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Motzkin Paths of Bounded Height with Two Forbidden Contiguous Subwords of Length Two

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Abstract

We revisit the topic of Motzkin excursions and meanders, which we consider in the context of forbidden patterns. We continue previous work of Asinowski, Banderier, Gittenberger, and Roitner, leading to matrix equations and also continued fractions. The enumeration is done by properly setting up bivariate generating functions that can be expanded using the kernel method.

Contents

1 Introduction

Motzkin excursions (Motzkin paths) consist of the steps $U = (1, 1)$, $D = (1, -1)$, $H = (1, 0)$, start at the origin, never go below the x-axis, and return to the x-axis. Motzkin meanders (partial Motzkin paths, prefixes of Motzkin paths) are similar, but do not necessarily return to the x-axis. The letter H stands for 'horizontal'; sometimes one finds the symbol F ('flat'). The example excursion (Figure 1) does not contain the (contiguous) pattern UD.

The notations ‘excursion’ and ‘meander’ were Philippe Flajolet’s favorites; see, e.g., [3]. In his honor, they are be used here.

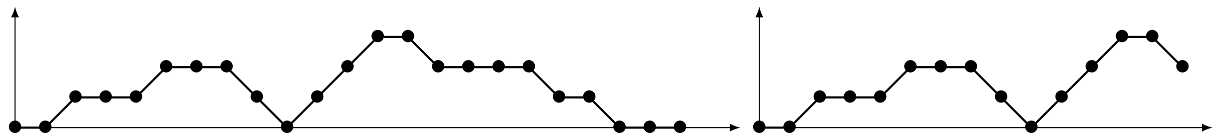


Figure 1: A Motzkin excursion and a Motzkin meander (ending at level 2).

This paper deals with *forbidden subpatterns*. In our analysis, we typically have *two* contiguous patterns that are forbidden. Our effort is a continuation of earlier work by Asinowski, Banderier, Gittenberger, Roitner (and perhaps others) [1, 2, 7]. What we did:

- We rederived appropriate generating functions.
- We worked out (almost?) all interesting cases explicitly.
- We considered versions of *bounded height*; the example excursion (Figure 1) has height 3, which is the largest ordinate value that occurred.
- We obtained generating functions for (restricted) Motzkin meanders ending on a prescribed level.
- We provided continued fraction expansions for the generating functions of Motzkin excursions. Note that, since the generating functions are always algebraic of order 2, there *must* be a periodic expansion. However, the expansions we obtained are motivated by the bounded height model and thus have a combinatorial meaning.
- Unlike other papers where one often reads ‘other cases may be done in a similar fashion’, here these ‘other cases’ are actually investigated.

What we did *not* do, although it would be possible:

- Derive generating functions of bounded restricted Motzkin meanders with prescribed end level.
- Consider questions of an asymptotic type, e.g., the average height of restricted Motzkin excursions.
- Allow the paths to go below the x-axis.

For the readers’ convenience, we prepared tables of the cases that were considered.

forbidden	UU, HH	UD, HH	DU, HH	DD, HH
excursion	A329666	A329676	A329666	A329666
meander	A329667	A329675	A329668	A329669

forbidden	UH, HU	DH, HD	UH, HD	DH, HU
excursion	A329701	A329701	A329702	A329701
meander	—	—	—	—

forbidden	UU, DD	UU, DU	UU, HD	UU, UD
excursion	A004149	A023431	A217282	A023431
meander	A308435	—	—	—

forbidden	UD, DD	DD, DU	UD, DU	UD, DH
excursion	A023431	A023431	A004149	A023431
meander	—	—	A308435	—

forbidden	DU, HD	DU, UH	HU, DD	UD, HU
excursion	A004149	A217282	A217282	—
meander	A308435	—	—	—

forbidden	UU, HU	DU, DH	DU, HU	UD, UH	UD, HD
excursion	degenerate	degenerate	degenerate	degenerate	degenerate

forbidden	HU, HD	HU, HH
excursion	degenerate	degenerate

Note that, even for sequences that were in OEIS, the description might not exactly match our application.

The motivation to provide such a quasi-complete list originates from Drew Sills' amazing rendition of Lucy Slater's account on Rogers-Ramanujan (type) identities [8].

2 Outline of the approach

- All of our examples consist of three states; since we take care of the current altitude of the path as well, we talk instead about layers. One layer is responsible for U-steps, leading into it; another one for H-steps; and another one for D-steps.

- We consistently use the letters f, g, h (and F, G, H) for the three layers; for each case, a nice drawing is provided. There are recursions for $f_i = f_i(z)$, $g_i = g_i(z)$, $h_i = h_i(z)$, according the possible steps leading into a layer.

- Double (bivariate) generating functions $F(u) = F(u, z) = \sum_{i\geq 0} f_i u^i$ etc. are introduced. The recursions translate into three functional equations for $F(u), G(u), H(u)$. They can be solved easily, but contain the quantities $F(0)$, $G(0)$, and $H(0)$. Substitution of $u = 0$ is not sufficient, but in each instance, there is a universal denominator that factorizes as $(u-r_1(z))(u-r_2(z))$. One of these factors is always 'bad', in the sense that $1/(u - r_1(z))$ does not have a power series expansion around $(0, 0)$. However, for combinatorial reasons, this cannot be, so the bad factor must have a bad counterpart in the numerator, and cancellation must occur. Our notation is consistent, so that $u - r_1(z)$ is the bad factor.

- Once the bad factors disappear, setting $u = 0$ is possible, and thus $F(0), G(0), H(0)$ can be computed. This procedure is typical for the kernel method, and many attractive examples have been collected in [5, 6]. The total is always denoted by $S(u) = F(u) + G(u) + H(u)$, and $[u^j]S(u)$ is the generating function of restricted Motzkin meanders ending at level j. So we get $S(0)$ (excursions), $S(1)$ meanders, and, as a bonus $[u^j]S(u)$.

- Since there is the denominator, which is quadratic in u, we obtain recursions of order 2 for f_j, g_j, h_j, s_j. In order to introduce the concept of *bounded height*, we replace f_j, g_j, h_j, s_j by 0 for $j > K$. The resulting recursion can always be written as a linear system, involving a $(K + 1) \times (K + 1)$ matrix. We solve for f_0, g_0, h_0, s_0 and use the notations $\phi_{K+1}, \gamma_{K+1}, \eta_{K+1}, \sigma_{K+1}$ for the results. This is always done by Cramer's rule, and expresses the solutions as a quotient of two determinants.

- After a warm-up example, we discuss the issue that the layer that describes a symbol D leading into it (say it is the third layer), contains a state h_K but it cannot be reached, since $f_{K+1}, g_{K+1}, h_{K+1}, s_{K+1}$, are *taboo*. So, strictly speaking, the bounded version of s_j is slightly incorrect, as it allows contributions that should not be allowed. So $s^{[K+1]} = f^{[K+1]} + g^{[K+1]} + h^{[K+1]}$ (ad hoc notation) is computed, but $f^{[K+1]} + g^{[K+1]} + h^{[K]}$ would be correct . However, each of the 3 constituents are computed individually.

- Since $(r_1/r_2)^K$ goes to 0 for large K, the bounded versions converge to the unbounded versions. Convergence is in terms of power series, in that more and more coefficients are correct.

- The $(K + 1) \times (K + 1)$ matrix has an irregular first row and column. Otherwise, it is a band matrix, and the determinant satisfies a recursion of order 2 (and thus allows for a Binet form). This recursion may be rewritten in terms of continued fractions—not uncommon when dealing with objects of bounded height.

3 DH and HD are forbidden, A329701

The generating functions relative to Figure 2 satisfy the system

$$\begin{aligned} f_i &= zf_i + zg_i, \;\; f_0 = \frac{z}{1-z}, \\ g_{i+1} &= zf_i + zg_i + zh_i, \;\; g_0 = 1, \\ h_i &= zg_{i+1} + zh_{i+1}, \end{aligned}$$

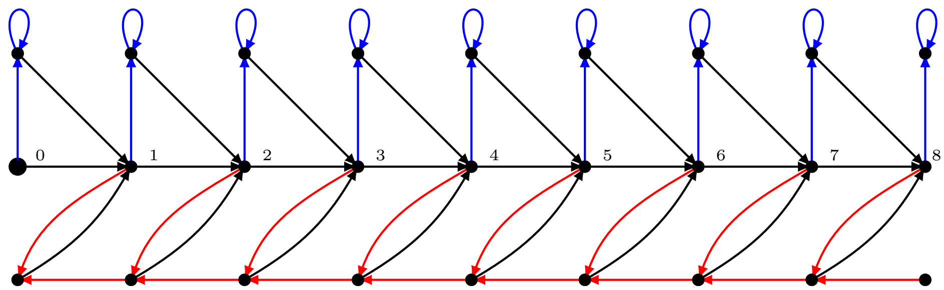


Figure 2: Graph (automaton) to recognize Motzkin paths with forbidden subwords DH and HD.

which translates into bivariate generating functions as follows:

$$\begin{aligned}F(u) &= zF(u) + zG(u),\\ G(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\ H(u) &= \frac{z}{u}\big(G(u) - 1\big) + \frac{z}{u}\big(H(u) - H(0)\big).\end{aligned}$$

Solving the system leads to

$$\begin{aligned}F(u) &= \frac{z(z^2u + z^2uH(0) - u + z)}{-z^3u - u + zu + zu^2 + z - z^2},\\ G(u) &= \frac{(1-z)(z^2u + z^2uH(0) - u + z)}{-z^3u - u + zu + zu^2 + z - z^2},\\ H(u) &= \frac{-z(-H(0) + zH(0) + zu + zuH(0))}{-z^3u - u + zu + zu^2 + z - z^2}.\end{aligned}$$

We factorize the denominator as $z(u - r_1)(u - r_2)$ with

$$\begin{aligned}r_1 &= \frac{1 - z + z^3 - W}{2z},\ r_2 = \frac{1 - z + z^3 + W}{2z},\\ W &= \sqrt{(1 + z - 2z^2 - z^3)(1 - 3z + 2z^2 - z^3)}.\end{aligned}$$

Dividing out the factor $(u - r_1)$ according to the kernel method[1] leads to the expressions

$$F(u) = \frac{z}{1 - z - ur_1},\ G(u) = \frac{1-z}{1 - z - ur_1},\ H(u) = \frac{r_1 - z}{z(1 - z - ur_1)},\ S(u) = \frac{r_1}{z(1 - z - ur_1)},$$

[1]Some authors prefer to multiply such an equation by the denominator, as in $z(u - r_1)(u - r_2)F(u) = z(z^2u + z^2uH(0) - u + z)$; by substituting $u = r_1$, one gets $0 = z(z^2r_1 + z^2r_1H(0) - r_1 + z)$ and can compute $H(0)$ from this. However, we stay here with the version I learnt from Knuth [4], also because I trained myself to use Maple and the operations rem and quo for remainder (resp., quotient). In our examples, there is not really a substantial difference.

whence

$$S(0)=\frac{r_1}{z(1-z)},\quad [u^j]S(u)=s_j=\frac{1}{z}\Big(\frac{r_1}{1-z}\Big)^{j+1};$$

the latter is the generating function of restricted Motzkin paths ending on level j. The generating function $S(0)$ of restricted Motzkin excursions is in the OEIS (A329701), but

$$S(1)=1+2z+5z^2+11z^3+26z^4+60z^5+142z^6+334z^7+794z^8+1885z^9+\cdots,$$

the generating function of meanders, is not.

Using the denominator $-z^3u-u+zu+zu^2+z-z^2$, we derive recursions

$$\begin{gathered}zs_{j-2}+(z-z^3-1)s_{j-1}+z(1-z)s_j=0,\ j\geq 2,\\ (z-z^3-1)s_0+z(1-z)s_1=-1,\end{gathered}$$

which, upon restricting ourselves to states not larger than K, can be written as a linear system

$$\begin{pmatrix}-1+z-z^3 & z(1-z) & 0 & \cdots & 0\\ z & -1+z-z^3 & z(1-z) & \cdots & 0\\ & z & -1+z-z^3 & z(1-z) & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & z & -1+z-z^3\end{pmatrix}\begin{pmatrix}s_0\\ s_1\\ s_2\\ \vdots\\ s_K\end{pmatrix}=\begin{pmatrix}-1\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

Denoting the determinant of the matrix without first row and column (a $K\times K$-matrix) by $\mathscr{D}_K$, we derive a recursion

$$\mathscr{D}_i=(z-z^3-1)\mathscr{D}_{i-1}-z^2(1-z)\mathscr{D}_{i-2},\ \mathscr{D}_0=1,\ \mathscr{D}_1=z-z^3-1$$

and, using $t_i=\mathscr{D}_{i-1}/\mathscr{D}_i$ and $\tau_i=-z^2(1-z)t_i$, ($\tau_0=0$),

$$t_i=\frac{1}{z-z^3-1-z^2(1-z)t_{i-1}},\quad \tau_i=\frac{z^2(1-z)}{1-z+z^3-\tau_{i-1}}.$$

There is a Binet formula for the determinant

$$\mathscr{D}_j=\frac{(-z)^{j+1}}{W}\big(r_1^{j+1}-r_2^{j+1}\big),$$

and the solution of the component s_0 may be achieved by Cramer's rule as a quotient of determinants:

$$\begin{aligned}s_0=\sigma_{K+1}&=\frac{-\mathscr{D}_K}{(z-z^3-1)\mathscr{D}_K-z^2(1-z)\mathscr{D}_{K-1}}\\ &=\frac{-1}{(z-z^3-1)-z^2(1-z)t_K}=\frac{1}{1-z+z^3-\tau_K}.\end{aligned}$$

So we have the continued fraction type expansion

$$\sigma_{K+1} = \frac{1}{1-z+z^3-\tau_K}, \quad \tau_K = \frac{z^2(1-z)}{1-z+z^3-\tau_{K-1}}, \ \tau_0 = 0.$$

Now we compute the individual quantities, according to restricted Motzkin paths ending in the first (resp., second, resp., third) layer $f_0 = \frac{z}{1-z}$ and $g_0 = 1$, regardless of K, so we concentrate on the third layer.

$$\begin{pmatrix} -1+z+z^2-z^3 & z(1-z) & 0 & \cdots & 0 \\ z & -1+z-z^3 & z(1-z) & \cdots & 0 \\ & z & -1+z-z^3 & z(1-z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Again by Cramer's rule,

$$h_0 = \eta_{K+1} = \frac{-z^2 \mathscr{D}_K}{(-1+z+z^2-z^3)\mathscr{D}_K - z^2(1-z)\mathscr{D}_{K-1}}, \quad \eta_0 = 0.$$

This formula produces $\eta_1 = \frac{z^2}{(1-z)(1-z^2)}$, which can be seen directly from the graph as a check. Further, we can see that $\sigma_1 = \frac{1}{(1-z)(1-z^2)}$, which also checks. Finally we derive continued fraction type formulas for the individual quantities ϕ_K, γ_K.

$$\begin{pmatrix} -z & (1-z)^2 & 0 & \cdots & 0 \\ z & -1+z-z^3 & z(1-z) & \cdots & 0 \\ & z & -1+z-z^3 & z(1-z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^3 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_K \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (z+1)(z-1)^2 & z(z-1) & 0 & \cdots & 0 \\ z & -1+z-z^3 & z(1-z) & \cdots & 0 \\ & z & -1+z-z^3 & z(1-z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{aligned} f_0 = \phi_{K+1} &= \frac{-z\mathscr{D}_K}{-z\mathscr{D}_K - z(1-z)^2\mathscr{D}_{K-1}} = \frac{1}{1+(1-z)^2 t_K} = \frac{z^2}{z^2-(1-z)\tau_K}, \ \phi_0 = \frac{z}{1-z}, \\ g_0 = \gamma_K &= 1, \\ h_0 = \eta_{K+1} &= \frac{z^2\mathscr{D}_K}{(z+1)(z-1)^2\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = \frac{z^2}{1-z-z^2+z^3-\tau_K}, \ \eta_0 = 0. \end{aligned}$$

4 UD and DU are forbidden, A004149

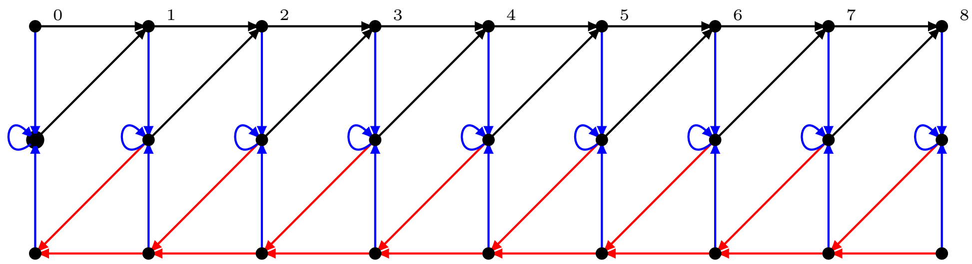


Figure 3: Graph (automaton) to recognize Motzkin paths with forbidden subwords UD and DU.

