = \frac{1}{k!}\left(xA_{r-1}(x)\frac{u^r}{r!}\right)^k = \frac{\big(\ln A^{(r)}(x,u)\big)^k}{k!}. \tag{25}$$

Proof. A permutation of $S_n^{(r),k}$ is a product of k disjoints circular permutation of lenght r, hence

$$A_{n,m,k}^{(r)} = \frac{1}{k!}\binom{n}{r}\cdots\binom{n-(k-1)r}{r}\sum_{m_1+\cdots+m_k=m} C_{r,m_1}\times\cdots\times C_{r,m_k}.$$

We get

$$A_{n,k}^{(r)}(x) = \frac{1}{k!}\frac{n!}{(r!)^k}(C_r(x))^k.$$

We deduce

$$A_k^{(r)}(x,u) = \frac{1}{k!}\frac{(C_r(x))^k u^{kr}}{(r!)^k} = \frac{1}{k!}\Big(xA_{r-1}(x)\frac{u^r}{r!}\Big)^k.$$

□

Theorem 23. *For a fixed integer $r \geq 2$, a q-analogue exponential generating function of the number $A_{n,m,k}^{(r)}$ has the closed form*

$$A^{(r)}(q,x,u) = \exp\left(qxA_{r-1}(x)\frac{u^r}{r!}\Big)\right.. \tag{26}$$

Proof. We immediately have the result by multiplying by q^k the Equation (25) and summing over k. □

Corollary 24. *For fixed integers $r \geq 2$ and n, we have*

$$\sum_{\pi\in\mathcal{S}_n^{r,k}} x^{e(\pi)}q^{c(\pi)} = \begin{cases} \dfrac{(rk)!}{(r!)^k\ k!}x^kq^kA_{r-1}(x)^k, & \textit{if } n = kr\ ; \\ 0 & \textit{otherwise.}\end{cases}$$

Corollary 25. *For fixed integers $r \geq 2$ and n, we have*

$$|\mathcal{S}_n^r| = \begin{cases} \dfrac{n!}{r^{n/r}\ (n/r)!}, & \text{if } r \mid n\ ; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 26. *For a fixed integer $r \geq 2$, we have*

$$\sum_{n\geq 0} |\mathcal{S}_n^r| \frac{u^n}{n!} = \exp\left(u^r/r\right).$$

Theorem 27. *For fixed integers $r \geq 2$ and s, we have*

$$\frac{\partial^s [A^{(r)}(x,u)]_s}{\partial u^s} = x^s \left(A_{r-1}(x) \frac{u^{r-1}}{(r-1)!} \right)^s A^{(r)}(x,u). \tag{27}$$

Proof. Since

$$[A^{(r)}_{n+s,m}]_s = \binom{n}{r-1} \cdots \binom{n-(s-1)(r-1)}{r-1} \sum_{m_1+\cdots+m_s+m_t=m} C_{r,m_1} \cdots C_{r,m_s} A^{(r)}_{n-s(r-1),m_t},$$

we get

$$[A^{(r)}_{n+s}(x)]_s = \binom{n}{r-1} \cdots \binom{n-(s-1)(r-1)}{r-1} (C_r(x))^s A^{(r)}_{n-s(r-1)}(x).$$

We deduce

$$\frac{\partial^s [A^{(r)}(x,u)]_s}{\partial u^s} = \Big(C_r(x) \frac{u^{r-1}}{(r-1)!} \Big)^s A^{(r)}(x,u) = x^s \Big(A_{r-1}(x) \frac{u^{r-1}}{(r-1)!} \Big)^s A^{(r)}(x,u).$$

□

Theorem 28. *For fixed integers $r \geq 2$, k and s, the partial differential exponential generating function of the excedance distribution over $((r),k,s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A^{(r)}_k(x,u)]_s}{\partial u^s} = x^s \left(A_{r-1}(x) \frac{u^{r-1}}{(r-1)!} \right)^s \frac{\left(\ln A^{(r)}(x,u)\right)^{k-s}}{(k-s)!}. \tag{28}$$

Proof. Let us consider the first s integers and their cycles

$$[A^{(r)}_{n+s,m,k}]_s = \binom{n}{r-1} \cdots \binom{n-(s-1)(r-1)}{r-1} \times \sum_{m_1+\cdots+m_s+m_t=m} C_{r,m_1} \cdots C_{r,m_s} A^{(r)}_{n-s(r-1),m_t,k-s}.$$

We get

$$\frac{\partial^s [A_k^{(r)}(x,u)]_s}{\partial u^s} = \left(C_r(x)\frac{u^{r-1}}{(r-1)!}\right)^s A_{k-s}^{(r)}(x,u) = \left(xA_{r-1}(x)\frac{u^{r-1}}{(r-1)!}\right)^s \frac{(\ln A^{(r)}(x,u))^{k-s}}{(k-s)!}.$$

□

Theorem 29. *For fixed integers $r \geq 2$ and s, a partial differential q-analogue exponential generating function of the excedance distribution over $((r),\star,s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A^{(r)}(q,x,u)]_s}{\partial u^s} = q^s x^s \left(A_{r-1}(x)\frac{u^{r-1}}{(r-1)!}\right)^s \left(A^{(r)}(x,u)\right)^q. \qquad (29)$$

Proof. From Equation (28), we get

$$\begin{aligned}\frac{\partial^s [A^{(r)}(q,x,u)]_s}{\partial u^s} &= \sum_{k\geq 0} x^s \left(A_{r-1}(x)\frac{u^{r-1}}{(r-1)!}\right)^s \frac{(\ln A^{(r)}(x,u))^{k-s}}{(k-s)!} q^{k-s} q^s,\\ &= q^s x^s \left(A_{r-1}(x)\frac{u^{r-1}}{(r-1)!}\right)^s \exp\left(q \ln A^{(r)}(x,u)\right).\end{aligned}$$

