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Identities and Hadamard Product of the Generalized Fibonacci, Lucas, Catalan, and Harmonic Numbers

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Abstract

We consider properties of the generalized Fibonacci and Lucas numbers. We obtain identities related to the generalized Fibonacci, Lucas, Catalan, and harmonic numbers. We also derive explicit expressions for the generating functions of the Hadamard products of the generalized Fibonacci, Lucas, Catalan, and harmonic numbers.

1 Introduction

The Hadamard product of generating functions is a subject of study in a variety of fields including combinatorics, probability theory, and mathematical physics. The Hadamard product [12] of the generating functions $A(x) = \sum_{n\geqslant 0} a_n x^n$ and $B(x) = \sum_{n\geqslant 0} b_n x^n$ is a generating function defined by

$$A(x) * B(x) = \sum_{n\geqslant 0} a_n b_n x^n.$$

It is well known [35, 21] that the Hadamard product of two rational generating functions is a rational function. However, there is no general method to obtain an explicit expression for the Hadamard product of two generating functions. Some studies have explored specific classes of generating functions. For example, Shapiro [29] provided a combinatorial proof of the Hadamard product of generating functions for Chebyshev polynomials given by

$$\frac{1}{1-ax-x^2} * \frac{1}{1-bx-x^2} = \frac{1-x^2}{1-abx-(2+a^2+b^2)x^2-abx^3+x^4}. \tag{1}$$

Kim [18, 19] generalized (1) to the case

$$\frac{1}{1-ax-x^2} * \frac{x^m}{1-bx-x^n}.$$

Kar [17] considered a rational generating function as a polynomial plus a sum of power series. Thus, he reduced the problem of finding the Hadamard product of two rational generating functions for the case of the Hadamard product

$$\frac{1}{(1-ax)^{m+1}} * \frac{1}{(1-bx)^{n+1}}.$$

Kar and Gessel [10] extended results to find the Hadamard product

$$\frac{x^i}{(1-ax)^{m+1}} * \frac{x^j}{(1-bx)^{n+1}}.$$

Potekhina and Tolovikov [25, 26] developed a method to obtain the Hadamard product for the case

$$\frac{1}{1-d_1x+d_2x^2+\cdots+d_nx^n} * \frac{x^k}{1-b_1x+b_2x^2+\cdots+b_mx^n}.$$

It is also known that if one of the generating functions is rational and the other is algebraic or D-finite, then their Hadamard product is, respectively, an algebraic or D-finite generating function [28, 34]. However, even in such cases, there are no general methods for finding explicit expressions for the Hadamard product. Combinatorial techniques can sometimes be used to derive explicit expressions for the Hadamard product in certain specific cases. For example, Boyadzhiev [4, p. 378, Equation (2.27)] found an explicit expression for the Hadamard product

$$\sum_{n>0} H_n n^m x^n,$$

where H_n are harmonic numbers.

Chen [6] obtained remarkable generating functions for the Hadamard product of central binomial numbers and harmonic numbers

$$\frac{1}{1-x}\log\left(\frac{1}{1-x}\right) * \frac{1}{\sqrt{1-4x}} = \frac{2}{\sqrt{1-4x}}\log\left(\frac{\sqrt{1-4x}+1}{2\sqrt{1-4x}}\right).$$

Barry [2] obtained an explicit expression for the Hadamard product of C_nF_{n+1} of the Catalan numbers and the generalized Fibonacci numbers

$$\frac{1-\sqrt{1-4x}}{2x} * \frac{1}{1-ax-bx^2} = \frac{1}{x}\sqrt{\frac{1-2ax-\sqrt{1-4ax-16bx^2}}{2(a^2-4b)}}.$$

The Hadamard product has many applications. Bragg [5] applied the Hadamard product to evaluate an extensive number of trigonometric integrals in terms of sums. Prodinger and Selkirk [27] used generating functions and the Hadamard product for getting an explicit expression of the partial sum of Tetranacci numbers. E. D. Leinartas and E. K. Leinartas [23] applied the Hadamard composition for solving some systems of difference equations. Zhilinskii [36] applied the symmetrized Hadamard product for construction of diagonal in polyad quantum number effective resonant vibrational Hamiltonians. Potekhina and Tolovikov [25] used the Hadamard product for calculation of distributions of statistics relating to oscillating random walks with integer steps.

In this paper based on the identity for the generalized Fibonacci F_n and Lucas L_n numbers

$$F_{kn+2m}\sum_{i=0}^{n}T_{n-i,i}=\sum_{i=0}^{n}T_{n-i,i}L_{ki+m}F_{k(n-i)+m},$$

where $T_{n,i}\in\mathbb{R}$ and the property $T_{n-i,i}=T_{i,n-i}$ is satisfied, we derive explicit expressions for the generating functions of the Hadamard products of the forms

$$F(x)*H(x)=\sum_{n>0}F_nH_nx^n$$

$$L(x)*H(x)=\sum_{n>0}F_nH_nx^n$$

$$F(x)*C(x)=\sum_{n>0}F_nC_nx^n$$

$$L(x)*C(x)=\sum_{n\geqslant 0}L_nC_nx^n,$$

where $H(x)$ is the generating function for harmonic numbers, $L(x)$ is the generating function for the generalized Lucas numbers, and $C(x)$ is the generating function for the Catalan numbers.