According to Figure 3, we have

$$\begin{aligned}f_{i+1} &= zf_i + zg_i,\ \ f_0 = 0,\\ g_i &= [i=0] + zf_i + zg_i + zh_i,\\ h_i &= zg_{i+1} + zh_{i+1}\end{aligned}$$

and as a consequence

$$\begin{aligned}F(u) &= zuF(u) + zuG(u),\\ G(u) &= 1 + zF(u) + zG(u) + H(u),\\ H(u) &= \frac{z}{u}(G(u) - G(0)) + \frac{z}{u}(H(u) - H(0)).\end{aligned}$$

We have $zu^2 + (-1 + z - z^2 - z^3)u + z = z(u - r_1)(u - r_2)$ with

$$r_1 = \frac{1 - z + z^2 + z^3 - W}{2z},\ \ r_2 = \frac{1 - z + z^2 + z^3 + W}{2z},\ \ W = \sqrt{(1 - z^4)(1 - 2z - z^2)}.$$

Dividing out the bad factor $u - r_1$ from the components of the solved system, we end up with

$$F(u) = -1 + \frac{1}{1 - ur_1},\ \ G(u) = \frac{r_1(1 - zu)}{z(1 - ur_1)},\ \ H(u) = \frac{r_1(1 - z) - z}{z^2(1 - ur_1)},\ \ S(u) = \frac{r_1 - z}{z^2(1 - ur_1)}.$$

From the denominator $zu^2 + (-1 + z - z^2 - z^3)u + z$ we derive the recursion

$$\begin{aligned}zs_{j-2} + (-1 + z - z^2 - z^3)s_{j-1} + zs_j = 0,\ \ j \geq 2,\\ (-1 + z - z^3)s_0 + zs_1 = -1 + z^2\end{aligned}$$

or, ignoring s_i with $i > K$, in matrix form

$$\begin{pmatrix} -1+z-z^3 & z & 0 & \cdots & 0 \\ z & -1+z-z^2-z^3 & z & \cdots & 0 \\ & z & -1+z-z^2-z^3 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^2-z^3 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1+z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

As always, we consider the $K \times K$ determinant after erasing the first row (resp., column) and derive a recursion (resp., solution) for it:

$$\mathscr{D}_K = (-1+z-z^2-z^3)\mathscr{D}_{K-1} - z^2\mathscr{D}_{K-2}, \quad \mathscr{D}_K = \frac{(-z)^{K+1}}{W}(r_1^{K+1} - r_2^{K+1}).$$

Further, introducing $t_i = \mathscr{D}_{i-1}/\mathscr{D}_i$ and $\tau_i = -z^2 t_i$ we derive the continued fraction type recursion

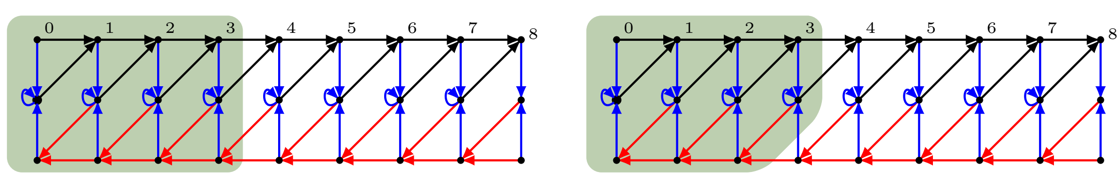
$$t_K = \frac{1}{-1+z-z^2-z^3-z^2 t_{K-1}}, \quad \tau_K = \frac{z^2}{1-z+z^2+z^3-\tau_{K-1}}, \ \tau_0 = 0.$$

The s_0-component of the system can be solved by Cramer's rule:

$$s_0 = \sigma_{K+1} = \frac{(-1+z^2)\mathscr{D}_K}{(-1+z-z^3)\mathscr{D}_K - z^2\mathscr{D}_{K-1}} = \frac{(-1+z^2)}{(-1+z-z^3)-z^2 t_K} = \frac{1-z^2}{1-z+z^3-\tau_K}.$$

Working with $s_j^{[K]}$ (states larger than K ignored) *alone* might lead to small anormalities. According to the Figures 4 (which shows the instance $K=3$), $f_j^{[K]} + g_j^{[K]} + h_j^{[K]}$ is considered. But it should really be $f_j^{[K]} + g_j^{[K]} + h_j^{[K-1]}$ as the K-state in the third layer cannot be reached.

To achieve this, we have to compute $f_j^{[K]}$, $g_j^{[K]}$, $h_j^{[K]}$ separately.



Now $f_0 = \phi_K = 0$, regardless of K. Further

$$\begin{pmatrix} -1+z-z^3 & z & 0 & \cdots & 0 \\ z & -1+z-z^2-z^3 & z & \cdots & 0 \\ & z & -1+z-z^2-z^3 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^2-z^3 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1 \\ z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and thus

$$g_0 = \gamma_{K+1} = \frac{-\mathscr{D}_K - z^2\mathscr{D}_{K-1}}{(-1+z-z^3)\mathscr{D}_K - z^2\mathscr{D}_{K-1}}, \ \gamma_0 = \frac{1}{1-z}.$$

Similarly,

$$\begin{pmatrix} -1+2z-z^2+z^4 & z(1-z) & 0 & \cdots & 0 \\ z & -1+z-z^2-z^3 & z & \cdots & 0 \\ & z & -1+z-z^2-z^3 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^2-z^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and thus

$$h_0 = \eta_{K+1} = \frac{-z^3 \mathscr{D}_K}{(-1+2z-z^2+z^4)\mathscr{D}_K - z^2(1-z)\mathscr{D}_{K-1}}, \ \eta_0 = 0.$$

The generating function of Motzkin excursions

$$S(0) = 1 + z + z^2 + 2z^3 + 4z^4 + 8z^5 + 16z^6 + 33z^7 + 69z^8 + 146z^9 + 312z^{10} + \cdots$$

is A004149 in the OEIS, and Motzkin meanders

$$S(1) = 1 + 2z + 4z^2 + 9z^3 + 20z^4 + 45z^5 + 102z^6 + 233z^7 + 535z^8 + 1234z^9 + 2857z^{10} \cdots$$

is A308435 in the OEIS.

5 DD and DU are forbidden, A023431

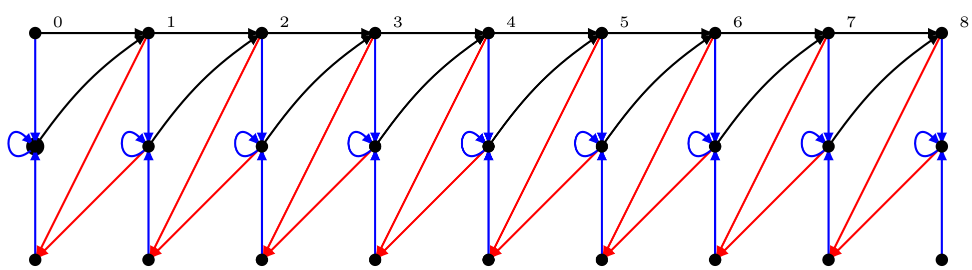


Figure 4: Graph (automaton) to recognize Motzkin paths with forbidden subwords DD and DU.

According to Figure 4, we find

$$\begin{aligned} f_{i+1} &= zf_i + zg_i, \ f_0 = 0, \\ g_i &= [i=0] + zf_i + zg_i + zh_i, \ i \geq 0, \\ h_i &= zf_{i+1} + zg_{i+1} \end{aligned}$$

and thus

$$\begin{aligned}
F(u) &= zuF(u) + zuG(u),\\
G(u) &= 1 + zF(u) + zG(u) + zH(u),\\
H(u) &= \frac{z}{u}F(u) + \frac{z}{u}(G(u) - G(0)).
\end{aligned}$$

According to the denominator $z^2 + (-1+z)u + zu^2 = z(u-r_1)(u-r_2)$ we find the simplified versions

$$F(u) = -1 + \frac{z}{z - ur_1},\ G(u) = 1 + \frac{r_1 - z^2}{z(z - ur_1)},\ H(u) = \frac{r_1^2}{z(z - ur_1)},\ S(u) = \frac{r_1(1+r_1)}{z(z - ur_1)}$$

with

$$r_1 = \frac{1 - z - W}{2z},\ r_2 = \frac{1 - z + W}{2z},\quad W := \sqrt{1 - 2z + z^2 - 4z^3}.$$

The denominator provides the recursion

$$\begin{gathered}
zs_{j-2} + (z-1)s_{j-1} + z^2 s_j = 0,\ j \geq 2,\\
-(1 - z - z^3)s_0 + z^2 s_1 = -1 - z^2
\end{gathered}$$

and in matrix form

$$\begin{pmatrix}
-1+z+z^3 & z^2 & 0 & \cdots & 0\\
z & z-1 & z^2 & \cdots & 0\\
 & z & z-1 & z^2 & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & z & z-1
\end{pmatrix}
\begin{pmatrix} s_0\\ s_1\\ s_2\\ \vdots\\ s_K \end{pmatrix}
=
\begin{pmatrix} -1-z^2\\ 0\\ 0\\ \vdots\\ 0 \end{pmatrix}$$

With the usual determinant we find

$$\mathscr{D}_K = (z-1)\mathscr{D}_{K-1} - z^3\mathscr{D}_{K-2} = \frac{(-z)^{K+1}}{W}(r_1^{K+1} - r_2^{K+1})$$

$$t_K = \frac{1}{(z-1) - z^3 t_{K-1}},\ z^3 t_K = \frac{z^3}{-1 + z - z^3 t_{K-1}}, \tau_K = \frac{z^3}{1 - z - \tau_{K-1}},\quad \tau_0 = 0$$

$$\begin{aligned}
s_0 = \sigma_{K+1} &= \frac{-(1+z^2)\mathscr{D}_K}{(-1+z+z^3)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{1+z^2}{1 - z - z^3 + z^3 t_K}\\
&= \frac{1+z^2}{1 - z - z^3 - \tau_K},\quad \sigma_0 = \frac{1}{1-z}.\quad \tau_K = \frac{-z^3\mathscr{D}_{K-1}}{\mathscr{D}_K}
\end{aligned}$$

Here it is again wise to compute the three components (with bound K) separately, since state K in the third layer cannot be reached. Now $f_0 = \phi_K$ is always zero, but

$$\begin{pmatrix} -1+z+z^3 & z^2 & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1 \\ z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Consequently,

$$\gamma_{K+1} = \frac{-\mathscr{D}_K - z^3\mathscr{D}_{K-1}}{(-1+z+z^3)\mathscr{D}_K - z^3\mathscr{D}_{K-1}}, \quad \gamma_0 = \frac{1}{1-z}.$$

Similarly,

$$\begin{pmatrix} -1+2z-z^2+z^3 & z^2(1-z) & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^2 \\ z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Consequently,

$$\eta_{K+1} = \frac{-z^2\mathscr{D}_K}{(-1+2z-z^2+z^3)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}}, \quad \eta_0 = 0.$$

The generating function $S(0)$ of excursions is A023431 in OEIS, and

$$\begin{aligned} S(1) &= \frac{1+z}{1-2z-z^2} + \frac{(z-1)r_1}{z(1-2z-z^2)} \\ &= 1 + 2z + 5z^2 + 12z^3 + 28z^4 + 66z^5 + 157z^6 + 374z^7 + 892z^8 + 2132z^9 + \cdots, \end{aligned}$$

the generating function of meanders, is not in OEIS.

Since

$$S(u) = \frac{r_1(1+r_1)}{z^2(1-\frac{ur_1}{z})} \quad \Longrightarrow \quad [u^j]S(u) = s_j = \frac{1+r_1}{z}\Big(\frac{r_1}{z}\Big)^{j+1};$$

the last quantity is the generating function of (restricted) Motzkin paths ending on level j.

6 UH and HU are forbidden, A329701

The following recursions can be read off the automaton (Figure 5), by considering the last step separately:

$$\begin{aligned} f_i &= zf_i + zg_i, \\ g_i &= [i=0] + zf_{i+1} + zg_{i+1} + zh_{i+1}, \\ h_{i+1} &= zg_i + zh_i, \ i \ge 0, \ h_0 = 0, \end{aligned}$$

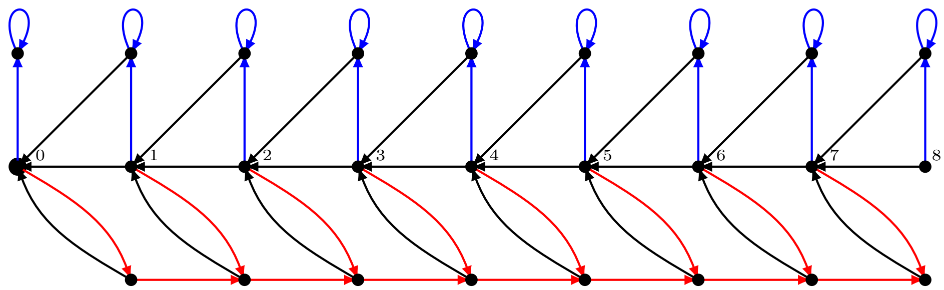


Figure 5: Graph (automaton) to recognize Motzkin paths with forbidden subwords UH and HU.

and using double generating functions

$$
\begin{aligned}
F(u) &= zF(u) + zG(u) = \frac{zG(u)}{1-z},\\
G(u) &= 1 + \frac{z}{u}\big(F(u) - \frac{z}{1-z}G(0)\big) + \frac{z}{u}\big(G(u) - G(0)\big) + \frac{z}{u}H(u),\\
H(u) &= zuG(u) + zuH(u).
\end{aligned}
$$

For $S = F + G + H$ we get

$$S = -\frac{(-u + zu + zG(0))\,(z^2u - 1)}{(uz^3 - zu + z^2u^2 - z + u - zu^2)\,(-1+z)}$$

We have the factorization $(uz^3 - zu + z^2u^2 - z + u - zu^2) = z(z-1)(u-r_1)(u-r_2)$ with

$$r_1 = \frac{1 - z + z^3 - W}{2z(1-z)},\ r_2 = \frac{1 - z + z^3 + W}{2z(1-z)},\ W = \sqrt{(1 + z - 2z^2 - z^3)(1 - 3z + 2z^2 - z^3)}.$$

After dividing out the factor $u - r_1$ from numerator and denominator and simplification,

$$F(u) = \frac{(1-zu)r_1}{1-u(1-z)r_1},\ G(u) = \frac{(1-zu)(1-z)r_1}{z(1-u(1-z)r_1)},\ H(u) = \frac{(1-z)ur_1}{1-u(1-z)r_1},$$
$$S(u) = \frac{(1-z^2u)r_1}{z(1-u(1-z)r_1)}.$$

From this we find

$$S(0) = 1 + z + 2z^2 + 2z^3 + 4z^4 + 5z^5 + 11z^6 + 17z^7 + 38z^8 + 67z^9 + \cdots,$$

which is A329701. The generating function for meanders is

$$S(1) = 1 + 2z + 3z^2 + 5z^3 + 9z^4 + 17z^5 + 34z^6 + 69z^7 + 144z^8 + 302z^9 + \cdots,$$

which is not in OEIS.

For $j \geq 1$,

$$[u^j]S(u) = [u^j]\frac{(1-z^2u)r_1}{z(1-u(1-z)r_1)} = \frac{(1-z)^j r_1^{j+1}}{z} - z(1-z)^{j-1}r_1^j$$

which is the generating function of restricted Motzkin paths ending at level j.

Now we move to paths of bounded height. We have

$$S = \frac{\text{numerator}}{z(z-1)u^2 + u(z^3 - z + 1) - z}$$

and a similar shape for F, G, H. This leads to a recursion: Set $s_j = [u^j]S$, then

$$\begin{aligned} z(z-1)s_j + (z^3 - z + 1)s_{j+1} - zs_{j+2} &= 0, \quad j \geq 1, \\ z(z-1)s_0 + (z^3 - z + 1)s_1 - zs_2 &= -z^2, \\ (z-1)s_0 + zs_1 &= -1, \end{aligned}$$

which is in matrix form

$$\begin{pmatrix} z-1 & z & 0 & \cdots & & 0 \\ z(z-1) & z^3 - z + 1 & -z & \cdots & & 0 \\ & z(z-1) & z^3 - z + 1 & -z & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & z(z-1) & z^3 - z + 1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1 \\ -z^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $\mathscr{D}_K$ be the determinant with K rows and columns, after deleting the first row and column. Expanding, we derive the recursion ($\mathscr{D}_0 = 1,\ \mathscr{D}_1 = 1 - z + z^3$)

$$\mathscr{D}_K = (z^3 - z + 1)\mathscr{D}_{K-1} + z^2(z-1)\mathscr{D}_{K-2} = \frac{z^{K+1}(1-z)^{K+1}}{W}(r_2^{K+1} - r_1^{K+1}).$$

Rewriting this, we get, with $t_i = \frac{\mathscr{D}_{i-1}}{\mathscr{D}_i}$, $\tau_i = z^2(z-1)t_i$, $\tau_0 = 0$

$$t_K = \frac{1}{1 - z + z^3 + z^2(z-1)t_{K-1}}, \quad \tau_K = \frac{z^2(z-1)}{1 - z + z^3 + \tau_{K-1}}.$$

Solving the system by Cramer's rule, we get the solution for s_0 (return to the x-axis) as

$$s_0 = \sigma_{K+1} = \frac{-\mathscr{D}_K + z^3\mathscr{D}_{K-1}}{(z-1)\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = \frac{-1 + z^3 t_{K-1}}{(z-1) - z^2(z-1)t_{K-1}}.$$

A simple computation (using Maple's convert-parfrac construction) shows

$$\sigma_{K+1} = \frac{z}{1-z} + \frac{1-z}{1 - z - \tau_K},$$

which is of the continued fraction type.