Hence, the result follows. □

5 r^+-cycle permutations

$\mathcal{S}_n^{r^+}$	the set of r^+-cycle permutations	
$A^r_{n,m}$	$\lvert\{\sigma \in \mathcal{S}_n^{r^+} : e(\sigma) = m\}\rvert$	Th. 30 Eq. (30)
	r-associated Stirling Eulerian number	Th. 31 Eq. (31)
$A^r_n(x)$	$\sum_m A^r_{n,m} x^m = \sum_{\sigma\in\mathcal{S}_n^{r^+}} x^{e(\sigma)}$	
$A^r(x,u)$	$\sum_n A^r_n(x)\frac{u^n}{n!}$ the egf	Th. 32 Eq. (32)
$\mathcal{S}_n^{r^+,k}$	the set of (r^+,k)-cycle permutations	
$A^r_{n,m,k}$	$\lvert\{\sigma \in \mathcal{S}_n^{r^+,k} : e(\sigma) = m\}\rvert$	
$A^r_{n,k}(x)$	$\sum_m A^r_{n,m,k} x^m = \sum_{\sigma\in\mathcal{S}_n^{r^+,k}} x^{e(\sigma)}$	
$A^r_k(x,u)$	$\sum_n A^r_{n,k}(x)\frac{u^n}{n!}$	Th. 34 Eq. (34)
$A^r(q,x,u)$	$\sum_k\sum_n A^r_{n,k}(x) q^k \frac{u^n}{n!}$ q-analogue of $A^r(x,u)$	Th. 35 Eq. (35)
$\mathcal{S}_n^{r^+,\star,s}$	the set of $(r^+,\star,s)$-cycle permutations	

$[A^r_{n,m}]_s$	$\lvert\{\sigma \in \mathcal{S}_n^{r^+,\star,s} : e(\sigma) = m\}\rvert$	
$[A^r_n(x)]_s$	$\sum_m [A^r_{n,m}]_s x^m = \sum_{\sigma\in\mathcal{S}_n^{r^+,\star,s}} x^{e(\sigma)}$	
$[A^r(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^+,\star,s}} x^{e(\sigma)} \dfrac{u^n}{n!}$	Th. 36 Eq. (36)
$[A^r_k(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r^+,k,s}} x^{e(\sigma)} \dfrac{u^n}{n!}$	Th. 37 Eq. (37)
$[A^r(q,x,u)]_s$	$\sum_k [A^r_k(x,u)]_s q^k$ q-analogue of $[A^r(x,u)]_s$	Th. 38 Eq. (38)

Table 4: Notation.

Theorem 30. *For a fixed integer r, for all integers n and m such that $0 \le m \le n$ and $n \ge r+1$, the r-associated Stirling Eulerian numbers satisfy the following formula*

$$A^r_{n,m} = mA^r_{n-1,m} + (n-m)A^r_{n-1,m-1} + \binom{n-1}{r} \sum_s C_{r+1,s} A^r_{n-r-1,m-s}. \tag{30}$$

Proof. Let $n > r$. Consider the integer n in an r^+-cycle permutation of $\mathcal{S}_n^{r^+}$.

1. If n belongs to a cycle of length $r+1$ with s excedances $(1 \le s < m)$, there are $\binom{n-1}{r}$ ways to choose the remaining r elements and $C_{r+1,s}$ ways to form the specified cycle. We then create a permutation with $m-s$ excedances, where the cycle lengths are greater than r using the remaining $n-r-1$ elements, which gives $A^r_{n-r-1,m-s}$ possibilities. Thus, the total number of possibilities is $\binom{n-1}{r} \sum_s C_{r+1,s} A^r_{n-r-1,m-s}$.

2. Otherwise, we consider τ as an r^+-cycle permutation in $S_{n-1}^{r^+}$:

   (a) If τ has m excedances, we insert n after an excedance. There are $mA^r_{n-1,m}$ possibilities.

   (b) If τ has $m-1$ excedances, we insert n after an anti-excedance. There are $(n-m)A^r_{n-1,m-1}$ possibilities.

□

Theorem 31. *For a fixed integer r, for all integers n and m such that $0 \le m \le n$ and $n \ge r+1$, we have*

$$A^r_{n+1,m} = \sum_{j=r}^{n} \binom{n}{j} \sum_{i=1}^{m} C_{j+1,i} A^r_{n-j,m-i}. \tag{31}$$

Proof. Let r be a fixed integer with $n \ge r$. Consider the integer $n+1$, which belongs to a cycle of length $j+1$ $(r \le j)$ with i $(1 \le i \le m)$ excedances. There are $\binom{n}{j}$ ways to choose these j elements and $C_{j+1,i}$ ways to form that cycle. We then form a permutation with $m-i$ excedances, where the cycle lengths are greater than r, using the remaining $n-j$ elements, resulting in $A^r_{n-j,m-i}$ possibilities. Hence, we obtain the result. □

Theorem 32. *For a fixed integer $r \geq 1$, the exponential generating function of the r-associated Stirling Eulerian numbers has the closed form*

$$A^r(x,u) = A(x,u)\exp\left(-u - x\sum_{j=1}^{r-1} A_j(x)\frac{u^{j+1}}{(j+1)!}\right). \tag{32}$$

Proof. From Equation (31), we get

$$\begin{aligned}
A^r_{n+1}(x) &= \sum_{j=r}^{n}\binom{n}{j}C_{j+1}(x)A^r_{n-j}(x) = \sum_{j=0}^{n}C_{j+1}(x)A^r_n(x) - \sum_{j=0}^{r-1}C_{j+1}(x)A^r_n(x),\\
&= \sum_{j=0}^{n}C_{j+1}(x)A^r_n(x) - A_n(x) - \sum_{j=1}^{r-1}C_{j+1}(x)A^r_n(x).
\end{aligned}$$

We deduce

$$(A^r(x,u))'_u = \frac{\big(A(x,u)\big)'_u}{A(x,u)}A^r(x,u) - A^r(x,u) - \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^j}{j!}A^r(x,u).$$

Thus,

$$\frac{(A^r(x,u))'_u}{A^r(x,u)} = \frac{\big(A(x,u)\big)'_u}{A(x,u)} - 1 - \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^j}{j!}.$$

Integrating by u, we have

$$\ln A^r(x,u) = \ln A(x,u) - u - \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^{j+1}}{(j+1)!}.$$

Hence

$$\begin{aligned}
A^r(x,u) &= A(x,u)\exp\Big(-u - \sum_{j=1}^{r-1}C_{j+1}(x)\frac{u^{j+1}}{(j+1)!}\Big),\\
&= A(x,u)\exp\Big(-u - x\sum_{j=1}^{r-1}A_j(x)\frac{u^{j+1}}{(j+1)!}\Big).
\end{aligned}$$

□

Corollary 33. *For a fixed integer r, the exponential generating function of the excedance distribution over circular r^+-cycle permutations has the closed form*

$$C^{r^+}(x,u) = \ln(A^r(x,u)). \tag{33}$$

Proof. It follows directly from Equation (12). □

Theorem 34. *For fixed integers r and k, the exponential generating function of the excedance distribution over (r^+, k)-cycle permutations has the closed form*

$$A_k^r(x,u) = \frac{\big(\ln(A^r(x,u))\big)^k}{k!}. \tag{34}$$

Proof. A permutation of $\mathcal{S}_n^{r^+,k}$ is a product of k disjoint circular r^+-cycle permutations. Hence,

$$A_{n,m,k}^r = \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k>r \\ i_1+\cdots+i_k=n}} \sum_{m_1+\cdots+m_k=m} (i_1,\dots,i_k)!\; C_{i_1,m_1}\cdots C_{i_k,m_k}.$$

The result follows directly. □

Theorem 35. *For a fixed integer r, a q-analogue exponential generating function of the r-associated Stirling Eulerian numbers has the closed form*

$$A^r(q,x,u) = \Big(A^r(x,u)\Big)^q. \tag{35}$$

Proof. From Equation (34), we get

$$\begin{aligned}\sum_k A_k^r(x,u)q^k &= \sum_k \frac{(\ln(A^r(x,u)))^k}{k!}q^k = \exp(\ln A^r(x,u)q), \\ &= \Big(A^r(x,u)\Big)^q.\end{aligned}$$

□

Theorem 36. *For a fixed integer $r \geq 1$, the partial differential exponential generating function of the excedance distribution over $(r^+, \star, s)$-cycle permutations has the closed form*

$$\frac{\partial^s[A^r(x,u)]_s}{\partial u^s} = x^s\left(A(x,u) - \sum_{n=0}^{r-1} A_n(x)\frac{u^n}{n!}\right)^s A^r(x,u). \tag{36}$$

Proof. Let us consider the first s integers and their cycles

$$[A_{n+s,m}^r]_s = \sum_{\substack{i_1,\dots,i_s\geq r \\ i_1+\cdots+i_s=n}} \sum_{m_1+\cdots+m_t=m} (i_1,\dots,i_s)!\; C_{i_1+1,m_1}\cdots C_{i_s+1,m_s} A^r_{n-\sum_{\ell=1}^s i_\ell, m_t}.$$

We get

$$[A_{n+s}^r(x)]_s = \sum_{\substack{i_1,\dots,i_s\geq r \\ i_1+\cdots+i_s=n}} (i_1,\dots,i_s)!\; x^s A_{i_1}(x)\cdots A_{i_s}(x) A^r_{n-\sum_{\ell\geq 1}^s i_\ell}(x).$$

We deduce

$$\frac{\partial^s A^r(x,u)}{\partial u^s} = x^s\Big(A(x,u) - \sum_{n=0}^{r-1} A_n(x)\frac{u^n}{n!}\Big)^s A^r(x,u).$$

□

Theorem 37. *For fixed integers $r \geq 1$, k and s, the partial differential exponential generating function of the excedance distribution over (r^+, k, s)-cycle permutations has the closed form*

$$\frac{\partial^s [A_k^r(x,u)]_s}{\partial u^s} = x^s \left(A(x,u) - \sum_{n=0}^{r-1} A_n(x) \frac{u^n}{n!} \right)^s \frac{\left(\ln A^r(x,u)\right)^{k-s}}{(k-s)!}. \tag{37}$$

Proof. A permutation of $\mathcal{S}_{n+s}^{r^+,k,s}$ is a product of k disjoint circular r^+-cycle permutations. Considering the first s integers and their cycles, we have

$$[A^r_{n+s,m,k}]_s = \sum_{\substack{i_1,\ldots,i_s \geq r \\ i_1+\cdots+i_s=n}} \sum_{m_1+\cdots+m_t=m} (i_1,\ldots,i_s)!\; C_{i_1+1,m_1} \cdots C_{i_s+1,m_s} A^r_{n-\sum_{\ell=1}^s i_\ell, m_t, k-s}.$$

The result follows directly. □

Theorem 38. *For fixed integers $r \geq 1$ and s, a partial differential q-analogue exponential generating function of the excedance distribution over $(r^+, \star, s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A^r(q,x,u)]_s}{\partial u^s} = q^s x^s \left(A(x,u) - \sum_{n=0}^{r-1} A_n(x) \frac{u^n}{n!} \right)^s \left(A^r(x,u)\right)^q. \tag{38}$$

Proof. The result follows directly from Equation (37). □

The following section contains the corollaries of the previous sections, obtained by setting different values for r and s in the q-analogues. Extensive results are provided for Brenti's work [3] and for the results of Ksavrelof and Zeng [9]. Various identities are formulated.

6 Excedance distribution by cycle over involutions, permutations and derangements

$\mathcal{S}_n^{0^+,k}$	the set of $(0^+,k)$-cycle permutations		
$A_{n,m,k}$	$\lvert\{\sigma \in \mathcal{S}_n^{0^+,k} : e(\sigma) = m\}\rvert$		
$A_{n,k}(x)$	$\sum_m A_{n,m,k} x^m = \sum_{\sigma \in \mathcal{S}_n^{0^+,k}} x^{e(\sigma)}$		
$A_k(x,u)$	$\sum_n A_{n,k}(x) \frac{u^n}{n!}$		Cor. 43 Eq. (40)
$A(q,x,u)$	$\sum_n \sum_{\sigma \in \mathcal{S}_n} x^{e(\sigma)} q^{c(\sigma)} \frac{u^n}{n!}$	q-analogue of $A(x,u)$	Cor. 44 Eq. (41)
$[A(x,u)]_s$	$\sum_n \sum_{\sigma \in \mathcal{S}_n^{0^+,\star,s}} x^{e(\sigma)} \frac{u^n}{n!}$		Cor. 45 Eq. (42)

$[A_k(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{0^+,k,s}} x^{e(\sigma)}\dfrac{u^n}{n!}$	Cor. 46 Eq. (43)
$[A(q,x,u)]_s$	$\sum_k [A_k(x,u)]_s q^k$ q-analogue of $[A(x,u)]_s$	Cor. 47 Eq. (44)

Table 5: Notation.