The recursion for the functions f_0 is in matrix form

$$\begin{pmatrix} 1-z-z^2+z^3 & -z & 0 & \cdots & & 0 \\ z(z-1) & z^3-z+1 & -z & \cdots & & 0 \\ & z(z-1) & z^3-z+1 & -z & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & z(z-1) & z^3-z+1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_K \end{pmatrix} = \begin{pmatrix} z \\ -z^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

whence

$$\begin{aligned} f_0 = \phi_{K+1} &= \frac{z\mathscr{D}_K - z^3\mathscr{D}_{K-1}}{(1-z-z^2+z^3)\mathscr{D}_K + z^2(z-1)\mathscr{D}_{K-1}} \\ &= \frac{z}{1-z} + \frac{z^3}{(1-z)(1-z^2-z^2t_K)}, \quad \phi_0 = \frac{z}{1-z}. \end{aligned}$$

Likewise

$$\begin{pmatrix} 1-z-z^2+z^3 & -z & 0 & \cdots & & 0 \\ z(z-1) & z^3-z+1 & -z & \cdots & & 0 \\ & z(z-1) & z^3-z+1 & -z & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & z(z-1) & z^3-z+1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} 1-z \\ z(z-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} g_0 = \gamma_{K+1} &= \frac{(1-z)\mathscr{D}_K + z^2(z-1)\mathscr{D}_{K-1}}{(1-z-z^2+z^3)\mathscr{D}_K + z^2(z-1)\mathscr{D}_{K-1}} \\ &= 1 + \frac{z^2}{1-z^2-z^2t_K}, \quad \gamma_0 = 1. \end{aligned}$$

This leads to $\phi_K = \gamma_K \frac{z}{1-z}$, which is also combinatorially clear. $h_0 = \eta_K = 0$ for any K, so no more computations are required.

7 UU and HH are forbidden, A329666

From Figure 6 we derive

$$\begin{aligned} f_j &= [j=0] + zf_{j+1} + zg_{j+1} + zh_{j+1}, \\ g_{j+1} &= zf_j + zh_j, \;\; g_0 = 0, \\ h_j &= zf_j + zg_j \end{aligned}$$

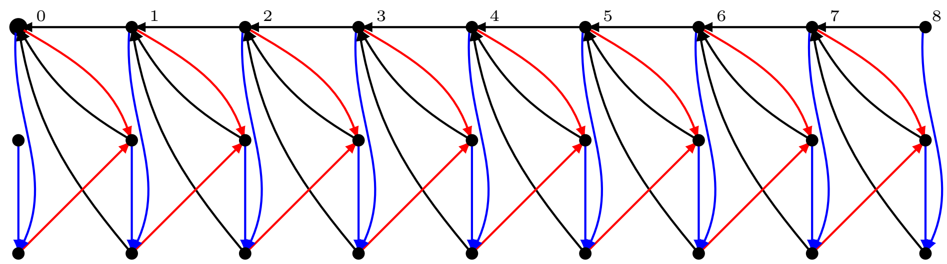


Figure 6: Graph (automaton) to recognize Motzkin paths with forbidden subwords UU and HH.

and

$$\begin{aligned}F(u) &= 1 + \frac{z}{u}[F(u) - F(0)] + \frac{z}{u}G(u) + \frac{z}{u}[H(u) - H(0)],\\ G(u) &= zuF(u) + zuH(u),\\ H(u) &= zF(u) + zG(u).\end{aligned}$$

Then

$$S(u) = F(u) + G(u) + H(u) = \frac{(1+z)(1+zu)(-u + z(1+z)F(0))}{z^2u^2 + (z^3 + z^2 - 1)u + z(1+z)}$$

Set $W = \sqrt{(1 - 2z - z^2 + z^3)(1 + 2z + 3z^2 + z^3)}$ and

$$r_1 = \frac{1 - z^2 - z^3 - W}{2z^2}, \quad r_1 = \frac{1 - z^2 - z^3 + W}{2z^2},$$

then $z^2u^2 + (z^3 + z^2 - 1)u + z(1+z) = z^2(u - r_1)(u - r_2)$. Dividing out $u - r_1$, we compute $F(0) = \dfrac{1 + zr_1}{1 - z^2 r_1} = \dfrac{r_1}{z(1+z)}$ and thus

$$S(0) = F(0) + H(0) = \frac{(1+z)(1+zr_1)}{1 - z^2 r_1} = \frac{r_1}{z}.$$

Using this, we get

$$F = \frac{(1+z)(1-z^2u)r_1}{z(1+z-uzr_1)}, \quad G = \frac{(1+z)ur_1}{1+z-uzr_1}, \quad H = \frac{(1+zu)r_1}{1+z-uzr_1}$$

$$S(u) = \frac{(1+z)^2(1+zu)r_1}{z(1+z-uzr_1)} = \frac{(1+z)(1+zu)r_1}{z(1-u\frac{zr_1}{1+z})}.$$

$$[u^j]S(u) = \Big(\frac{z}{1+z}\Big)^{j-1} r_1^{j+1} + z\Big(\frac{z}{1+z}\Big)^{j-2} r_1^j, \ j \ge 1.$$

The denominator leads to the recursion

$$\begin{aligned} z^2 s_{j-2} + (z^3+z^2-1)s_{j-1} + z(1+z)s_j &= 0, \quad j \ge 3, \\ z^2 s_0 + (z^3+z^2-1)s_1 + z(1+z)s_2 &= -z(1+z), \\ -s_0 + z(1+z)s_1 &= -1-z, \end{aligned}$$

which is in matrix form

$$\begin{pmatrix} -1 & z(1+z) & 0 & \cdots & & 0 \\ z^2 & z^3+z^2-1 & z(1+z) & \cdots & & 0 \\ & z^2 & z^3+z^2-1 & z(1+z) & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & z^2 & z^3+z^2-1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1-z \\ -z(1+z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $\mathscr{D}_K$ be the determinant with K rows and columns, after deleting the first row and column. Expanding, we derive the recursion

$$\mathscr{D}_K = (z^3+z^2-1)\mathscr{D}_{K-1} - z^3(1+z)\mathscr{D}_{K-2}, \ \mathscr{D}_0 = 1, \mathscr{D}_1 = z^3+z^2-1.$$

It follows that

$$\mathscr{D}_K = \frac{(-z^2)^{K+1}}{W}(r_1^{i+1} - r_2^{i+1}).$$

Further

$$s_0 = \frac{-(1+z)\mathscr{D}_K + z^2(1+z)^2\mathscr{D}_{K-1}}{-\mathscr{D}_K - z^3(1+z)\mathscr{D}_{K-1}} = (1+z)\frac{\mathscr{D}_K - z^2(1+z)\mathscr{D}_{K-1}}{\mathscr{D}_K + z^3(1+z)\mathscr{D}_{H-1}}$$

with the abbreviations $t_i = \frac{\mathscr{D}_{i-1}}{\mathscr{D}_i}$ and $\tau_i = -(1+z)t_i$

$$s_0 = \sigma_{K+1} = (1+z)\frac{1 - z^2(1+z)t_K}{1 + z^3(1+z)t_K} = -\frac{1+z}{z} + \frac{(1+z)^2}{z}\frac{1}{1 - z^3\tau_K}$$

as well as

$$t_K = \frac{1}{z^3+z^2-1-z^3(1+z)t_{K-1}} \quad \text{and} \quad \tau_K = \frac{1+z}{1-z^2-z^3-z^3\tau_{K-1}}, \ \tau_0 = 0.$$

Now we move to the restricted Motzkin paths according to the three layers, computing them separately.

$$\begin{pmatrix} -1+z^2+2z^3+z^4 & z(1+z) & 0 & \cdots & & 0 \\ z^2 & z^3+z^2-1 & z(1+z) & \cdots & & 0 \\ & z^2 & z^3+z^2-1 & z(1+z) & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & z^2 & z^3+z^2-1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_K \end{pmatrix} = \begin{pmatrix} -1 \\ z^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Therefore

$$\begin{aligned} f_0 = \phi_{K+1} &= \frac{-\mathscr{D}_K - z^3(1+z)\mathscr{D}_{K-1}}{(-1+z^2+2z^3+z^4)\mathscr{D}_K - z^3(1+z)\mathscr{D}_{K-1}} \\ &= 1 + \frac{z^2(1+z)^2}{1-z^2-2z^3+z^3(1+z)t_K},\ \phi_0 = 1. \end{aligned}$$

Further, $g_0 = \gamma_K = 0$ for all K, so we can move to the instance of the third layer.

$$\begin{pmatrix} -1 & z(1+z) & 0 & \cdots & & 0 \\ z^2 & z^3+z^2-1 & z(1+z) & \cdots & & 0 \\ & z^2 & z^3+z^2-1 & z(1+z) & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & z^2 & z^3+z^2-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z \\ -z^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Therefore

$$\begin{aligned} h_0 = \eta_{K+1} &= \frac{-z\mathscr{D}_K + z^3(1+z)\mathscr{D}_{K-1}}{-\mathscr{D}_K - z^3\mathscr{D}_{K-1}} \\ &= -1 - z + \frac{1+2z}{1+z^3 t_K},\ \eta_0 = z. \end{aligned}$$

The generating function of Motzkin excursions

$$S(0) = 1 + z + z^2 + 3z^3 + 4z^4 + 7z^5 + 15z^6 + 26z^7 + 50z^8 + 102z^9 + 196z^{10} + \cdots$$

is A329666 in OEIS; the generating function of Motzkin meanders

$$S(1) = 1 + 2z + 3z^2 + 6z^3 + 11z^4 + 21z^5 + 42z^6 + 83z^7 + 167z^8 + 341z^9 + 697z^{10} + \cdots$$

is A329667 in OEIS.

8 DU and HH are forbidden, A329666

The recursions are

$$\begin{aligned} f_{i+1} &= zf_i + zh_i, \quad f_0 = 1, \\ g_i &= zf_{i+1} + zg_{i+1} + zh_{i+1}, \\ h_i &= zf_i + zg_i \end{aligned}$$

$$\begin{aligned} F(u) &= 1 + zuF(u) + zuH(u), \\ G(u) &= \frac{z}{u}\big(F(u) - F(0)\big) + \frac{z}{u}\big(G(u) - G(0)\big) + \frac{z}{u}\big(H(u) - H(0)\big), \\ H(u) &= zF(u) + zG(z) \end{aligned}$$

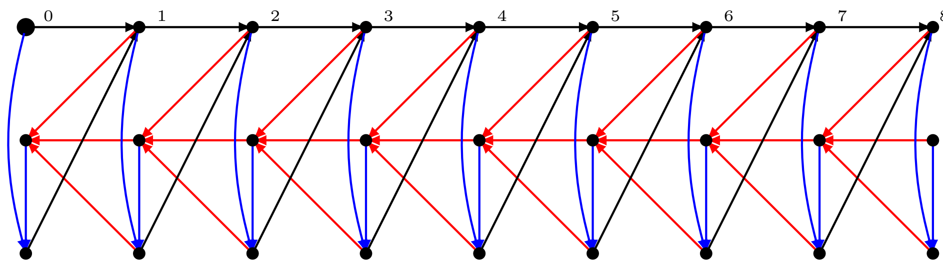


Figure 7: Graph (automaton) to recognize Motzkin paths with forbidden subwords DU and HH.

Decomposing the common denominator as

$$z(z+1)+(-z^3-1-z^2)u+z(z+1)u^2 = z(z+1)(u-r_1)(u-r_2)$$

with

$$W=\sqrt{(1+2z+3z^2+z^3)(1-2z-z^2+z^3)}$$

and

$$r_1=\frac{1+z^2+z^3-W}{2z(1+z)},\quad r_2=\frac{1+z^2+z^3+W}{2z(1+z)},$$

we obtain by dividing out $u-r_1$ and simplifying

$$F(u)=\frac{1}{1-ur_1},\ G(u)=\frac{r_1-z(1+z)}{z^2(1-ur_1)},\ H(u)=\frac{r_1-z}{z(1-ur_1)},\ S(u)=\frac{(1+z)(r_1-z)}{z^2(1-ur_1)}.$$

This leads to (excursions)

$$S(0)=1+z+z^2+3z^3+4z^4+7z^5+15z^6+26z^7+50z^8+102z^9+196z^{10}+392z^{11}+\cdots,$$

which is sequence A329666. Further

$$S(1)=1+2z+4z^2+9z^3+18z^4+38z^5+81z^6+171z^7+366z^8+787z^9+1693z^{10}+\cdots,$$

and this generating function of meanders is sequence A329668 in OEIS. And we get the generating function of Motzkin excursions ending at level j:

$$s_j=[u^j]\frac{(1+z)(r_1-z)}{z^2(1-ur_1)}=\frac{(1+z)(r_1-z)}{z^2}r_1^j.$$

Now we move to paths of bounded height (indices $>K$ cannot occur)

$$\begin{aligned} z(1+z)s_1-s_0&=-1-z,\\ z(1+z)s_j-(1+z^2+z^3)s_{j-1}+z(1+z)s_{j-2}&=0,\ j\ge 2, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} -1 & z(1+z) & 0 & \cdots & 0 \\ z(1+z) & -z^3-z^2-1 & z(1+z) & \cdots & 0 \\ & z(1+z) & -z^3-z^2-1 & z(1+z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & -z^3-z^2-1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1-z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

As always, let $\mathscr{D}_k$ denote the determinant, after removing the first row (resp., column). Then

$$\mathscr{D}_K = (-z^3-z^2-1)\mathscr{D}_{K-1} - z^2(1+z)^2\mathscr{D}_{K-2},\ \mathscr{D}_0 = 1, \mathscr{D}_1 = -z^3-z^2-1.$$

From this, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^2(1+z)^2 t_K$

$$t_K = \frac{1}{-z^3-z^2-1-z^2(1+z)^2 t_{K-1}}, \quad \tau_K = \frac{z^2(1+z)^2}{1+z^2+z^3-\tau_{K-1}},\ \tau_0 = 0.$$

Of course, $\mathscr{D}_K$ has also a Binet form:

$$\mathscr{D}_K = \frac{(-z(z+1))^{K+1}}{W}(r_1^{K+1} - r_2^{K+1})$$

$$\tau_i := -z^2(1+z)^2\frac{\mathscr{D}_{i-1}}{\mathscr{D}_i}, \quad \tau_i = \frac{z^2(1+z)^2}{1+z^2+z^3-\tau_{i-1}},\ \tau_0 = 0$$

Let us also consider the second and third layer in matrix form ($f_0^{[K]} = 1$) for any K:

$$\begin{pmatrix} -1+z^3+z^4 & z(1+z) & 0 & \cdots & 0 \\ z(1+z) & -z^3-z^2-1 & z(1+z) & \cdots & 0 \\ & z(1+z) & -z^3-z^2-1 & z(1+z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & -z^3-z^2-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -z^2(z+1)^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & z(1+z) & 0 & \cdots & 0 \\ z(1+z) & -z^3-z^2-1 & z(1+z) & \cdots & 0 \\ & z(1+z) & -z^3-z^2-1 & z(1+z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & -z^3-z^2-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By Cramer's method:

$$s_0 = \sigma_{K+1} = \frac{-(1+z)\mathscr{D}_K}{-\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}},\ \sigma_0 = 1+z$$

$$g_0 = \gamma_{K+1} = \frac{-z^2(1+z)^2\mathscr{D}_K}{(-1+z^3+z^4)\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}},\ \gamma_0 = 0$$

$$h_0 = \eta_{K+1} = \frac{-z\mathscr{D}_K}{-\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}},\ \eta_0 = z$$

Let us rewrite it in continued fraction style:

$$\sigma_{K+1} = \frac{1+z}{1+z^2(1+z)^2 t_K} = \frac{1+z}{1-\tau_K},$$
$$\gamma_{K+1} = \frac{-z^2(1+z)^2}{(-1+z^3+z^4)-z^2(1+z)^2 t_{K-1}} = \frac{z^2(1+z)^2}{1-z^3-z^4-\tau_K},$$
$$\eta_{K+1} = \frac{-z}{-1-z^2(1+z)^2 t_{K-1}} = \frac{z}{1-\tau_{K-1}}.$$

9 DD and HH are forbidden, A329666

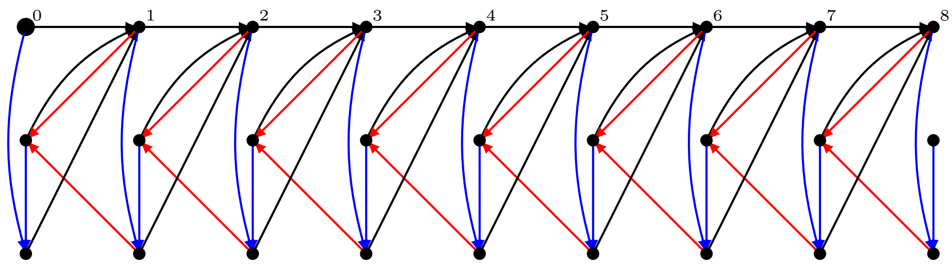


Figure 8: Graph (automaton) to recognize Motzkin paths with forbidden subwords DD and HH.