Corollary 39. *A q-analogue exponential generating function of the excedance distribution over 2^--cycle permutations (involutions) has the closed form*

$$A^{2^-}(q,x,u) = \exp(q(u + xu^2/2)). \tag{39}$$

Corollary 40. *For a fixed integer n, we have*

$$\sum_{\pi\in\mathcal{I}_{2n}} x^{e(\pi)}(q)^{c(\pi)} = \sum_{j=0}^{n} \frac{(2n)!}{2^{n-k}(2k)!\,(n-k)!} q^{n+k}x^{n-k},$$
$$\sum_{\pi\in\mathcal{I}_{2n+1}} x^{e(\pi)}(q)^{c(\pi)} = \sum_{j=0}^{n} \frac{(2n+1)!}{2^{n-k}(2k+1)!\,(n-k)!} q^{n+k+1}x^{n-k}.$$

Corollary 41. *A q-analogue exponential generating function of the excedance distribution over 2-cycle permutations (involution derangements) has the closed form*

$$A^{(2)}(q,x,u) = \exp(qxu^2/2).$$

Corollary 42. *For a fixed integer n, we have*

$$\sum_{\pi\in\mathcal{I}_{2n}\cap\mathcal{D}_{2n}} x^{e(\pi)}q^{c(\pi)} = \sum_{j=0}^{n} \frac{(2n)!}{2^n n!} q^n x^n.$$

Corollary 43. *For a fixed integer k, the exponential generating function of the excedance distribution over $(0^+,k)$-cycle permutations has the closed form*

$$A_k(x,u) = \sum_{n\geq 0}\sum_{\sigma\in\mathcal{S}_n^{0^+,k}} x^{e(\sigma)}\frac{u^n}{n!} = \frac{\big(\ln(A(x,u))\big)^k}{k!}. \tag{40}$$

Proof. The proof follows from Theorem 34. □

Corollary 44 (Brenti)**.** *A q-analogue exponential generating function of the excedance distribution by cycle over permutations has the closed form*

$$A(q,x,u) = \Big(A(x,u)\Big)^q. \tag{41}$$

(See [3, Proposition 7.3])

Proof. The proof follows from Theorem 35. □

Corollary 45. *For a fixed integer s, the partial differential exponential generating function of the excedance distribution over $(0^+, \star, s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A(x,u)]_s}{\partial u^s} = \Big(A(x,u)\exp(u(x-1))\Big)^s A(x,u). \tag{42}$$

Proof. The proof follows from Theorem 36. □

Corollary 46. *For fixed integers k and s, the partial differential exponential generating function of the excedance distribution over $(0^+, k, s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A_k(x,u)]_s}{\partial u^s} = \Big(A(x,u)\exp(u(x-1))\Big)^s \frac{\big(\ln A(x,u)\big)^{k-s}}{(k-s)!}. \tag{43}$$

Proof. It comes from Theorem 37. □

Corollary 47. *A partial differential q-analogue exponential generating function of the excedance distribution over $(0^+, \star, s)$-cycle permutations has the closed form*

$$\frac{\partial^s [A(q,x,u)]_s}{\partial u^s} = q^s\Big(A(x,u)\exp(u(x-1))\Big)^s \Big(A(x,u)\Big)^q. \tag{44}$$

Proof. We immediately have the result by multiplying the equation (43) by q^k and summing over k. □

Corollary 48.

$$\sum_{\sigma \in S_{n+s}^{0^+,\star,s}} x^{e(\sigma)}(-s)^{c(\sigma)} = (-1)^s s^{n+s}(x-1)^n.$$

Proof. From Corollary 47, we have that

$$\frac{\partial^s [A(q,x,u)]_s}{\partial u^s} = q^s\Big(A(x,u)\exp(u(x-1))\Big)^s \Big(A(x,u)\Big)^q,$$

setting $q = -s$, gives

$$\frac{\partial A(-s,x,u)}{\partial u} = (-s)^s\Big(A(x,u)\exp(u(x-1))\Big)^s \Big(A(x,u)\Big)^{-s}.$$

Hence,

$$\frac{\partial^s [A(q,x,u)]_s}{\partial u^s} = (-s)^s \exp(su(x-1)). \tag{45}$$

Identifying the coefficients of $\dfrac{u^n}{n!}$ in Equation (45) gives the desired proof. □

Corollary 49.

$$\sum_{\sigma \in S_{n+s}^{0+,\star,s}} x^{e(\sigma)}(-s-1)^{c(\sigma)} = (-1)^s(s+1)^s\big(xs^n - (s+1)^n\big)(x-1)^{n-1}.$$

Proof. From Corollary 47, we have

$$\frac{\partial^s[A(q,x,u)]_s}{\partial u^s} = q^s\Big(A(x,u)\exp(u(x-1))\Big)^s\Big(A(x,u)\Big)^q.$$

Setting $q = -s-1$, we have

$$\begin{aligned}\frac{\partial^s[A(-s-1,x,u)]_s}{\partial u^s} &= (-s-1)^s[A(x,u)\exp(u(x-1))]^s(A(x,u))^{-s-1},\\ &= (-s-1)^s\frac{\exp(su(x-1))}{A(x,u)},\\ &= (-s-1)^s\frac{\exp(su(x-1))(x-\exp(u(x-1)))}{x-1},\\ &= \frac{(-s-1)^s}{x-1}\Big(x\exp(su(x-1)) - \exp((s+1)u(x-1))\Big).\end{aligned}$$

Hence,

$$\frac{\partial^s[A(-s-1,x,u)]_s}{\partial u^s} = \frac{(-s-1)^s}{(x-1)}\Big(x\exp(su(x-1)) - \exp((s+1)u(x-1))\Big). \tag{46}$$

Identifying the coefficients of $\frac{u^n}{n!}$ in Equation (46) gives the desired proof. □

Remark 50. If we set $s = 1$ in Corollary 48 and Corollary 49, we get Brenti's results [3, Corollary 7.4].

$$\begin{aligned}\sum_{\pi \in \mathcal{S}_n} x^{e(\pi)}(-1)^{c(\pi)} &= -(x-1)^{n-1},\\ \sum_{\pi \in \mathcal{S}_n} x^{e(\pi)}(-2)^{c(\pi)} &= 2(2^{n-1}-x)(x-1)^{n-2}.\end{aligned}$$

Corollary 51. *A partial differential q-analogue exponential generating function of the excedance distribution over* $(1^+,\star,s)$*-cycle permutations (derangements) has the closed form*

$$\frac{\partial^s[A^1(q,x,u)]_s}{\partial u^s} = q^s x^s\Big(A(x,u)-1\Big)^s\Big(A(x,u)\exp(-u)\Big)^q. \tag{47}$$

Proof. We immediately have the result by setting $r = 1$ in equation (38). □

Corollary 52. *For fixed integers n and s, we have*

$$\sum_{\sigma\in\mathcal{S}_{n+s}^{1^+,\star,s}} x^{e(\sigma)}(-s)^{c(\sigma)} = (-1)^s\Big(\frac{sx}{x-1}\Big)^s \sum_{\substack{i_1\geq 1,\ldots,i_s\geq 1\\ i_1+\cdots+i_s=n}} (i_1,\ldots,i_s)!\ \prod_{j=1}^{s}(x^{i_j}-1). \tag{48}$$

Proof. From Corollary 51 Equation (47), setting $q=-s$, we have

$$\begin{aligned}
\frac{\partial^s[A^1(-s,x,u)]_s}{\partial u^s} &= (-s)^s x^s (A(x,u)-1)^s (A^1(x,u))^{-s},\\
&= (-s)^s x^s \Big(\frac{A(x,u)-1}{A(x,u)}\Big)^s \exp(su),\\
&= (-s)^s x^s (1-1/A(x,u))^s \exp(su),\\
&= (-s)^s x^s \Big(1-\frac{x-\exp((x-1)u)}{x-1}\Big)^s \exp(su),\\
&= \Big(\frac{-sx}{x-1}\Big)^s (-\exp(u)+\exp(xu))^s,\\
&= \Big(\frac{-sx}{x-1}\Big)^s \Big(\sum_{n\geq 1}(-1+x^n)\frac{u^n}{n!}\Big)^s.
\end{aligned}$$

Hence,

$$\frac{\partial^s[A^1(-s,x,u)]_s}{\partial u^s} = (-1)^s\big(\frac{sx}{x-1}\big)^s \sum_{\substack{i_1\geq 1,\ldots,i_s\geq 1\\ i_1+\cdots+i_s=n}} (i_1,\ldots,i_s)!\ \prod_{j=1}^{s}(x^{i_j}-1)\frac{u^n}{n!}. \tag{49}$$

Identifying the coefficients of $\dfrac{u^n}{n!}$ in Equation (49) gives the desired result. □

Remark 53. If we set $s=1$ in Equation (48), we get Ksavrelof and Zeng's result [9]

$$\sum_{\sigma\in\mathcal{D}_n} x^{e(\sigma)}(-1)^{c(\sigma)} = -x-x^2-\cdots-x^{n-1}. \tag{50}$$

Corollary 54. *For fixed integers n and s, we have*

$$\sum_{\sigma\in\mathcal{S}_{n+s}^{1^+,\star,s}} x^{e(\sigma)}(-s-1)^{c(\sigma)} = (-1)^s\frac{(s+1)^s x^s}{(x-1)^s}\sum_{\ell=0}^{n}\left(\frac{x-x^\ell}{x-1}\sum_{\substack{i_1\geq 1,\ldots,i_s\geq 1\\ i_1+\cdots+i_s=n-\ell}}(i_1,\ldots,i_s)!\prod_{j=1}^{s}(x^{i_j}-1)\right).$$

Proof. From Corollary 51 Equation (47), setting $q = -s-1$, we have

$$\begin{aligned}
\frac{\partial^s [A^1(-s-1,x,u)]_s}{\partial u^s} &= (-s-1)^s x^s (A(x,u)-1)^s (A^1(x,u))^{-s-1},\\
&= (-s-1)^s x^s \Big(\frac{A(x,u)-1}{A(x,u)}\Big)^s \exp((s+1)u) A(x,u)^{-1},\\
&= (-s-1)^s x^s (1-1/A(x,u))^s \exp(su) A(x,u)^{-1}\exp(u),\\
&= (-s-1)^s x^s \Big(1-\frac{x-\exp((x-1)u)}{x-1}\Big)^s \exp(su)\frac{x-\exp((x-1)u)}{x-1} e^u,\\
&= \frac{(-(s+1)x)^s}{(x-1)^{s+1}}(-\exp(u)+\exp(xu))^s (x\exp(u)-\exp(xu)),\\
&= \frac{(-(s+1)x)^s}{(x-1)^s}\Big(\sum_{n\geq 1}(-1+x^n)\frac{u^n}{n!}\Big)^s \Big(\sum_{n\geq 0}\frac{(x-x^n)}{x-1}\frac{u^n}{n!}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial^s [A^1(-s-1,x,u)]_s}{\partial u^s} &= (-1)^s \frac{(s+1)^s x^s}{(x-1)^s}\Big(\sum_{\substack{i_1\geq 1,\ldots,i_s\geq 1\\ i_1+\cdots+i_s=n}} (i_1,\ldots,i_s)! \prod_{j=1}^{s}(x^{i_j}-1)\frac{u^n}{n!}\Big)\\
&\quad \Big(1+\sum_{n\geq 1}\frac{(x-x^n)}{x-1}\frac{u^n}{n!}\Big),\\
&= (-1)^s \frac{(s+1)^s x^s}{(x-1)^s}\sum_{n\geq 0}\Big(\sum_{\ell=0}^{n}\frac{(x-x^\ell)}{x-1} \qquad (51)\\
&\quad \Big(\sum_{\substack{i_1\geq 1,\ldots,i_s\geq 1\\ i_1+\cdots+i_s=n-\ell}} (i_1,\ldots,i_s)! \prod_{j=1}^{s}(x^{i_j}-1)\Big)\Big)\frac{u^n}{n!}. \qquad (52)
\end{aligned}$$

Identifying the coefficients of $\frac{u^n}{n!}$ in Equation (52) gives the desired result. □

Corollary 55. *For all integer $n \geq 1$, we have*

$$\sum_{\sigma\in\mathcal{D}_n} x^{e(\sigma)}(-2)^{c(\sigma)} = \frac{-2x(-2^{n-1}x - 2^{n-1}x^{n-1} + (x+1)^n)}{(x-1)^2}. \qquad (53)$$