From Figure 8 we derive the recursions

$$\begin{aligned} f_{i+1} &= zf_i + zg_i + zh_i, \quad f_0 = 1,\\ g_i &= zf_{i+1} + zh_{i+1},\\ h_i &= zf_i + zg_i, \end{aligned}$$

and

$$\begin{aligned} F(u) &= 1 + zuF(u) + zuG(u) + zuH(u),\\ G(u) &= \frac{z}{u}[F(u)-F(0)] + \frac{z}{u}[H(u)-H(0)],\\ H(u) &= zF(u) + zG(z). \end{aligned}$$

With $W = \sqrt{(1+2z+3z^2+z^3)(1-2z-z^2+z^3)}$ and

$$r_1 = \frac{1-z^2-z^3-W}{2z(1+z)}, \quad r_2 = \frac{1-z^2-z^3+W}{2z(1+z)},$$

the denominator (after solving the system) factors:

$$z^2 + (z^2-1+z^3)u + z(1+z)u^2 = z(1+z)(u-r_1)(u-r_2)$$

After diving out $u - r_1$ and simplifying,

$$F(u) = \frac{z}{z - u(1+z)r_1}, \; G(u) = \frac{r_1 - z^2}{z(z - u(1+z)r_1)}, \; H(u) = \frac{r_1}{z - u(1+z)r_1},$$
$$S(u) = \frac{(1+z)r_1}{z(z - u(1+z)r_1)}.$$

We can read off to get the generating function of Motzkin excursions ending at level j:

$$[u^j]S(u) = [u^j]\frac{(1+z)r_1}{z^2(1 - \frac{(1+z)r_1 u}{z})} = \frac{(1+z)r_1}{z^2}\Big(\frac{(1+z)r_1}{z}\Big)^j = \frac{(1+z)^{j+1}r_1^{j+1}}{z^{j+2}}.$$

The generating function

$$S(0) = 1 + z + z^2 + 3z^3 + 4z^4 + 7z^5 + 15z^6 + 26z^7 + 50z^8 + 102z^9 + 196z^{10} + \cdots$$

is A329666 in OEIS;

$$S(1) = 1 + 2z + 4z^2 + 10z^3 + 23z^4 + 54z^5 + 129z^6 + 307z^7 + 733z^8 + 1757z^9 + 4213z^{10} + \cdots$$

is A329669 in OEIS.

Now we move to restricted paths (indices $> K$ are not allowed):

$$z(1+z)s_{j-2} + (-1 + z^2 + z^3)s_{j-1} + z^2 s_j = 0, \; j \ge 2,$$
$$(-1 + z^2 + z^3)s_0 + z^2 s_1 = -(1+z),$$

or in matrix form

$$\begin{pmatrix} z^3+z^2-1 & z^2 & 0 & \cdots & 0 \\ z(1+z) & z^3+z^2-1 & z^2 & \cdots & 0 \\ & z(1+z) & z^3+z^2-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & z^3+z^2-1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1-z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now $f_0 = \phi_K = 1$ for all K, so we can move to the next layer:

$$z(1+z)g_{j-2} + (-1 + z^2 + z^3)g_{j-1} + z^2 g_j = 0, \; j \ge 2,$$
$$(-1 + z^2 + 2z^3 + z^4)g_0 + z^2 g_1 = -(1+z)^2 z^2.$$

$$\begin{pmatrix} -1+z^2+2z^3+z^4 & z^2 & 0 & \cdots & 0 \\ z(1+z) & z^3+z^2-1 & z^2 & \cdots & 0 \\ & z(1+z) & z^3+z^2-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & z^3+z^2-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -z^2(1+z)^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Finally

$$z(1+z)h_{j-2} + (-1+z^2+z^3)h_{j-1} + z^2h_j = 0,\ j \ge 2,$$
$$(-1+z^2+z^3)h_0 + z^2h_1 = -z.$$

$$\begin{pmatrix} z^3+z^2-1 & z^2 & 0 & \cdots & 0 \\ z(1+z) & z^3+z^2-1 & z^2 & \cdots & 0 \\ & z(1+z) & z^3+z^2-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & z^3+z^2-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Finally we move to continued fractions, with the usual determinant, after deleting the first row (resp., column).

$$\mathscr{D}_K = (z^3+z^2-1)\mathscr{D}_{K-1} - z^3(1+z)\mathscr{D}_{K-2}$$

and, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^3(1+z)t_K$

$$t_K = \frac{1}{-1+z^2+z^3-z^3(1+z)t_{K-1}}, \quad \tau_K = \frac{z^3(1+z)}{1-z^2-z^3-\tau_{K-1}},\ \tau_0 = 0.$$

Using this, we finally get

$$\begin{aligned} s_0 = \sigma_{K+1} &= \frac{-(1+z)\mathscr{D}_K}{(-1+z^2+z^3)\mathscr{D}_K - z^3(1+z)\mathscr{D}_{K-1}} \\ &= \frac{1+z}{1-z^2-z^3-\tau_K}, \quad \sigma_0 = 1+z, \\ g_0 = \gamma_{K+1} &= \frac{-z^2(1+z)^2\mathscr{D}_K}{(-1+z^2+2z^3+z^4)\mathscr{D}_K - z^3(1+z)\mathscr{D}_{K-1}} \\ &= \frac{z^2(1+z)^2}{1-z^2-2z^3-z^4-\tau_K},\ \gamma_0 = 0, \\ h_0 = \eta_{K+1} &= \frac{-z\mathscr{D}_K}{(-1+z^2+z^3)\mathscr{D}_K - z^3(1+z)\mathscr{D}_{K-1}} \\ &= \frac{z}{1-z^2-z^3-\tau_K}, \quad \eta_0 = z. \end{aligned}$$

10 UH and HD are forbidden, A329702

The Figure 9 leads to the recursions

$$\begin{aligned} f_i &= zf_i + zg_i, \\ g_i &= [i=0] + zg_{i+1} + zh_{i+1}, \\ h_{i+1} &= zf_i + zg_i + zh_i,\ h_0 = 0 \end{aligned}$$

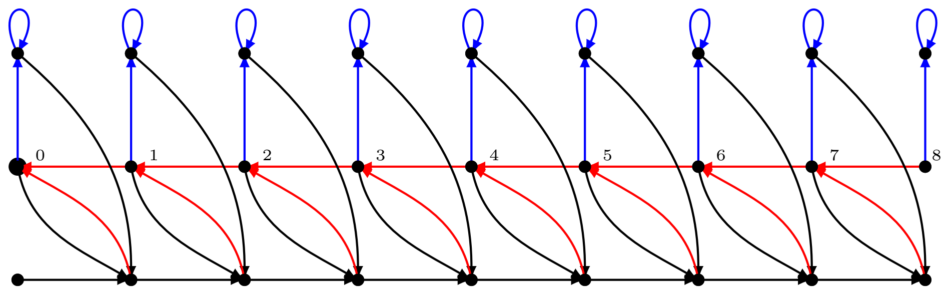


Figure 9: Graph (automaton) to recognize Motzkin paths with forbidden subwords UH and HD.

and functional equations

$$\begin{aligned}
F(u) &= zF(u) + zG(u) \Rightarrow F(u) = \frac{zG(u)}{1-z}\\
G(u) &= 1 + \frac{z}{u}(G(u) - G(0)) + \frac{z}{u}H(u),\\
H(u) &= zuF(u) + zuG(u) + zuH(u).
\end{aligned}$$

The denominator of the solution(s) is $z(z-1) + (-z+1-z^3)u + z(z-1)u^2 = z(z-1)(u-r_1)(u-r_2)$ with

$$r_1 = \frac{1-z-z^3-W}{2z(1-z)},\ r_2 = \frac{1-z-z^3+W}{2z(1-z)},\ W = \sqrt{(1+z-2z^2-z^3)(1-3z+2z^2-z^3)}$$

Dividing out $u - r_1$ and simplifying,

$$F(u) = \frac{(1-zu)r_1}{(1-z)(1-ur_1)},\ G(u) = \frac{(1-zu)r_1}{z(1-ur_1)},\ H(u) = \frac{ur_1}{(1-z)(1-ur_1)},$$
$$S(u) = \frac{r_1}{z(1-z)(1-ur_1)}.$$

The series

$$S(0) = 1 + z + 2z^2 + 3z^3 + 6z^4 + 10z^5 + 20z^6 + 36z^7 + 73z^8 + 139z^9 + 286z^{10} + 567z^{11} + \cdots$$

is A329702 in OEIS

$$S(1) = 1 + 2z + 4z^2 + 8z^3 + 17z^4 + 36z^5 + 78z^6 + 169z^7 + 371z^8 + 816z^9 + 1809z^{10} + \cdots$$

is not in OEIS. Also,

$$s_j = [u^j]S(u) = [u^j]\frac{r_1}{z(1-z)(1-ur_1)} = \frac{r_1^{j+1}}{z(1-z)},$$

which is the generating function of restricted Motzkin paths ending at level j.

From the denominator, we derive recursions

$$z(z-1)s_j + (1-z-z^3)s_{j-1} + z(z-1)s_{j-2} = 0,\ j \ge 2,$$
$$(1-z-z^3)s_0 + z(z-1)s_1 = 1,$$

which is in matrix form

$$\begin{pmatrix} 1-z-z^3 & z(z-1) & 0 & \cdots & 0 \\ z(z-1) & 1-z-z^3 & z(z-1) & \cdots & 0 \\ & z(z-1) & 1-z-z^3 & z(z-1) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & 1-z-z^3 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The recursions for the first (resp., second) layer in matrix form are

$$\begin{pmatrix} 1-z-z^2 & z(z-1) & 0 & \cdots & 0 \\ z(z-1) & 1-z-z^3 & z(z-1) & \cdots & 0 \\ & z(z-1) & 1-z-z^3 & z(z-1) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & 1-z-z^3 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_K \end{pmatrix} = \begin{pmatrix} z \\ -z^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-z-z^2 & z(z-1) & 0 & \cdots & 0 \\ z(z-1) & 1-z-z^3 & z(z-1) & \cdots & 0 \\ & z(z-1) & 1-z-z^3 & z(z-1) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & 1-z-z^3 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} 1-z \\ z(z-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

For the third layer we get $h_0^{[K]} = 0$ for all K without computations. Let $\mathscr{D}_K$ be the usual determinant. Then

$$\begin{aligned} \mathscr{D}_K &= (1-z-z^3)\mathscr{D}_{K-1} - z^2(z-1)^2\mathscr{D}_{K-2},\ \mathscr{D}_0 = 1,\ \mathscr{D}_1 = 1-z-z^3 \\ &= \frac{z^{K+1}(1-z)^{K+1}}{W}(r_2^{K+1} - r_1^{K+1}). \end{aligned}$$

Further, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$, $\tau_K = z^2(z-1)^2 t_K$

$$t_K = \frac{1}{1-z-z^3-z^2(z-1)^2 t_{K-1}}, \quad \tau_K = \frac{z^2(z-1)^2}{1-z-z^3-\tau_{K-1}},\ \tau = 0.$$

Then we can express the solutions in continued fraction form:

$$
\begin{aligned}
s_0 = \sigma_{K+1} &= \frac{(1-z-z^3)\mathscr{D}_K}{(1-z-z^3)\mathscr{D}_K - z^2(z-1)^2\mathscr{D}_{K-1}} = \frac{1-z-z^3}{1-z-z^3-\tau_K},\\
f_0 = \phi_{K+1} &= \frac{z\mathscr{D}_K + z^3(z-1)\mathscr{D}_{K-1}}{(1-z-z^2)\mathscr{D}_K - z^2(z-1)^2\mathscr{D}_{K-1}}\\
&= \frac{z+z^3(z-1)t_K}{1-z-z^2-z^2(z-1)^2t_K} = \frac{z}{1-z} + \frac{z^3}{(1-z)(1-z-z^2-\tau_K)},\\
g_0 = \gamma_{K+1} &= \frac{(1-z)\mathscr{D}_K - z^2(z-1)^2\mathscr{D}_{K-1}}{(1-z-z^2)\mathscr{D}_K - z^2(z-1)^2\mathscr{D}_{K-1}}\\
&= \frac{1-z-z^2(z-1)^2t_K}{1-z-z^2-z^2(z-1)^2t_K} = 1 + \frac{z^2}{1-z-z^2-\tau_K}.
\end{aligned}
$$

11 DH and HU are forbidden, A329701

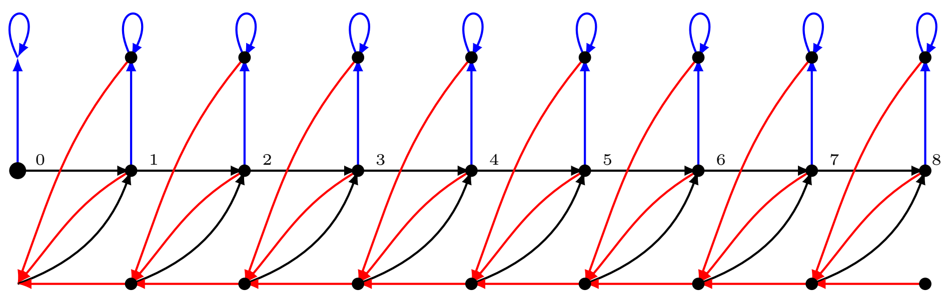


Figure 10: Graph (automaton) to recognize Motzkin paths with forbidden subwords DH and HU.