Proof. From Corollary 51 Equation (47), setting $s = 1$ and $q = -2$, we get

$$\begin{aligned}
\frac{\partial A^1(-2,x,u)}{\partial u} &= -2x(A(x,u)-1)(A^1(x,u))^{-2},\\
&= -2x(A(x,u)-1)\left(A(x,u)\exp(-u)\right)^{-2},\\
&= -2x\left(1/A(x,u) - 1/A(x,u)^2\right)\exp(2u),\\
&= -2x\left(\frac{x-\exp((x-1)u)}{x-1} - \frac{(x-\exp((x-1)u))^2}{(x-1)^2}\right)\exp(2u).
\end{aligned}$$

That is,

$$\frac{\partial A^1(-2,x,u)}{\partial u} = \frac{-2x}{(x-1)^2}\Big(-x\exp(2u) - \exp(2xu) + (x+1)\exp((x+1)u)\Big). \tag{54}$$

Identifying the coefficients of $\frac{u^n}{n!}$ in Equation (54) gives the desired proof. □

Remark 56. Note that $\frac{-2x(-2^{n-1}x - 2^{n-1}x^{n-1} + (x+1)^n)}{(x-1)^2} \in \mathbb{Z}[x]$. Below we find a few values of Equation (53).

1. $\sum_{\sigma\in\mathcal{D}_2} x^{e(\sigma)}(-2)^{c(\sigma)} = -2x$.
2. $\sum_{\sigma\in\mathcal{D}_3} x^{e(\sigma)}(-2)^{c(\sigma)} = -2x(x+1)$.
3. $\sum_{\sigma\in\mathcal{D}_4} x^{e(\sigma)}(-2)^{c(\sigma)} = -2x(x-1)^2$.
4. $\sum_{\sigma\in\mathcal{D}_5} x^{e(\sigma)}(-2)^{c(\sigma)} = -2x(x+1)(x^2-10x+1)$.

7 s-fixed-points-permutations

$\mathcal{F}_n^s$	the set of s-fixed-points-permutations	
$\mathcal{F}_n^{k,s}$	the set of s-fixed-points-permutations with k cycles	
$[F(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{F}_n^s} x^{e(\sigma)}\frac{u^n}{n!}$	Th. 57 Eq. (55)
$[F_k(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{F}_n^{k,s}} x^{e(\sigma)}\frac{u^n}{n!}$	Th. 58 Eq. (56)
$[F(q,x,u)]_s$	$\sum_k [F_k(x,u)]_s q^k$ q-analogue of $[F(x,u)]_s$	Th. 59 Eq. (57)

Table 6: Notation.

Theorem 57. *A partial differential exponential generating function of the excedance distribution over s-fixed-points-permutations has the closed form*

$$\frac{\partial^s [F(x,u)]_s}{\partial u^s} = \Big(A(x,u)\exp(u(x-1))\Big)^s A(x,u)\exp(-u). \tag{55}$$

Proof. Since

$$[F_{n+s,m}]_s = \sum_{\substack{i_1,\dots,i_s\geq 0\\ i_1+\cdots+i_s=n}} (i_1,\dots,i_s)! \sum_{m_1+\cdots+m_s+m_t=m} C_{i_1+1,m_1}\cdots C_{i_s+1,m_s} A^1_{n-\sum_{j=1}^s i_j, m_t},$$

we get

$$[F_{n+s}(x)]_s = \sum_{\substack{i_1,\ldots,i_s\geq 0\\ i_1+\cdots+i_s=n}} (i_1,\ldots,i_s)!\; C_{i_1+1}(x)\cdots C_{i_s+1}(x) A^1_{n-\sum_{j=1}^s i_j}(x).$$

We deduce

$$\begin{aligned}\frac{\partial^s [F(x,u)]_s}{\partial u^s} &= \Big(\frac{\partial}{\partial u} C(x,u)\Big)^s A^1(x,u),\\ &= \Big(A(x,u)\exp\left(u(x-1)\right)\Big)^s A(x,u)\exp(-u).\end{aligned}$$

□

Theorem 58. *Let $k \geq s$ be fixed integers. The partial differential exponential generating function of the excedance distribution over s-fixed-points-permutations having k cycles has the closed form*

$$\frac{\partial^s [F_k(x,u)]_s}{\partial u^s} = \Big(A(x,u)\exp(u(x-1))\Big)^s \frac{\Big(\ln\left(A(x,u)\exp\left(-u\right)\right)\Big)^{k-s}}{(k-s)!}. \tag{56}$$

Proof. Let us consider the first s integers, we have

$$[F_{n+s,m,k}]_s = \sum_{\substack{i_1,\ldots,i_s\geq 0\\ i_1+\cdots+i_s=n}} (i_1,\ldots,i_s)! \sum_{m_1+\cdots+m_s+m_t=m} C_{i_1+1,m_1}\cdots C_{i_s+1,m_s} A^1_{n-\sum_{l=1}^s i_l, m_t, k-s}.$$

We get

$$[F_{n+s,m}(x)]_s = \sum_{\substack{i_1,\ldots,i_s\geq 0\\ i_1+\cdots+i_s=n}} (i_1,\ldots,i_s)!\; C_{i_1+1}(x)\cdots C_{i_s+1}(x) A^1_{n-\sum_{l=1}^s i_l, k-s}(x).$$

Hence,

$$\begin{aligned}\frac{\partial^s [F_k(x,u)]_s}{\partial u^s} &= \left(\frac{\partial C(x,u)}{\partial u}\right)^s A^1_{k-s}(x,u),\\ &= \Big(A(x,u)\exp(u(x-1))\Big)^s \frac{\Big(\ln\left(A(x,u)\exp\left(-u\right)\right)\Big)^{k-s}}{(k-s)!}.\end{aligned}$$

□

Theorem 59. *A partial differential q-analogue exponential generating function of the excedance distribution over s-fixed-points-permutations has the closed form*

$$\frac{\partial^s [F(q,x,u)]_s}{\partial u^s} = q^s\Big(A(x,u)\exp(u(x-1))\Big)^s \Big(A(x,u)\Big)^q \exp(-qu). \tag{57}$$

Proof. We immediately have the result by multiplying by q^k and summing over k the right member of Equation (56) . □

Corollary 60. *For fixed integers n and s, we have*

$$\sum_{\pi \in \mathcal{F}_n^s} x^{e(\pi)}(-s)^{c(\pi)} = (-1)^s s^n x^{n-s}.$$

Corollary 61. *For fixed integers n and s, we have*

$$\sum_{\pi \in \mathcal{F}_n^s} x^{e(\pi)}(-s-1)^{c(\pi)} = (-1)^s (s+1)^s \frac{x(sx+1)^{n-s} - (s+1)^{n-s} x^{n-s}}{x-1}.$$

8 Generalizations

$\mathcal{S}_n^{(r_1)\cdots(r_s)}$	the set of $(r_1)\cdots(r_s)$-cycle permutations	
$\mathcal{S}_n^{r_1\cdots r_s}$	the set of $r_1\cdots r_s$-cycle permutations	
$\mathcal{S}_n^{r_1\cdots r_s(r_{s+1})\cdots(r_\ell)}$	the set of $r_1\cdots r_s(r_{s+1})\cdots(r_\ell)$-cycle permutations	
$\mathcal{S}_n^{r_1\cdots r_s(r_{s+1})\cdots(r_\ell),k}$	$\{\sigma \in \mathcal{S}_n^{r_1\cdots r_s(r_{s+1})\cdots(r_\ell)} : c(\sigma) = k\}$	
$\mathcal{S}_n^{r_1^*\cdots r_\ell^*}$	the set of $r_1^*\cdots r_\ell^*$-cycle permutations	
$[A_{n+s,m,k}^{r_1\cdots r_s}]_s$	$\lvert\{\sigma \in \mathcal{S}_{n+s}^{r_1\cdots r_s,\star,s} : e(\sigma) = m\}\rvert$	
$[A_{n+s,k}^{r_1\cdots r_s}(x)]_s$	$\sum_{\sigma\in\mathcal{S}_{n+s}^{r_1\cdots r_s,k,s}} x^{e(\sigma)}$	
$[A^{r_1\cdots r_s}(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r_1\cdots r_s}} x^{e(\sigma)} \frac{u^n}{n!}$	Th. 62 Eq. (58)
$[A_k^{r_1\cdots r_s}(x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r_1\cdots r_s},k} x^{e(\sigma)} \frac{u^n}{n!}$	Th. 63 Eq. (59)
$[A^{r_1\cdots r_s}(q,x,u)]_s$	q-analogue of $[A^{r_1\cdots r_s}(x,u)]_s$	Th. 64 Eq. (60)
$A^{(r_1)\cdots(r_\ell)}(x,u)$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{(r_1)\cdots(r_\ell)}} x^{e(\sigma)} q^{c(\sigma)} \frac{u^n}{n!}$	Th. 65 Eq. (61)
$[A^{r_1\cdots r_s(r_{s+1})\cdots(r_\ell)}(q,x,u)]_s$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r_1\cdots r_s(r_{s+1})\cdots(r_\ell),\star,s}} x^{e(\sigma)} q^{c(\sigma)} \frac{u^n}{n!}$	Th. 66 Eq. (62)
$A^{r_1^*\cdots r_\ell^*}(q,x,u)$	$\sum_n \sum_{\sigma\in\mathcal{S}_n^{r_1^*\cdots r_\ell^*}} x^{e(\sigma)} q^{c(\sigma)} \frac{u^n}{n!}$	Cor. 67 Eq. (63)

Table 7: Notation.

Theorem 62. *For a fixed integer s, the exponential generating function of the excedance distribution over $(r_1\cdots r_s,\star,s)$-cycle permutations has the differential closed form*

$$\frac{\partial^s [A^{r_1\cdots r_s}(x,u)]_s}{\partial u^s} = x^s \left(A_{r_1-1}(x) \frac{u^{r_1-1}}{(r_1-1)!} \cdots A_{r_s-1}(x) \frac{u^{r_s-1}}{(r_s-1)!} \right) A(x,u). \qquad (58)$$

Proof. Consider the first s integers in a permutation in $\mathcal{S}_{n+s}^{r_1\cdots r_s,\star,s}$ and their cycles. We have

$$[A_{n+s,m}^{r_1\cdots r_s}]_s = \binom{n}{r_1-1,\ldots,r_s-1} \sum_{m_1+\cdots+m_s+m_t=m} C_{r_1,m_1}\cdots C_{r_s,m_s} A_{n-\sum_{\ell=1}^{s}(r_\ell-1),m_t}.$$

The result follows directly. □

Theorem 63. *The partial differential exponential generating function of the excedance distribution over $(r_1\cdots r_s,k,s)$-cycle permutations has the differential closed form*

$$\frac{\partial^s[A_k^{r_1\cdots r_s}(x,u)]_s}{\partial u^s} = x^s\left(A_{r_1-1}(x)\frac{u^{r_1-1}}{(r_1-1)!}\cdots A_{r_s-1}(x)\frac{u^{r_s-1}}{(r_s-1)!}\right)\frac{\big(\ln A(x,u)\big)^{k-s}}{(k-s)!}. \quad (59)$$

Proof. Let us consider again the first s integers in permutations of $\mathcal{S}_{n+s}^{r_1\cdots r_s,k}$, we have

$$[A_{n+s,m,k}^{r_1\cdots r_s}]_s = \binom{n}{r_1-1,\ldots,r_s-1} \sum_{m_1+\cdots+m_s+m_t=m} C_{r_1,m_1}\cdots C_{r_s,m_s} A_{n-\sum_{\ell=1}^{s}(r_\ell-1),m_t,k-s}.$$

The result follows directly. □

Theorem 64. *For a fixed integer s, a partial differential $q-$analogue exponential generating function of the excedance distribution over $r_1\cdots r_s$-cycle permutations has the closed form*

$$\frac{\partial^s[A^{r_1\cdots r_s}(q,x,u)]_s}{\partial u^s} = q^s x^s\left(A_{r_1-1}(x)\frac{u^{r_1-1}}{(r_1-1)!}\cdots A_{r_s-1}(x)\frac{u^{r_s-1}}{(r_s-1)!}\right)\Big(A(x,u)\Big)^q. \quad (60)$$

Proof. We immediately get the result by multiplying by q^k and summing over k the right member of Equation (59). □

Theorem 65. *Let ℓ be a fixed integer. A q-analogue exponential generating function of the excedance distribution over $(r_1)\cdots(r_\ell)$-cycle permutations has the closed form*

$$A^{(r_1)\cdots(r_\ell)}(q,x,u) = \exp\left(qx\Big(A_{r_1-1}(x)\frac{u^{r_1}}{r_1!}+\cdots+A_{r_\ell-1}(x)\frac{u^{r_\ell}}{r_\ell!}\Big)\right). \quad (61)$$