The corresponding recursions are

$$
\begin{aligned}
f_j &= zf_j + zg_j,\\
g_{j+1} &= zg_j + zh_j,\ \ g_0 = 1,\\
h_j &= zf_{j+1} + zg_{j+1} + zh_{j+1},
\end{aligned}
$$

and further

$$
\begin{aligned}
F(u) &= zF(u) + zG(u),\\
G(u) &= 1 + zuG(u) + zuH(u),\\
H(u) &= \frac{z}{u}\big(F(u)-F(0)\big) + \frac{z}{u}\big(G(u)-1\big) + \frac{z}{u}\big(H(u)-H(0)\big),
\end{aligned}
$$

Solving the system there is a denominator which factors:

$$z(z-1)+(-z^3-z+1)u+z(z-1)u^2=z(z-1)(u-r_1)(u-r_2)$$

with

$$r_1=\frac{1-z+z^3-W}{2z(1-z)},\quad r_2=\frac{1-z+z^3+W}{2z(1-z)},$$
$$W=\sqrt{(1-3z+2z^2-z^3)(1+z-2z^2-z^3)}.$$

Dividing out the factor $u-r_1$ and simplifying, we find

$$F=\frac{z}{(1-z)(1-ur_1)},\ G=\frac{1}{1-ur_1},\ H=\frac{r1-z}{z(1-ur_1)},$$
$$S=\frac{(1-z)r_1+z^2}{z(1-z)(1-ur_1)}.$$

The generating function of excursions

$$S(0)=1+z+2z^2+2z^3+4z^4+5z^5+11z^6+17z^7+38z^8+67z^9+148z^{10}+\cdots$$

the generating function of meanders is

$$S(1)=1+2z+4z^2+7z^3+14z^4+28z^5+60z^6+128z^7+281z^8+615z^9+1365z^{10}+\cdots$$

is not in OEIS. We can also compute the generating function of Motzkin paths ending at level j:

$$[u^j]S(u)=[u^j]\frac{(1-z)r_1+z^2}{z(1-z)(1-ur_1)}=\frac{(1-z)r_1+z^2}{z(1-z)}r_1^j=\frac{r_1^{j+1}}{z}+\frac{zr_1^j}{1-z}.$$

Now we move to height restricted paths:

$$-z(1-z)s_{j-2}+(1-z-z^3)s_{j-1}-z(1-z)s_j=0,\ j\ge 2,$$
$$(1-z)s_0+z(z-1)s_1=1,$$

and in matrix form

$$\begin{pmatrix} 1-z & z(z-1) & 0 & \cdots & 0\\ z(z-1) & 1-z-z^3 & z(z-1) & \cdots & 0\\ & z(z-1) & 1-z-z^3 & z(z-1) & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & z(z-1) & 1-z-z^3 \end{pmatrix}\begin{pmatrix} s_0\\ s_1\\ s_2\\ \vdots\\ s_K\end{pmatrix}=\begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix}$$

The next two quantities need no computation, since $f_0^{[K]} = \frac{z}{1-z}$ and $g_0^{[K]} = 1$ for all K. Finally

$$\begin{pmatrix} 1-z-z^2 & z(z-1) & 0 & \cdots & 0 \\ z(z-1) & 1-z-z^3 & z(z-1) & \cdots & 0 \\ & z(z-1) & 1-z-z^3 & z(z-1) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & 1-z-z^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $\mathscr{D}_K$ the usual determinant, then $\mathscr{D}_K = (1-z-z^3)\mathscr{D}_{K-1} - z^2(z-1)^2\mathscr{D}_{K-2}$ and

$$\mathscr{D}_j = \frac{z^{j+1}(1-z)^{j+1}}{W}(r_2^{j+1} - r_1^{j+1}), \quad \mathscr{D}_0 = 1,\ \mathscr{D}_1 = 1-z-z^3.$$

Furthermore, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = z^2(z-1)^2 t_K$,

$$t_K = \frac{1}{1-z-z^3-z^2(z-1)^2 t_{K-1}} \Longrightarrow \tau_K = \frac{z^2(z-1)^2}{1-z-z^3-\tau_{K-1}},\ \tau_0 = 0.$$

Finally

$$s_0^{[K+1]} = \sigma_0 = \frac{\mathscr{D}_K}{(1-z)\mathscr{D}_K - z^2(z-1)^2\mathscr{D}_{K-1}} = \frac{1}{1-z-\tau_K},\ s_0^{[0]} = \frac{1}{1-z},$$
$$h_0^{[K+1]} = \eta_0 = \frac{z^2\mathscr{D}_K}{(1-z-z^2)\mathscr{D}_K - z^2(z-1)^2\mathscr{D}_{K-1}} = \frac{z^2}{1-z-z^2-\tau_K},\ h_0^{[0]} = 0.$$

12 UU and DD are forbidden, A004149

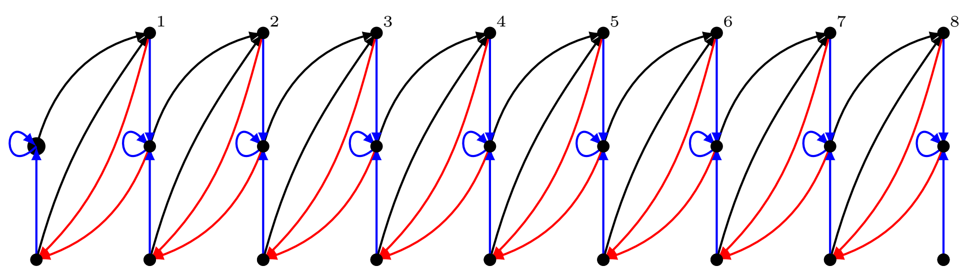


Figure 11: Graph (automaton) to recognize Motzkin paths with forbidden subwords UU and DD.

The recursions are

$$\begin{aligned} f_{i+1} &= zg_i + zh_i,\ f_0 = 0, \\ g_i &= [i=0] + zf_i + zg_i + zh_i,\ i \geq 0, \\ h_i &= zf_{i+1} + zg_{i+1}, \end{aligned}$$

leading to the functional equations

$$\begin{aligned}
F(u) &= zuG(u) + zuH(u),\\
G(u) &= 1 + zF(u) + zG(u) + zH(u),\\
H(u) &= \frac{z}{u}F(u) + \frac{z}{u}(G(u) - G(0)).
\end{aligned}$$

Solving the system, a denominator pops up which can be factored:

$$z^2 + z^3u - u + zu + z^2u + z^2u^2 = z^2(u - r_1)(u - r_2)$$

with

$$r_1 = \frac{1 - z - z^2 - z^3 - W}{2z^2},\ r_2 = \frac{1 - z - z^2 - z^3 + W}{2z^2}, \quad W = \sqrt{(1 - z^4)(1 - 2z - z^2)}.$$

Dividing out the factor $u - r_1$ and simplifying, we find

$$F(u) = \frac{ur_1}{z(1 - ur_1)},\ G(u) = \frac{r_1 + z}{z(1 - ur_1)},\ H(u) = \frac{(1 - z)r_1 - z^2}{z^2(1 - ur_1)},\ S(u) = \frac{(1 + zu)r_1}{z^2(1 - ur_1)}.$$

The sequence of excursions

$$S(0) = 1 + z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + 33z^6 + 69z^7 + 146z^8 + 312z^9 + 673z^{10} + \cdots$$

is A308435 in OEIS; the sequence of meanders

$$S(1) = 1 + 2z + 4z^2 + 9z^3 + 20z^4 + 45z^5 + 102z^6 + 233z^7 + 535z^8 + 1234z^9 + 2857z^{10} + \cdots$$

is A308435 in OEIS. We can also compute the generating function of Motzkin paths ending at level $j \ge 1$:

$$s_j = [u^j]S(u) = [u^j]\frac{(1 + zu)r_1}{z^2(1 - ur_1)} = \frac{r_1^{j+1}}{z^2} + \frac{r_1^j}{z}.$$

Now we move to the restricted versions, where indices $> K$ cannot occur.

From the denominator we derive the recursion

$$\begin{aligned}
z^2s_{j-2} + (z^3 + z^2 + z - 1)s_{j-1} + z^2s_j &= 0,\ j \ge 3,\\
z^2s_0 + (z^3 + z^2 + z - 1)s_1 + z^2s_2 &= -z,\\
(z^2 + z - 1)s_0 + z^2s_1 &= -1,
\end{aligned}$$

or in matrix form

$$\begin{pmatrix}
z^2 + z - 1 & z^2 & 0 & \cdots & 0\\
z^2 & z^3 + z^2 + z - 1 & z^2 & \cdots & 0\\
 & z^2 & z^3 + z^2 + z - 1 & -z & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & z^2 & z^3 + z^2 + z - 1
\end{pmatrix}
\begin{pmatrix} s_0\\ s_1\\ s_2\\ \vdots\\ s_K \end{pmatrix}
=
\begin{pmatrix} -1\\ -z\\ 0\\ \vdots\\ 0 \end{pmatrix}$$

The sequence for the first layer does not need computations, since we always have $f_0^{[K]}$, any K.

$$\begin{pmatrix} z^2+z-1 & z^2 & 0 & \cdots & 0 \\ z^2 & z^3+z^2+z-1 & z^2 & \cdots & 0 \\ & z^2 & z^3+z^2+z-1 & -z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z^3+z^2+z-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1+z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 2z-1 & z^2(1-z) & 0 & \cdots & 0 \\ z^2 & z^3+z^2+z-1 & z^2 & \cdots & 0 \\ & z^2 & z^3+z^2+z-1 & -z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z^3+z^2+z-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now we express the solutions in terms of continued fractions. With the usual determinant,

$$\mathscr{D}_j = (z^3+z^2+z-1)\mathscr{D}_{j-1} - z^4\mathscr{D}_{j-2} = \frac{(-z^2)^{j+1}}{W}(r_1^{j+1} - r_2^{j+1})$$

Further, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^4 t_K$,

$$t_K = \frac{1}{z^3+z^2+z-1-z^4 t_{K-1}}, \quad \tau_K = \frac{z^4}{1-z-z^2-z^3-\tau_{K-1}}.$$

Now we can express the various solutions in continued fraction form:

$$\begin{aligned} s_0 = \sigma_{K+1} &= \frac{-\mathscr{D}_K + z^3\mathscr{D}_{K-1}}{(z^2+z-1)\mathscr{D}_K - z^4\mathscr{D}_{K-1}} \\ &= \frac{-1+z^3 t_K}{z^2+z-1-z^4 t_{K-1}} = -\frac{1}{z}\frac{1-z^2}{z(1-z-z^2-\tau_K)},\ \sigma_0 = \frac{1}{1-z}, \\ g_0 = \gamma_{K+1} &= \frac{(-1+z^2)\mathscr{D}_K}{(z^2+z-1)\mathscr{D}_K - z^4\mathscr{D}_{K-1}} \\ &= \frac{-1+z^2}{z^2+z-1-z^4 t_K} = \frac{1-z^2}{1-z-z^2-\tau_K},\ \gamma_0 = \frac{1}{1-z} \\ h_0 = \eta_{K+1} &= \frac{-z^2\mathscr{D}_K}{(2z-1)\mathscr{D}_K - z^4(1-z)\mathscr{D}_{K-1}} \\ &= \frac{-z^2}{(2z-1)-z^4(1-z)t_K} = \frac{z^2}{1-2z-(1-z)\tau_K},\ \eta_0 = 0. \end{aligned}$$

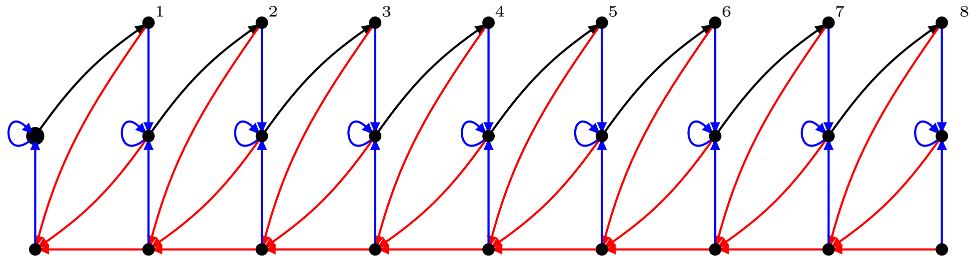


Figure 12: Graph (automaton) to recognize Motzkin paths with forbidden subwords UU and DU.

13 UU and DU are forbidden, A023431

The recursions are

$$\begin{aligned} f_{i+1} &= zg_i, \;\; f_0 = 0, \\ g_i &= [i=0] + zf_i + zg_i + zh_i, \;\; i \ge 0, \\ h_i &= zf_{i+1} + zg_{i+1} + zh_{i+1}, \end{aligned}$$

and the according functional equations

$$\begin{aligned} F(u) &= zuG(u), \\ G(u) &= 1 + zF(u) + zG(u) + zH(u), \\ H(u) &= \frac{z}{u}(F(u) - F(0)) + \frac{z}{u}(G(u) - G(0)) + \frac{z}{u}(H(u) - H(0)); \end{aligned}$$

the denominator factors

$$z^2u^2 + (z-1)u + z = z^2(u - r_1)(u - r_2),$$

with

$$r_1 = \frac{1 - z - W}{2z^2} \; r_2 = \frac{1 - z + W}{2z^2}, \quad W = \sqrt{1 - 2z + z^2 - 4z^3}.$$

Then, dividing out the factor $u - r_1$ and simplifying, we find

$$F = \frac{ur_1}{1 - uzr_1}, \; G = \frac{r_1}{z(1 - uzr_1)}, \; H = \frac{r_1(1-z) - z}{z^2(1 - uzr_1)}, \; S = \frac{(1 + uz^2)r_1 - z}{z^2(1 - uzr_1)}.$$

The series of excursions (A023431 in OEIS) is

$$S(0) = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 26z^6 + 52z^7 + 104z^8 + 212z^9 + 438z^{10} + \cdots$$

and meanders (not in the OEIS)

$$S(1) = 1 + 2z + 4z^2 + 8z^3 + 16z^4 + 33z^5 + 69z^6 + 145z^7 + 307z^8 + 655z^9 + 1405z^{10} + \cdots.$$

Now we move to Motzkin paths of bounded height, provided by the denominator:

$$\begin{aligned} z^2 s_{j-2} + (z-1)s_{j-1} + z s_j &= 0,\ j \geq 3, \\ z^2 s_0 + (z-1)s_1 + z s_2 &= -z, \\ (z-1)s_0 + z s_1 &= -1, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} z-1 & z & 0 & \cdots & 0 \\ z^2 & z-1 & z & \cdots & 0 \\ & z^2 & z-1 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z-1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1 \\ -z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since $f_0^{[K]} = 0$ for any K, we move to the next one:

$$\begin{aligned} z^2 s_{j-2} + (z-1)s_{j-1} + z s_j &= 0,\ j \geq 3 \\ z^2 s_0 + (z-1)s_1 + z s_2 &= -z, \\ (z-1)s_0 + z s_1 &= -1, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} z-1 & z & 0 & \cdots & 0 \\ z^2 & z-1 & z & \cdots & 0 \\ & z^2 & z-1 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-2z+z^2-z^3 & z(z-1) & 0 & \cdots & 0 \\ z^2 & z-1 & z & \cdots & 0 \\ & z^2 & z-1 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

With the usual determinant, we derive the recursion

$$\mathscr{D}_K = (z-1)\mathscr{D}_{K-1} - z^3\mathscr{D}_{K-2} = \frac{(-z^2)^{K+1}}{W}(r_1^{K+1} - r_2^{K+1})$$

and also, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^3 t_K$

$$t_K = \frac{1}{-1+z-z^3 t_{K-1}}, \quad \tau_K = \frac{z^3}{1-z-\tau_{K-1}}$$

Now we can express the solutions:

$$\begin{aligned}
s_0 = \sigma_{K+1} &= \frac{-\mathscr{D}_K + z^2\mathscr{D}_{K-1}}{(z-1)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{-1+z^2t_K}{z-1-z^3t_K}\\
&= -\frac{1}{z} + \frac{1}{z(1-z-\tau_K)}, \quad \sigma_0 = \frac{1}{1-z}\\
g_0 = \gamma_{K+1} &= \frac{-\mathscr{D}_K}{(z-1)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{1}{1-z-\tau_{K-1}}, \quad \gamma_0 = \frac{1}{1-z},\\
h_0 = \eta_{K+1} &= \frac{z^2\mathscr{D}_K}{(1-2z+z^2-z^3)\mathscr{D}_K - z^3(z-1)\mathscr{D}_{K-1}}\\
&= \frac{z^2}{1-2z+z^2-z^3-z^3(z-1)t_K} = \frac{z^2}{1-2z+z^2-z^3-(1-z)\tau_K}, \quad \eta_0 = 0.
\end{aligned}$$

14 UU and HD are forbidden, A217282

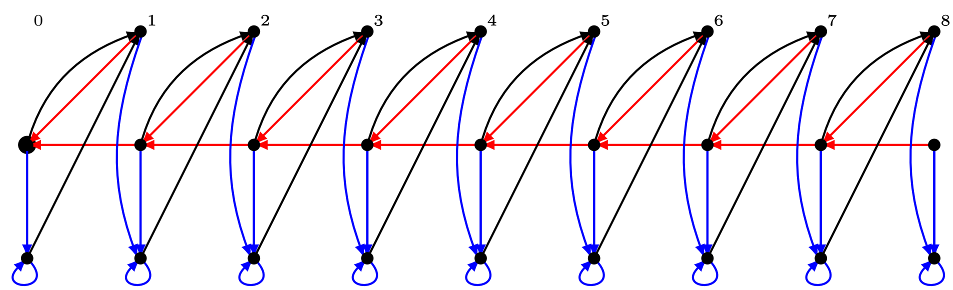


Figure 13: Graph (automaton) to recognize Motzkin paths with forbidden subwords UU and HD.