Proof. A permutation σ in $\mathcal{S}_n^{(r_1)\cdots(r_\ell),k}$ is the product of (r_j,k_j)-cycle permutations ($j = 1,\cdots,\ell$) such that $\sum_j k_j = k$. Thus,

$$A_{n,m,k}^{(r_1)\cdots(r_\ell)} = \sum_{i_1+\cdots+i_\ell=n}\ \sum_{m_1+\cdots+m_\ell=m}\ \sum_{k_1+\cdots+k_\ell=k} (i_1,\ldots,i_\ell)!\ A_{i_1,m_1,k_1}^{(r_1)}\cdots A_{i_\ell,m_\ell,k_\ell}^{(r_\ell)}.$$

It follows that,

$$\begin{aligned}A^{(r_1)\cdots(r_\ell)}(q,x,u) &= A^{(r_1)}(q,x,u)\cdots A^{(r_\ell)}(q,x,u),\\ &= \exp\Big(qxA_{r_1-1}(x)\frac{u^{r_1}}{r_1!}\Big)\cdots\exp\Big(qxA_{r_\ell-1}(x)\frac{u^{r_\ell}}{r_\ell!}\Big).\end{aligned}$$

This gives the result. □

Theorem 66. *Let ℓ be a fixed integer. Let $(r_1, \ldots, r_\ell)$ be the sequence such that $r_i > 1,\ i = 1, \ldots, s$ and $1 < r_{s+1} < \cdots < r_\ell$. A partial differential q-analogue exponential generating function of the excedance distribution over $r_1 \cdots r_s(r_{s+1}) \cdots (r_\ell)$-cycle permutations has the closed form*

$$\frac{\partial^s [A^{r_1 \cdots r_s(r_{s+1}) \cdots (r_\ell)}(q,x,u)]_s}{\partial u^s} = q^s x^s (A_{r_1-1}(x)\frac{u^{r_1-1}}{(r_1-1)!} \cdots A_{r_s-1}(x)\frac{u^{r_s-1}}{(r_s-1)!}) \left(A^{(r_{s+1})\cdots(r_\ell)}(x,u)\right)^q. \quad (62)$$

Proof. Consider the first s integers.

$$[A^{r_1 \cdots r_s(r_{s+1}) \cdots (r_\ell)}_{n+s,m,k}]_s = \binom{n}{r_1-1, \ldots, r_{s-1}} \sum_{m_1+\cdots+m_s+m_t=m} C_{r_1,m_1} \cdots C_{r_s,m_s} \times A^{(r_{s+1})\cdots(r_\ell)}_{n-\sum_{i=1}^{s} r_i-1, m_t, k-s}.$$

Therefore, we obtain

$$\begin{aligned}\frac{\partial^s [A^{r_1 \cdots r_s(r_{s+1}) \cdots (r_\ell)}(q,x,u)]_s}{\partial u^s} &= q^s C_{r_1}(x)\frac{u^{r_1-1}}{(r_1-1)!} \cdots C_{r_s}(x)\frac{u^{r_s-1}}{(r_s-1)!} \times \\ &\left(A^{(r_{s+1})\cdots(r_\ell)}(x,u)\right)^q, \\ &= q^s x^s \left(A_{r_1-1}(x)\frac{u^{r_1-1}}{(r_1-1)!} \cdots A_{r_s-1}(x)\frac{u^{r_s-1}}{(r_s-1)!}\right) \times \\ &\left(A^{(r_{s+1})\cdots(r_\ell)}(x,u)\right)^q.\end{aligned}$$

□

Corollary 67. *For a fixed integer ℓ, let $(r_1, \ldots, r_\ell)$ be a sequence such that $1 < r_1 < \cdots < r_\ell$. A q-analogue exponential generating function of the excedance distribution over $r_1^\star \cdots r_\ell^\star$-cycle permutations has the closed form*

$$A^{r_1^\star \cdots r_\ell^\star}(q,x,u) = A(q,x,u) \exp\left(-qx \sum_{i=1}^{\ell} A_{r_i-1} \frac{u^{r_i}}{r_i!}\right). \quad (63)$$

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References

[1] P. Appell, Développements en série entière de $(1+ax)^{1/x}$, *Arch. Math. Phys.* **65** (1880), 171–175.

[2] H. Belbachir and I. E. Bousbaa, Associated Lah numbers and r-Stirling numbers, arxiv preprint arXiv:1404.5573 [math.CO], 2014. Available at https://arxiv.org/abs/1404.5573.

[3] F. Brenti, A class of q-symmetric functions arising from plethysm, *J. Combin. Theory Ser. A* **91** (2000), 137–170.

[4] A. Z. Broder, The r-Stirling numbers, *Discrete Math.* **49** (1984), 241–259.

[5] L. Carlitz, On some polynomials of Tricomi, *Boll. Unione Mat. Ital.* **13** (1958), 58–64.

[6] L. Carlitz, Note on Nörlund's polynomials $B_n^{(z)}$: *Proc. Amer. Math. Soc.* **11** (1990), 447–451.

[7] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel, 1974.

[8] D. Foata and M. P. Schützenberger, *Théorie Géométrique des Polynômes Eulériens*, Lecture Notes in Mathematics, Vol. 138, Springer, 1970.

[9] G. Ksavrelof and J. Zeng, Two involutions for signed excedance numbers, *Sém. Lothar. Combin.* **49** (2003), B49e.

[10] J. Ma, S. M. Ma, Y. Yeh, and Z. Xu, The cycle descent statistic on permutations, *Electron. J. Combin.* **23** (2016), #P4.20.

[11] R. Mantaci and F. Rakotondrajao, Exceedingly deranging!, *Adv. Appl. Math.* **30** (2003), 177–188.

[12] F. Rakotondrajao, k-fixed-points-permutations, *INTEGERS* **7** (2007), #A36.

[13] J. Riordan, *An Introduction to Combinatorial Analysis*, Princeton University Press, 1978.

[14] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[15] F. G. Tricomi, A class of non-orthogonal polynomials related to those of Laguerre, *J. Anal. Math.* **1** (1951), 209–231.

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