The recursions that can be read off from Figure 13 are

$$\begin{aligned}
f_{i+1} &= zg_i + zh_i,\ f_0 = 0,\\
g_i &= [i=0] + zf_{i+1} + zg_{i+1},\ i \ge 0,\\
h_i &= zf_i + zg_i + zh_i,
\end{aligned}$$

and the corresponding functional equations

$$\begin{aligned}
F(u) &= zuG(u) + zuH(u),\\
G(u) &= 1 + \frac{z}{u}F(u) + \frac{z}{u}(G(u) - G(0)),\\
H(u) &= zF(u) + zG(u) + zH(u).
\end{aligned}$$

The factorization of the ubiquitous denominator:

$$z^2u^2 - (z+1)(-1+z)^2u - z(-1+z) = z^2(u-r_1)(u-r_2),$$

with

$$r_1 = \frac{1-z-z^2+z^3-W}{2z^2},\ r_2 = \frac{1-z-z^2+z^3+W}{2z^2}, \quad W = \sqrt{1-2z-z^2+3z^4-2z^5+z^6}$$

After dividing out $u - r_1$ and simplifying, we find

$$F(u) = \frac{ur_1}{1-z-uzr_1},\ G(u) = \frac{(1-z-z^2u)r_1}{z(1-z-uzr_1)},\ H(u) = \frac{(1+zu)r_1}{1-z-uzr_1},\ S(u) = \frac{(1+zu)r_1}{z(1-z-uzr_1)}.$$

The sequence of excursions

$$S(0) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 9z^5 + 16z^6 + 30z^7 + 57z^8 + 110z^9 + 216z^{10} + \cdots,$$

is A217282 in OEIS, and

$$S(1) = 1 + 2z + 4z^2 + 8z^3 + 16z^4 + 33z^5 + 68z^6 + 142z^7 + 298z^8 + 629z^9 + 1334z^{10} + \cdots,$$

the series of Motzkin meanders is not in OEIS. The enumeration of Motzkin paths ending at level $j \ge 1$ is

$$\begin{aligned}[u^j]S(u) = [u^j]\frac{(1+zu)r_1}{z(1-z)\big(1-\frac{uzr_1}{1-z}\big)} &= \frac{1}{z(1-z)}\Big(\frac{zr_1}{1-z}\Big)^j + \frac{1}{1-z}\Big(\frac{zr_1}{1-z}\Big)^{j-1} r_1\\ &= \frac{1}{z^2}\Big(\frac{zr_1}{1-z}\Big)^{j+1} + \frac{1}{z}\Big(\frac{zr_1}{1-z}\Big)^j.\end{aligned}$$

Now we move to the restricted paths.

$$\begin{aligned} z^2s_{j-2} - (-1+z)^2(z+1)s_{j-1} + z(1-z)s_j = 0,\ j \ge 3\\ z^2s_0 - (-1+z)^2(z+1)s_1 + z(1-z)s_0 = -z,\\ (z-1)s_0 + z(1-z) = -1 \end{aligned}$$

or in matrix form

$$\begin{pmatrix} z-1 & z(1-z) & 0 & \cdots & 0\\ z^2 & -(-1+z)^2(z+1) & z(1-z) & \cdots & 0\\ & z^2 & -(-1+z)^2(z+1) & z(1-z) & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & z^2 & -(-1+z)^2(z+1) \end{pmatrix} \begin{pmatrix} s_0\\ s_1\\ s_2\\ \vdots\\ s_K \end{pmatrix} = \begin{pmatrix} -1\\ -z\\ 0\\ \vdots\\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+z+z^2 & z(1-z) & 0 & \cdots & 0\\ z^2 & -(-1+z)^2(z+1) & z(1-z) & \cdots & 0\\ & z^2 & -(-1+z)^2(z+1) & z(1-z) & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & z^2 & -(-1+z)^2(z+1) \end{pmatrix} \begin{pmatrix} g_0\\ g_1\\ g_2\\ \vdots\\ g_K \end{pmatrix} = \begin{pmatrix} -1+z\\ z^2\\ 0\\ \vdots\\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+z & z(1-z) & 0 & \cdots & 0 \\ z^2 & -(-1+z)^2(z+1) & z(1-z) & \cdots & 0 \\ & z^2 & -(-1+z)^2(z+1) & z(1-z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & -(-1+z)^2(z+1) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z \\ -z^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

For the usual determinant we derive a recursion and an explicit Binet form:

$$\mathscr{D}_K = -(1-z)^2(z+1)\mathscr{D}_{K-1} - z^3(1-z)\mathscr{D}_{K-2} = \frac{(-z^2)^{K+1}}{W}(r_1^{K+1} - r_2^{K+1}).$$

Alternatively, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^3 t_K$

$$t_K = \frac{1}{-z^3 + z + z^2 - 1 - z^3(1-z)t_{K-1}} \Longrightarrow \tau_K = \frac{z^3}{1 - z - z^2 + z^3 - (1-z)\tau_{K-1}}, \ \tau_0 = 0.$$

Finally, $\phi_0 = 0$ for all K, and

$$\begin{aligned} s_0 = \sigma_{K+1} &= \frac{-\mathscr{D}_K + z^2(1-z)\mathscr{D}_{K-1}}{(z-1)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}} = \frac{-1 + z^2(1-z)t_K}{(z-1) - z^3(1-z)t_K} \\ &= -\frac{1}{z} + \frac{1}{z(1-z)(1-\tau_K)}, \quad \sigma_0 = \frac{1}{1-z}, \\ g_0 = \gamma_{K+1} &= \frac{(-1+z)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}}{(-1+z+z^2)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}} = \frac{(-1+z) - z^3(1-z)t_K}{(-1+z+z^2) - z^3(1-z)t_K} \\ &= 1 + \frac{z^2}{1 - z - z^2 - (1-z)\tau_K}, \quad \gamma_0 = 1, \\ h_0 = \eta_{K+1} &= \frac{-z\mathscr{D}_K + z^3(1-z)\mathscr{D}_{K-1}}{(z-1)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}} = \frac{-z + z^3(1-z)t_K}{(z-1) - z^3(1-z)t_K} \\ &= -1 + \frac{1}{(1-z)(1-\tau_K)}, \quad \eta_0 = \frac{z}{1-z}. \end{aligned}$$

15 UU and UD are forbidden, A023431

The recursions that can be read off from Figure 14 are

$$\begin{aligned} f_{i+1} &= zg_i + zh_i, \ f_0 = 0, \\ g_i &= [i=0] + zf_i + zg_i + zh_i, \ i \geq 0, \\ h_i &= zg_{i+1} + zh_{i+1}, \end{aligned}$$

and the corresponding functional equations

$$\begin{aligned} F(u) &= zuG(u) + zuH(u), \\ G(u) &= 1 + zF(u) + zG(u) + zH(u), \\ H(u) &= \frac{z}{u}(G(u) - G(0)) + \frac{z}{u}(H(u) - H(0)). \end{aligned}$$

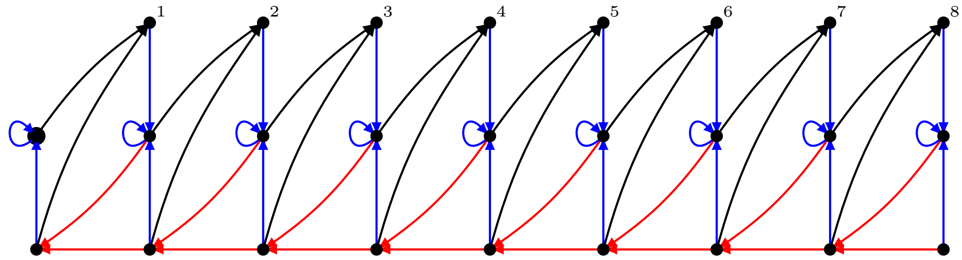


Figure 14: Graph (automaton) to recognize Motzkin paths with forbidden subwords UU and UD.

Solving this, there is a denominator $z + (-1+z)u + z^2u^2 = z^2(u - r_1)(u - r_2)$ with

$$r_1 = \frac{1 - z - W}{2z^2},\ r_2 = \frac{1 - z + W}{2z^2}, \quad W = \sqrt{1 - 2z + z^2 - 4z^3}.$$

Dividing the factor $u - r_1$ out of the solutions and simplifying, we get

$$F(u) = \frac{ur_1}{1 - uzr_1},\ G(u) = \frac{1 + r_1}{1 - uzr_1},\ H(u) = \frac{(1-z)r_1 - z}{z(1 - uzr_1)},\ S(u) = \frac{(1 + zu)r_1}{z(1 - uzr_1)}.$$

The sequence of excursions

$$S(0) = 1 + z + z^2 + 2z^3 + 4z^4 + 7z^5 + 13z^6 + 26z^7 + 52z^8 + 104z^9 + 212z^{10} + \cdots$$

is A023431 in OEIS, the sequence of meanders

$$S(1) = 1 + 2z + 3z^2 + 6z^3 + 12z^4 + 24z^5 + 49z^6 + 102z^7 + 214z^8 + 452z^9 + 962z^{10} + \cdots$$

is not in OEIS.

From the denominator we derive the recursion

$$\begin{aligned} z^2 s_{j-2} + (z-1)s_{j-1} + zs_j = 0,\ j \geq 3 \\ z^2 s_0 + (z-1)s_1 + zs_2 = -z, \\ (-1 + z - z^2)s_0 + zs_1 = -1, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} -1+z-z^2 & z & 0 & \cdots & 0 \\ z^2 & z-1 & z & \cdots & 0 \\ & z^2 & z-1 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z-1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1 \\ -z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\phi_K = 0$ for any K, so we further get

$$\begin{pmatrix} -1+z-z^2 & z & 0 & \cdots & 0 \\ z^2 & z-1 & z & \cdots & 0 \\ & z^2 & z-1 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1-z^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+2z-z^2+z^3 & z(1-z) & 0 & \cdots & 0 \\ z^2 & z-1 & z & \cdots & 0 \\ & z^2 & z-1 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z^2 & z-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The traditional determinant satisfies

$$\mathscr{D}_K = (z-1)\mathscr{D}_{K-1} - z^3\mathscr{D}_{K-2} = \frac{(-z^2)^{K+1}}{W}(r_2^{K+1} - r_1^{K+1}).$$

Further, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^3 t_K$

$$t_K = \frac{1}{(z-1) - z^3 t_{K-1}}, \Longrightarrow \tau_K = \frac{z^3}{1-z-\tau_{K-1}}, \quad \tau_0 = 0.$$

Now we can express the solutions in continued fraction form:

$$\begin{aligned} s_0 = \sigma_{K+1} &= \frac{-\mathscr{D}_K + z^2\mathscr{D}_{K-1}}{(-1+z-z^2)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{-1+z^2 t_K}{(-1+z-z^2) - z^3 t_K} \\ &= -\frac{1}{z} + \frac{1+z^2}{z(1-z+z^2-\tau_K)}, \quad \sigma_0 = \frac{1}{1-z}, \\ g_0 = \gamma_{K+1} &= \frac{-(1+z^2)\mathscr{D}_K}{(-1+z-z^2)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{1+z^2}{1-z+z^2-\tau_K}, \quad \gamma_0 = \frac{1}{1-z}, \\ h_0 = \eta_{K+1} &= \frac{-z^3\mathscr{D}_K}{(-1+2z-z^2+z^3)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}} \\ &= \frac{z^3}{1-2z+z^2-z^3-(1-z)\tau_K}, \quad \eta_0 = 0. \end{aligned}$$

16 UD and DD are forbidden, A023431

The recursion related to Figure 15 are

$$\begin{aligned} f_{i+1} &= zf_i + zg_i + zh_i, \ f_0 = 0, \\ g_i &= [i=0] + zf_i + zg_i + zh_i, \ i \geq 0 \\ h_i &= zg_{i+1} \end{aligned}$$

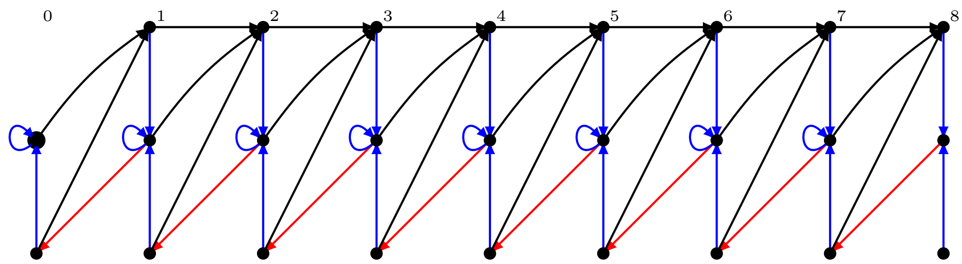


Figure 15: Graph (automaton) to recognize Motzkin paths with forbidden subwords UD and DD.

and the system of functional equations is

$$\begin{aligned}
F(u) &= zuF(u) + zuG(u) + zuH(u),\\
G(u) &= 1 + zF(u) + zG(u) + zH(u),\\
H(u) &= \frac{z}{u}(G(u) - G(0)).
\end{aligned}$$

The ubiquitous denominator is $zu^2 + (z-1)u + z^2 = z(u - r_1)(u - r_2)$ with

$$r_1 = \frac{1 - z - W}{2z},\ r_2 = \frac{1 - z + W}{2z},\quad W = \sqrt{1 - 2z + z^2 - 4z^3}.$$

Dividing out the factor $u - r_1$ and simplifying, we end up with

$$F(u) = \frac{ur_1}{z - ur_1},\ G(u) = 1 + \frac{r_1}{z - ur_1},\ H(u) = \frac{(1-z)r_1 - z^2}{z(z - ur_1)},\ S(u) = \frac{r_1}{z(z - ur_1)}.$$

We find

$$S(0) = 1 + z + z^2 + 2z^3 + 4z^4 + 7z^5 + 13z^6 + 26z^7 + 52z^8 + 104z^9 + 212z^{10} + \cdots$$

which is A023431 in OEIS and

$$S(1) = 1 + 2z + 4z^2 + 9z^3 + 21z^4 + 49z^5 + 115z^6 + 272z^7 + 646z^8 + 1538z^9 + 3670z^{10} + \cdots$$

which is not in OEIS. Furthermore

$$[u^j]S(u) = [u^j]\frac{r_1}{z(z - ur_1)} = [u^j]\frac{r_1}{z^2(1 - \frac{ur_1}{z})} = \frac{1}{z}\Big(\frac{r_1}{z}\Big)^{j+1}$$

Since, before diving out the factor, we have

$$S(u) = \frac{-u - z - uz^2G(0) + zG(0)}{-u + zu^2 + z^2 + zu},$$

we get a recursion from it:

$$zs_{j-2} + (z-1)s_{j-1} + z^2 s_j = 0,\ j \geq 2,$$
$$(z-1)s_0 + z^2 s_1 = -1,$$

which is in matrix form

$$\begin{pmatrix} z-1 & z^2 & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The usual determinant is

$$\mathscr{D}_K = (z-1)\mathscr{D}_{K-1} - z^3 \mathscr{D}_{K-2} = \frac{(-z)^{K+1}}{W}(r_1^{K+1} - r_2^{K+1}).$$

With $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^3 t_K$

$$t_K = \frac{1}{z-1-z^3 t_{K-1}} \implies \tau_K = \frac{z^3}{1-z-\tau_{K-1}},\ \tau_0 = 0$$

$$\begin{pmatrix} z-1 & z^2 & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1 \\ z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+2z-z^2+z^3 & z^2(1-z) & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

From this, we can express the solutions (return to the x-axis) in continued fraction form

$$s_0 = \sigma_{K+1} = \frac{-\mathscr{D}_K}{(z-1)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{1}{1-z+z^3 t_H} = \frac{1}{1-z-\tau_H},\ \sigma_0 = \frac{1}{1-z},$$
$$g_0 = \gamma_{K+1} = \frac{-\mathscr{D}_K - z^3\mathscr{D}_{K-1}}{(z-1)\mathscr{D}_K - z^3\mathscr{D}_{K-1}} = \frac{-1-z^3 t_K}{z-1-z^3 t_K} = 1 + \frac{z}{1-z-\tau_K},\ \gamma_0 = \frac{1}{1-z},$$
$$h_0 = \eta_{K+1} = \frac{-z^3\mathscr{D}_K}{(-1+2z-z^2+z^3)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}} = \frac{z^3}{1-2z+z^2-z^3-(1-z)\tau_K},$$
$$\eta_0 = 0.$$

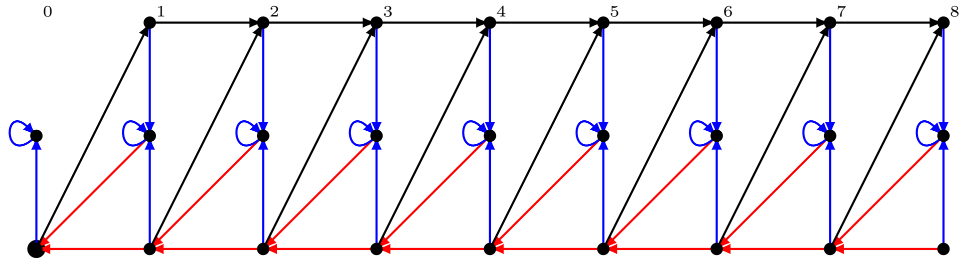


Figure 16: Graph (automaton) to recognize Motzkin paths with forbidden subwords UD and HU.

17 UD and HU are forbidden, not in OEIS

The recursion related to Figure 16 are

$$\begin{aligned} f_{i+1} &= zf_i + zh_i,\ f_0 = 0, \\ g_i &= zf_i + zg_i + zh_i,\ i \ge 0 \\ h_i &= [i = 0] + zg_{i+1} + zh_{i+1} \end{aligned}$$

and the system of functional equations is

$$\begin{aligned} F(u) &= zuF(u) + zuH(u), \\ G(u) &= zF(u) + zG(u) + zH(u), \\ H(u) &= 1 + \frac{z}{u}(G(u) - G(0)) + \frac{z}{u}(H(u) - H(0)). \end{aligned}$$

The ubiquitous denominator is

$$z(z-1)u^2 + (1-z)(1+z^2)u - z = z(z-1)(u-r_1)(u-r_2)$$

with

$$r_1 = \frac{(1-z)(1+z^2) - W}{2z(1-z)},\ r_2 = \frac{(1-z)(1+z^2) + W}{2z(1-z)},$$
$$W = \sqrt{(1-z)(1-z-2z^2-2z^3+z^4-z^5)}.$$

Dividing out the factor $u - r_1$ and simplifying, we end up with

$$F(u) = \frac{1}{1-u(1-z)r_1}\ \ G(u) = \frac{r_1}{1-u(1-z)r_1},\ H(u) = \frac{(1-z)r_1 - z}{z(1-u(1-z)r_1)},$$
$$S(u) = \frac{r_1}{z(1-u(1-z)r_1)}.$$

We find

$$S(0) = 1 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 9z^6 + 16z^7 + 30z^8 + 57z^9 + 110z^{10} + \cdots$$

which is not in OEIS and

$$S(1) = 1 + 2z + 3z^2 + 5z^3 + 9z^4 + 17z^5 + 33z^6 + 65z^7 + 130z^8 + 263z^9 + 537z^{10} + \cdots$$

which is also not in OEIS. Furthermore

$$[u^j]S(u) = [u^j]\frac{r_1}{z(1-u(1-z)r_1)} = \frac{r_1}{z}\big((1-z)r_1\big)^j = \frac{1}{z(1-z)}\big((1-z)r_1\big)^{j+1}.$$

Since, before diving out the factor, we have

$$S(u) = \frac{u + z + uz^2G(0) - zG(0)}{z(z-1)u^2 + (1-z)(1+z^2)u - z},$$

we get a recursion from it:

$$\begin{gathered} z(z-1)s_{j-2} + (1-z)(1+z^2)s_{j-1} - zs_j = 0, \; j \geq 2, \\ (z-1)(1+z^2)s_0 + zs_1 = -1 \end{gathered}$$

or in matrix form

$$\begin{pmatrix} (z-1)(1+z^2) & z & 0 & \cdots & 0 \\ z(z-1) & (1-z)(1+z^2) & -z & \cdots & 0 \\ & z(z-1) & (1-z)(1+z^2) & -z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & (1-z)(1+z^2) \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The quantity $\phi_K = 0$ for all K, so let us concentrate on the remaining two.

$$\begin{pmatrix} (z-1)(1+z^2) & z & 0 & \cdots & 0 \\ z(z-1) & (1-z)(1+z^2) & -z & \cdots & 0 \\ & z(z-1) & (1-z)(1+z^2) & -z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & (1-z)(1+z^2) \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} z^3+z-1 & z & 0 & \cdots & 0 \\ z(z-1) & (1-z)(1+z^2) & -z & \cdots & 0 \\ & z(z-1) & (1-z)(1+z^2) & -z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & (1-z)(1+z^2) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The usual determinant is

$$\mathscr{D}_K = (z-1)\mathscr{D}_{K-1} - z^3\mathscr{D}_{K-2} = \frac{(-z)^{K+1}}{W}(r_2^{K+1} - r_1^{K+1}).$$

With $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^3 t_K$

$$t_K = \frac{1}{z-1-z^3 t_{K-1}} \implies \tau_K = \frac{z^3}{1-z-\tau_{K-1}},\ \tau_0 = 0.$$

$$\begin{pmatrix} z-1 & z^2 & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -1 \\ z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+2z-z^2+z^3 & z^2(1-z) & 0 & \cdots & 0 \\ z & z-1 & z^2 & \cdots & 0 \\ & z & z-1 & z^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & z-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z^3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

From this, we can express the solutions (return to the x-axis) in continued fraction form

$$\begin{aligned} s_0 = \sigma_{K+1} &= \frac{-\mathscr{D}_K}{(z-1)(1+z^2)\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = \frac{-1}{(z-1)(1+z^2) - z^2(z-1)t_K} \\ &= \frac{z}{(1-z)(z+z^3-\tau_K)},\ \sigma_0 = \frac{1}{1-z}, \\ g_0 = \gamma_{K+1} &= \frac{-z\mathscr{D}_K}{(z-1)(1+z^2)\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = z\sigma_{K+1}, \\ h_0 = \eta_{K+1} &= \frac{-z^3\mathscr{D}_K}{(-1+2z-z^2+z^3)\mathscr{D}_K - z^3(1-z)\mathscr{D}_{K-1}} \\ &= \frac{-z^3}{(-1+2z-z^2+z^3) - z^3(1-z)t_K} = \frac{z^3}{1-2z+z^2-z^3-(1-z)\tau_K},\ \eta_0 = 0. \end{aligned}$$

18 DU and HD are forbidden, A004149

The recursion related to Figure 17 are

$$\begin{aligned} f_{i+1} &= zf_i + zh_i,\ f_0 = 1, \\ g_i &= zf_{i+1} + zg_{i+1},\ i \geq 0, \\ h_i &= zf_i + zg_i + zh_i, \end{aligned}$$

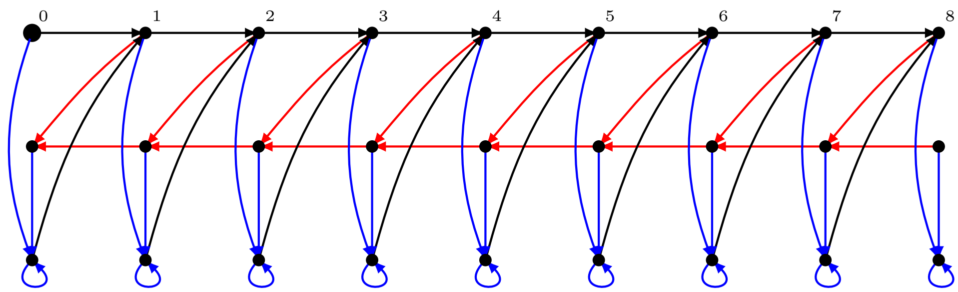


Figure 17: Graph (automaton) to recognize Motzkin paths with forbidden subwords DU and HD.

and the system of functional equations is

$$\begin{aligned}F(u) &= 1 + zuF(u) + zuH(u),\\ G(u) &= \frac{z}{u}(G(u) - G(0)) + \frac{z}{u}(H(u) - H(0)),\\ H(u) &= zF(u) + zG(u) + zH(u).\end{aligned}$$

The ubiquitous denominator is $zu^2 + (-1 + z - z^2 - z^3)u + z = z(u - r_1)(u - r_2)$ with

$$r_1 = \frac{1 - z + z^2 + z^3 - W}{2z}, \; r_2 = \frac{1 - z + z^2 + z^3 - W}{2z}, \quad W = \sqrt{(1 - z^4)(1 - 2z - z^2)}.$$

Dividing out the factor $u - r_1$ and simplifying, we end up with

$$F(u) = \frac{1}{1 - ur_1}, \; G(u) = \frac{(1 - z)r_1 - z}{z^2(1 - ur_1)}, \; H(u) = \frac{r_1 - z}{z(1 - ur_1)}, \; S(u) = \frac{r_1 - z}{z^2(1 - ur_1)}.$$

We find

$$S(0) = 1 + z + z^2 + 2z^3 + 4z^4 + 8z^5 + 16z^6 + 33z^7 + 69z^8 + 146z^9 + 312z^{10} + \cdots$$

which is A004149 in OEIS and

$$S(1) = 1 + 2z + 4z^2 + 9z^3 + 20z^4 + 45z^5 + 102z^6 + 233z^7 + 535z^8 + 1234z^9 + 2857z^{10} + \cdots$$

1,2,4,9,20,45,102,233 which is A308435 in OEIS. Furthermore

$$[u^j]S(u) = [u^j]\frac{r_1 - z}{z^2(1 - ur_1)} = \frac{r_1^{j+1}}{z^2} - \frac{r_1^j}{z}.$$

Since, before diving out the factor, we have

$$S(u) = \frac{\text{numerator}}{zu^2 + (-1 + z - z^2 - z^3)u + z},$$

we get a recursion from it:

$$zs_{j-2} + (-1 + z - z^2 - z^3)s_{j-1} + zs_j = 0, \; j \geq 2,$$
$$(-1 + z - z^3)s_0 + zs_1 = z^2 - 1,$$

which is in matrix form

$$\begin{pmatrix} -1+z-z^3 & z & 0 & \cdots & 0 \\ z & -1+z-z^2-z^3 & z & \cdots & 0 \\ & z & -1+z-z^2-z^3 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^2-z^3 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} z^2-1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The usual determinant is

$$\mathscr{D}_K = (-1 + z - z^2 - z^3)\mathscr{D}_{K-1} - z^2\mathscr{D}_{K-2} = \frac{(-z)^{K+1}}{W}(r_1^{K+1} - r_2^{K+1}).$$

With $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^2 t_K$,

$$t_K = \frac{1}{-1 + z - z^2 - z^3 - z^2 t_{K-1}} \implies \tau_K = \frac{z^2}{1 - z + z^2 + z^3 - \tau_{K-1}}, \; \tau_0 = 0.$$

Since $\phi_K = 1$ for all K, we concentrate on the remaining two:

$$\begin{pmatrix} -1+2z-z^2+z^4 & z(1-z) & 0 & \cdots & 0 \\ z & -1+z-z^2-z^3 & z & \cdots & 0 \\ & z & -1+z-z^2-z^3 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^2-z^3 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_K \end{pmatrix} = \begin{pmatrix} -z^3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+z-z^3 & z & 0 & \cdots & 0 \\ z & -1+z-z^2-z^3 & z & \cdots & 0 \\ & z & -1+z-z^2-z^3 & z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z & -1+z-z^2-z^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} z(z^2-1) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

From this, we can express the solutions (return to the x-axis) in continued fraction form

$$s_0 = \sigma_{K+1} = \frac{(z^2-1)\mathscr{D}_K}{(-1+z-z^3)\mathscr{D}_K - z^2\mathscr{D}_{K-1}} = \frac{(z^2-1)}{(-1+z-z^3) - z^2 t_K}$$
$$= \frac{1-z^2}{1 - z + z^3 - \tau_K}, \quad \sigma_0 = \frac{1}{1-z},$$

$$
\begin{aligned}
g_0 = \gamma_{K+1} &= \frac{-z^3\mathscr{D}_K}{(-1+2z-z^2+z^4)\mathscr{D}_K - z^2(1-z)\mathscr{D}_{K-1}}\\
&= \frac{z^3}{1-2z+z^2-z^4-(1-z)\tau_K},\ \ \gamma_0 = 0,\\
h_0 = \eta_{K+1} &= \frac{z(z^2-1)\mathscr{D}_K}{(-1+z-z^3)\mathscr{D}_K - z^2\mathscr{D}_{K-1}}\\
&= \frac{z(z^2-1)}{(-1+z-z^3)-z^2t_K} = \frac{z(1-z^2)}{1-z+z^3-\tau_K},\ \ \eta_0 = \frac{z}{1-z}.
\end{aligned}
$$

19 DU and UH are forbidden, A217282

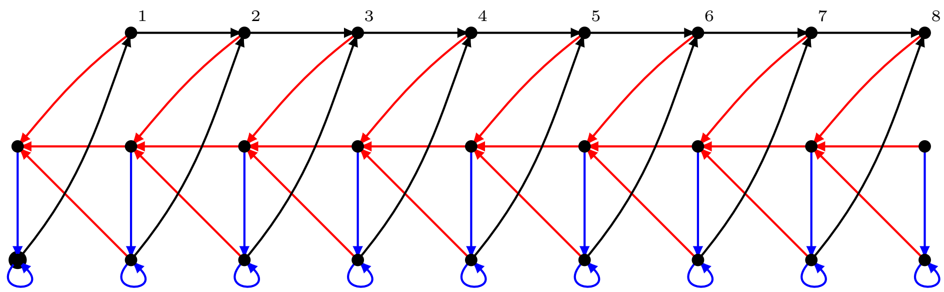


Figure 18: Graph (automaton) to recognize Motzkin paths with forbidden subwords DU and UH.

The recursion related to Figure 18 are

$$
\begin{aligned}
f_{i+1} &= zf_i + zh_i,\ \ f_0 = 0,\\
g_i &= zf_{i+1} + zg_{i+1} + zh_{i+1},\ \ i \ge 0\\
h_i &= 1 + zg_i + zh_i
\end{aligned}
$$

and the system of functional equations is

$$
\begin{aligned}
F(u) &= zuF(u) + zuH(u),\\
G(u) &= \frac{z}{u}(F(u)-F(0)) + \frac{z}{u}(G(u)-G(0)) + \frac{z}{u}(H(u)-H(0)),\\
H(u) &= 1 + zG(u) + zH(u).
\end{aligned}
$$

The ubiquitous denominator is $zu^2 + (-1+z-z^2-z^3)u + z = z(z-1)(u-r_1)(u-r_2)$ with

$$
r_1 = \frac{1-z+z^2-z^3-W}{2z(1-z)},\ \ r_2 = \frac{1-z+z^2-z^3-W}{2z(1-z)},
$$
$$
W = \sqrt{(1-z)(1-z-2z^2-2z^3+z^4-z^5)}.
$$

Dividing out the factor $u-r_1$ and simplifying, we end up with

$$F(u)=\frac{ur_1}{1-u(1-z)r_1},\ G(u)=\frac{(1-z)r_1-z}{z^2(1-u(1-z)r_1)},\ H(u)=\frac{(1-zu)r_1}{z(1-u(1-z)r_1)},$$
$$S(u)=\frac{r_1-z}{z^2(1-u(1-z)r_1)}.$$

We find

$$S(0)=1+z+2z^2+3z^3+5z^4+9z^5+16z^6+30z^7+57z^8+110z^9+216z^{10}+\cdots$$

which is A217282 in OEIS and

$$S(1)=1+2z+4z^2+7z^3+13z^4+25z^5+49z^6+98z^7+198z^8+404z^9+831z^{10}+\cdots$$

which is not in OEIS. Furthermore

$$[u^j]S(u)=[u^j]\frac{r_1-z}{z^2(1-u(1-z)r_1)}=\frac{(1-z)^jr_1^{j+1}}{z^2}-\frac{(1-z)^jr_1^j}{z},\quad j\ge 1.$$

Since, before diving out the factor, we have

$$S(u)=\frac{\textsf{numerator}}{-z-(z-1)(1+z^2)u+z(z-1)u^2},$$

we get a recursion from it:

$$z(z-1)s_{j-2}+(1-z+z^2-z^3)s_{j-1}-zs_j=0,\ j\ge 2,$$
$$(z-1)s_0+zs_1=-1,$$

which is in matrix form

$$\begin{pmatrix} z-1 & z & 0 & \cdots & 0\\ z(z-1) & 1-z+z^2-z^3 & -z & \cdots & 0\\ & z(z-1) & 1-z+z^2-z^3 & -z & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & z(z-1) & 1-z+z^2-z^3 \end{pmatrix}\begin{pmatrix} s_0\\ s_1\\ s_2\\ \vdots\\ s_K\end{pmatrix}=\begin{pmatrix} -1\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

Likewise,

$$\begin{pmatrix} z^3+z-1 & z & 0 & \cdots & 0\\ z(z-1) & 1-z+z^2-z^3 & -z & \cdots & 0\\ & z(z-1) & 1-z+z^2-z^3 & -z & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & z(z-1) & 1-z+z^2-z^3 \end{pmatrix}\begin{pmatrix} g_0\\ g_1\\ g_2\\ \vdots\\ g_K\end{pmatrix}=\begin{pmatrix} -z^2\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

and

$$\begin{pmatrix} z^3+z-1 & z & 0 & \cdots & 0 \\ z(z-1) & 1-z+z^2-z^3 & -z & \cdots & 0 \\ & z(z-1) & 1-z+z^2-z^3 & -z & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(z-1) & 1-z+z^2-z^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -1 \\ -z \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The usual determinant is

$$\mathscr{D}_K = (1-z+z^2-z^3)\mathscr{D}_{K-1} + z^2(z-1)\mathscr{D}_{K-2} = \frac{(z(1-z))^{K+1}}{W}(r_2^{K+1} - r_1^{K+1}).$$

With $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = z^2(1-z)t_K$,

$$t_K = \frac{1}{1-z+z^2-z^3+z^2(z-1)t_{K-1}} \implies \tau_K = \frac{z^2(1-z)}{1-z+z^2-z^3-\tau_{K-1}}, \quad \tau_0 = 0.$$

From this, we can express the solutions (return to the x-axis) in continued fraction form

$$\begin{aligned} s_0 = \sigma_{K+1} &= \frac{-\mathscr{D}_K}{(z-1)\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = \frac{1}{1-z-\tau_K}, \quad \sigma_0 = \frac{1}{1-z}, \\ g_0 = \gamma_{K+1} &= \frac{-z^2\mathscr{D}_K}{(z^3+z-1)\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = \frac{z^2}{1-z-z^3-\tau_K}, \ \gamma_0 = 0, \\ h_0 = \eta_{K+1} &= \frac{-\mathscr{D}_K + z^2\mathscr{D}_{K-1}}{(z^3+z-1)\mathscr{D}_K - z^2(z-1)\mathscr{D}_{K-1}} = \frac{-1+z^2t_K}{(z^3+z-1)-z^2(z-1)t_K} \\ &= \frac{1}{1-z} + \frac{z^3}{(1-z)(1-z-z^3-\tau_K)}, \ \eta_0 = \frac{z}{1-z}. \end{aligned}$$

20 UD and HH are forbidden, A329676

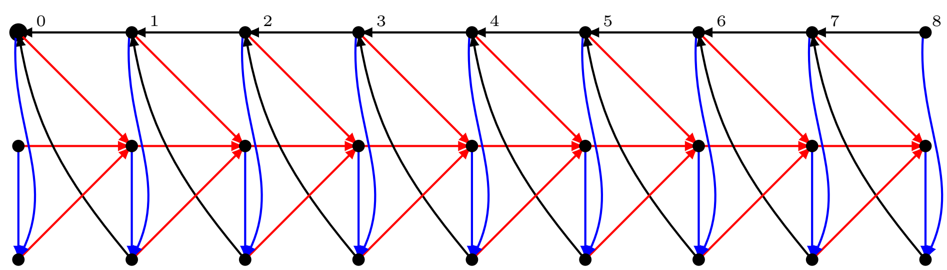


Figure 19: Graph (automaton) to recognize Motzkin paths with forbidden subwords UD and HH.

We derive from Figure 19 the recursions

$$\begin{aligned}
f_i &= [i=0] + zf_{i+1} + zh_{i+1},\\
g_{i+1} &= zf_i + zg_i + zh_i,\\
h_i &= zf_i + zg_i,
\end{aligned}$$

and

$$\begin{aligned}
F(u) &= 1 + \frac{z}{u}(F(u)-F(0)) + \frac{z}{u}(H(u)-H(0)),\\
G(u) &= zuF(u) + zuG(u) + zuH(u),\\
H(u) &= zF(u) + zG(u).
\end{aligned}$$

and set $S = F+G+H$ and $W := \sqrt{(1+2z+3z^2+z^2)(1-2z-z^2+z^3)}$. With

$$r_1 = \frac{1+z^2+z^3-W}{2z(1+z)}, \quad r_2 = \frac{1+z^2+z^3+W}{2z(1+z)}$$

we have the factorization of the common numerator:

$$z(1+z) + (-1-z^3-z^2)u + zu^2(1+z) = z(1+z)(u-r_1)(u-r_2).$$

Then

$$S(u) = \frac{(1+z)(zS(0)-u)}{z(1+z)(u-r_1)(u-r_2)} = \frac{zS(0)-u}{z(u-r_1)(u-r_2)}$$

Set $u=0$, divide out $u-r_1$, then $S(0) = \frac{1}{zr_2}$ and further

$$F(u) = \frac{(1-z(1+z)u)r_1}{z(1+z)(1-ur_1)},\ G(u) = \frac{ur_1}{1-ur_1},\ H(u) = \frac{r_1}{(1+z)(1-ur_1)}$$

$$S(u) = \frac{r_1}{z(1-ur_1)} \Longrightarrow [u^j]S(u) = s_j = \frac{r_1^{j+1}}{z}.$$

Using the denominator, we find recursions

$$\begin{aligned}
z(1+z)s_{j-2} - (1+z^2+z^3)s_{j-1} + z(1+z)s_j &= 0,\ j \geq 2,\\
-s_0(1+z^2+z^3) + z(1+z)s_1 &= -1-z.
\end{aligned}$$

In matrix form:

$$\begin{pmatrix}
-z^3-z^2-1 & z(1+z) & 0 & \cdots & 0\\
z(1+z) & -z^3-z^2-1 & z(1+z) & \cdots & 0\\
 & z(1+z) & -z^3-z^2-1 & z(1+z) & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & z(1+z) & -z^3-z^2-1
\end{pmatrix}
\begin{pmatrix} s_0\\ s_1\\ s_2\\ \vdots\\ s_K \end{pmatrix}
=
\begin{pmatrix} -1-z\\ 0\\ 0\\ \vdots\\ 0 \end{pmatrix}$$

Perhaps a comment that we made before should be repeated here. While it is convenient to compute $s_0^{[K+1]} = f_0^{[K+1]} + g_0^{[K+1]} + h_0^{[K+1]}$ directly, it is slightly incorrect (and irrelevant for large K): One arrives in states from the first layer only with a D-step, in states from the second layer with a U-step, and in states from the third layer with an H-step. Consequently, the state $K+1$ in the first layer cannot be reached, and the correct version would be $f_0^{[K]} + g_0^{[K+1]} + h_0^{[K+1]}$. Therefore we compute the respective quantities also independently.

We have in matrix form:

$$\begin{pmatrix} z^4+z^3-1 & z(1+z) & 0 & \cdots & 0 \\ z(1+z) & -z^3-z^2-1 & z(1+z) & \cdots & 0 \\ & z(1+z) & -z^3-z^2-1 & z(1+z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & -z^3-z^2-1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_K \end{pmatrix} = \begin{pmatrix} -1 \\ z(1+z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The quantity $g_0^{[K]} = 0$ for all K, so we can move to the third layer. Again, we have in matrix form

$$\begin{pmatrix} -z^3-z^2-1 & z(1+z) & 0 & \cdots & 0 \\ z(1+z) & -z^3-z^2-1 & z(1+z) & \cdots & 0 \\ & z(1+z) & -z^3-z^2-1 & z(1+z) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & z(1+z) & -z^3-z^2-1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $\mathscr{D}_K$ be the determinant with K rows and columns, after deleting the first row and column. Expanding, we derive the recursion ($\mathscr{D}_0 = 1$, $\mathscr{D}_1 = -z^3 - z^2 - 1$)

$$\mathscr{D}_K = (-z^3 - z^2 - 1)\mathscr{D}_{K-1} - z^2(1+z)^2\mathscr{D}_{K-2} = \mathscr{D}_K = \frac{(-z(1+z))^{K+1}}{W}(r_1^{K+1} - r_2^{K+1}).$$

Using Cramer's rule as always,

$$\begin{aligned} f_0 = \phi_{K+1} &= \frac{-\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}}{(-1+z^3+z^4)\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}}, \ \phi_0 = 1, \\ h_0 = \eta_{K+1} &= \frac{-z\mathscr{D}_K}{(-z^3-z^2-1)\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}}, \ \eta_0 = z, \\ s_0 = \sigma_{K+1} &= \frac{-(1+z)\mathscr{D}_K}{(-z^3-z^2-1)\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}}, \ \sigma_0 = 1+z. \end{aligned}$$

Finally we move to expansions of the continuous fraction type. From the recursion for the determinants, we have, with $t_K = \mathscr{D}_{K-1}/\mathscr{D}_K$ and $\tau_K = -z^2(1+z)^2 t_K$,

$$t_K = \frac{-1}{1+z^2+z^3+z^2(1+z)^2 t_{K-1}}, \quad -z^2(1+z)^2 t_K = \frac{z^2(1+z)^2}{1+z^2+z^3+z^2(1+z)^2 t_{K-1}},$$

and

$$\tau_K = \frac{z^2(1+z)^2}{1+z^2+z^3-\tau_{K-1}}, \quad \tau_0 = 0.$$

Furthermore,

$$\begin{aligned}
\sigma_{K+1} &= \frac{-(1+z)\mathscr{D}_K}{(-z^3-z^2-1)\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}} = \frac{1+z}{1+z^2+z^3-\tau_K},\\
\phi_{K+1} &= \frac{-\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}}{(-1+z^3+z^4)\mathscr{D}_K - z^2(1+z)^2\mathscr{D}_{K-1}}\\
&= \frac{-z^2(1+z)^2 t_K}{(-1+z^3+z^4) - z^2(1+z)^2 t_K} = 1 - \frac{1-z^3-z^4}{1-z^3-z^4-\tau_K},\\
\sigma_{K+1} &= \frac{-(1+z)}{(-z^3-z^2-1) - z^2(1+z)^2 t_K} = \frac{1+z}{1+z^2+z^3-\tau_K}.
\end{aligned}$$

The generating function

$$S(0) = 1 + z + z^3 + 2z^4 + 2z^5 + 5z^6 + 10z^7 + 16z^8 + 34z^9 + 68z^{10} + \cdots$$

is in OEIS as sequence A329676; the sequence

$$S(1) = 1 + z + z^3 + 2z^4 + 2z^5 + 5z^6 + 10z^7 + 16z^8 + 34z^9 + 68z^{10} + \cdots$$

is sequence A329675 in OEIS.

21 HU and DD are forbidden, A217282

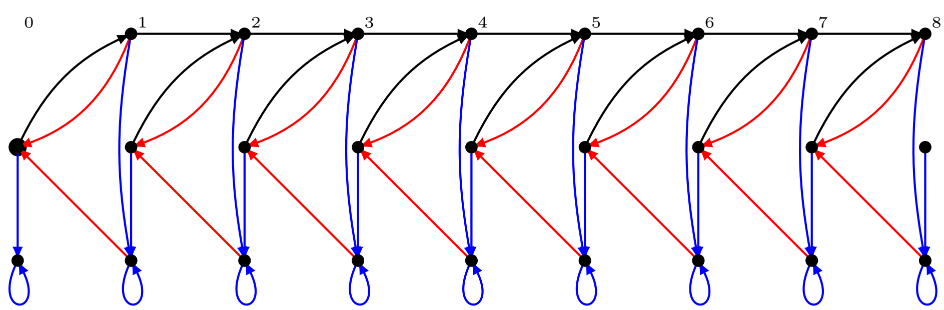


Figure 20: Graph (automaton) to recognize Motzkin paths with forbidden subwords HU and DD.

The following recursions can be read off the automaton (Figure 20), by considering the last step separately:

$$\begin{aligned} f_{i+1} &= zf_i + zg_i,\ \ f_0 = 0,\\ g_i &= [i=0] + zf_{i+1} + zh_{i+1},\\ h_i &= zf_i + zg_i + zh_i, \end{aligned}$$

and using double generating functions

$$\begin{aligned} F(u) &= zuF(u) + zuG(u),\\ G(u) &= 1 + \frac{z}{u}\big(F(u) - F(0)\big) + \frac{z}{u}\big(H(u) - H(0)\big),\\ H(u) &= zF(u) + zG(u) + zH(u). \end{aligned}$$

We have the factorization $-z^2 + (z+1)(1-z)^2u + z(z-1)u^2 = z(z-1)(u-r_1)(u-r_2)$ with

$$r_1 = \frac{1-z-z^2+z^3-W}{2z(1-z)}, \quad r_2 = \frac{1-z-z^2+z^3+W}{2z(1-z)},$$
$$W = \sqrt{(1-z)(1-z-2z^2-2z^3+z^4-z^5)}.$$

Solving the system of functional equations and dividing out the factor $u-r_1$ we get after simplification

$$F(u) = \frac{u(1-z)r_1}{z-u(1-z)r_1},\ G(u) = \frac{(1-z)(1-zu)r_1}{z(z-u(1-z)r_1)},\ H(u) = \frac{r_1}{z-u(1-z)r_1},$$
$$S(u) = \frac{r_1}{z(z-u(1-z)r_1)}.$$

From this we find

$$S(0) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 9z^5 + 16z^6 + 30z^7 + 57z^8 + 110z^9 + 216z^{10} + \cdots,$$

which is A217282. The generating function for meanders is

$$S(1) = 1 + 2z + 4z^2 + 8z^3 + 16z^4 + 33z^5 + 69z^6 + 146z^7 + 312z^8 + 671z^9 + 1451z^{10} + \cdots,$$

which is not in OEIS. Furthermore, for $j \ge 1$,

$$[u^j]S(u) = [u^j]\frac{r_1}{z^2(1-\frac{u(1-z)r_1}{z})} = \frac{(1-z)^j r_1^{j+1}}{z^{j+2}},$$

which is the generating function of restricted Motzkin paths ending at level j.

For $S = F + G + H$ we get

$$S = \frac{\textsf{numerator}}{-z^2 + (z+1)(1-z)^2u + z(z-1)u^2},$$

and from this a recursion

$$z(z-1)s_{j-2}+(z+1)(1-z)^2s_{j-1}-z^2s_j=0, \quad j\ge 2,$$
$$-(z+1)(1-z)^2s[0]+z^2s[1]=-1,$$

which is in matrix form

$$\begin{pmatrix}-(z+1)(1-z)^2 & z^2 & 0 & \cdots & & 0\\ z(z-1) & (z+1)(1-z)^2 & -z^2 & \cdots & & 0\\ & z(z-1) & (z+1)(1-z)^2 & -z^2 & & 0\\ \vdots & \ddots & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & & z(z-1) & (z+1)(1-z)^2\end{pmatrix}\begin{pmatrix}s_0\\ s_1\\ s_2\\ \vdots\\ s_K\end{pmatrix}=\begin{pmatrix}-1\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

We have $f_0=\phi_K=0$ for all K. Furthermore,

$$\begin{pmatrix}z^2+z-1 & z^2 & 0 & \cdots & & 0\\ z(z-1) & (z+1)(1-z)^2 & -z^2 & \cdots & & 0\\ & z(z-1) & (z+1)(1-z)^2 & -z^2 & & 0\\ \vdots & \ddots & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & & z(z-1) & (z+1)(1-z)^2\end{pmatrix}\begin{pmatrix}g_0\\ g_1\\ g_2\\ \vdots\\ g_K\end{pmatrix}=\begin{pmatrix}z-1\\ z(z-1)\\ 0\\ \vdots\\ 0\end{pmatrix}$$

and

$$\begin{pmatrix}-(z+1)(-1+z)^2 & z^2 & 0 & \cdots & & 0\\ z(z-1) & (z+1)(1-z)^2 & -z^2 & \cdots & & 0\\ & z(z-1) & (z+1)(1-z)^2 & -z^2 & & 0\\ \vdots & \ddots & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & & z(z-1) & (z+1)(1-z)^2\end{pmatrix}\begin{pmatrix}h_0\\ h_1\\ h_2\\ \vdots\\ h_K\end{pmatrix}=\begin{pmatrix}-z\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

Let $\mathscr{D}_K$ be the determinant with K rows and columns, after deleting the first row and column. Expanding, we derive the recursion ($\mathscr{D}_0=1$, $\mathscr{D}_1=(z+1)(1-z)^2$)

$$\mathscr{D}_K=(z+1)(1-z)^2\mathscr{D}_{K-1}+z^3(z-1)\mathscr{D}_{K-2}=\frac{z^{K+1}(1-z)^{K+1}}{W}(r_2^{K+1}-r_1^{K+1}).$$

Rewriting this, we get, with $t_K=\frac{\mathscr{D}_{K-1}}{\mathscr{D}_K}$, $\tau_K=z^3(1-z)t_K$, $\tau_0=0$

$$t_K=\frac{1}{1-z-z^2+z^3+z^3(z-1)t_{K-1}}, \quad \tau_K=\frac{z^3(1-z)}{1-z+z^3-\tau_{K-1}}.$$

Solving the system by Cramer's rule, we get the solution for s_0 (return to the x-axis) as

$$\begin{aligned}s_0=\sigma_{K+1}&=\frac{-\mathscr{D}_K}{(-1+z+z^2-z^3)\mathscr{D}_K-z^3(z-1)\mathscr{D}_{K-1}}\\ &=\frac{-1}{(-1+z+z^2-z^3)-z^3(z-1)t_K}=\frac{1}{1-z-z^2+z^3-\tau_K},\ \sigma_0=\frac{1}{1-z},\end{aligned}$$

$$\begin{aligned}
g_0 = \gamma_{K+1} &= \frac{(z-1)\mathscr{D}_K - z^3(z-1)\mathscr{D}_{K-1}}{(-1+z+z^2)\mathscr{D}_K - z^3(z-1)\mathscr{D}_{K-1}} = \frac{(z-1) - z^3(z-1)t_K}{(-1+z+z^2) - z^3(z-1)t_K}\\
&= 1 + \frac{z^2}{1-z-z^2-\tau_K}, \ \gamma_0 = 1,\\
h_0 = \eta_{K+1} &= \frac{-z\mathscr{D}_K}{(z^2-z^3-1+z)\mathscr{D}_K - z^3(z-1)\mathscr{D}_{K-1}}\\
&= \frac{-z}{(z^2-z^3-1+z) - z^3(z-1)t_K} = \frac{z}{1-z-z^2+z^3-\tau_K}, \ \eta_0 = \frac{z}{1-z}.
\end{aligned}$$

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