2 Identity for the generalized Fibonacci and Lucas numbers

The Fibonacci and Lucas numbers are among the most studied objects in combinatorics. A vast number of publications and monographs are devoted to those numbers [1, 14, 20]. Numerous generalizations of the Fibonacci and Lucas numbers have also been proposed [3, 7, 15, 9, 8, 16, 31]. Siebeck [30] was one of the first to study generalized Fibonacci numbers. He examined the recurrence of the form $N_r=aN_{r-1}+cN_{r-2}$ with initial conditions $N_0=0$, $N_1=1$. He also wrote the following generating function

$$\frac{x}{1-ax-cx^2}$$

and the recursive formula

$$N_{n+m} = cN_{n-1}N_m + N_{m+1}N_n. \tag{2}$$

Lucas then proposed a generalization, which became known as the Lucas sequence. He studied the recurrence relation of the form

$$X_{n+2} = PX_{n+1} - QX_n,$$

where P, Q are nonzero rational integers, and introduced two sequences $U_n(P, Q)$ with $U_0 = 0$ $U_1 = 1$ and $V_n(P, Q)$ with $V_0 = 2$, $V_1 = P$. He obtained the identities

$$U_{n+m} = U_{m+1}U_n - QU_{n-1}U_m \tag{3}$$

$$2U_{n+m} = V_nU_m + U_nV_m.$$

Here P and Q are present in general form. By comparing the identities (2) and (3), we find that they are completely the same. Kalman and Mena [16] made a further similar generalization and proposed a generalization for the Fibonacci numbers of the form

$$A_{n+2} = aA_{n+1} + bA_n,$$

where $a, b \in \mathbb{R}$. For fixed a and b they denoted the set of all such sequences by $\mathcal{R}(a, b)$. With initial values $A_0 = 0$, $A_1 = 1$ the resulting sequence (a, b) was referred to as the Fibonacci sequence; similarly, with $A_0 = 2$, $A_1 = b$, they defined the Lucas sequence (a, b). Then the Fibonacci sequence was written as F_n, and the Lucas sequence as L_n when the values a and b were not explicitly specified (a, b). They also wrote the identities for the generalized Fibonacci and Lucas numbers

$$L_n = F_{n+1} + bF_{n-1}.$$

In addition, they presented the following identity accomplished for the generalized Fibonacci numbers

$$F_{n+m} = bF_{n-1}F_m + F_nF_{m+1}. \tag{4}$$

This identity is completely coincides with the identities (2) and (3) obtained earlier. Hereafter, we adopt the notation by Kalman and Mena [16]. The generating function for these generalized Fibonacci numbers is

$$F(x) = \frac{x}{1 - ax - bx^2},$$

where $a, b \in \mathbb{R}$.

The generating function for these generalized Lucas numbers is

$$L(x) = \frac{2 - a\,x}{1 - ax - bx^2},$$

where $a, b \in \mathbb{R}$.

By setting the values a and b, we obtain a particular sequence. For example, in the case $\mathcal{R}(1,1)$ the sequence F_n corresponds to the Fibonacci numbers and L_n to the Lucas numbers. Similarly, for $\mathcal{R}(2,1)$ F_n represents the Pell numbers and L_n the Pell-Lucas numbers. Table 2 presents the sequences F_n and L_n for different values of a and b, as they occur in the *On-Line Encyclopedia of Integer Sequences* OEIS [32].

a	b	F_n	L_n
1	1	A000045	A000032
2	1	A000129	A002203
3	1	A006190	A006497
4	1	A001076	A014448
5	1	A052918	A087130
6	1	A005668	A085447
7	1	A054413	A086902
8	1	A041025	A086594
9	1	A099371	A087798
1	2	A001045	A014551
1	3	A006130	A075118
2	2	A002605	A080040
2	3	A015518	A102345
3	2	A007482	A206776
3	3	A030195	A172012

Table 1: The generalized Fibonacci and Lucas numbers as sequences in the OEIS.

Next, we present several identities that will be applied to obtain the generating functions in the following sections.

Proposition 1. *The following identity for the generalized Fibonacci and Lucas numbers is true*

$$2F_{n+m} = F_m L_n + L_m F_n. \tag{5}$$

Proof. Based on (4) we obtain

$$F_{n+m} = bF_{n-1}F_m + F_n F_{m+1}$$

and

$$F_{n+m} = bF_n F_{m-1} + F_{n+1}F_m.$$

By adding and grouping, we obtain

$$2F_{n+m} = bF_n F_{m-1} + F_{n+1}F_m + bF_{n-1}F_m + F_n F_{m+1},$$

$$2F_{n+m} = \left(bF_n F_{m-1} + F_n F_{m+1}\right) + \left(F_{n+1}F_m + bF_{n-1}F_m\right),$$

$$2F_{n+m} = F_n\big(bF_{m-1} + F_{m+1}\big) + F_m\big(F_{n+1} + bF_{n-1}\big).$$

Using

$$L_n = F_{n+1} + bF_{n-1},$$

we obtain the sought-for formula

$$2F_{n+m} = F_m L_n + L_m F_n.$$

$\square$

As a consequence of this statement, where $n = m$, we obtain the identity

$$F_{2\,n} = F_n L_n. \tag{6}$$

We consider an identity that connects the sequences F_n, L_n and $T_{n,i}$. Let $T_{n,i} \in \mathbb{R}$, $n, i \in \mathbb{N}$ and suppose that $T_{n,i}$ satisfies the property $T_{i,n-i} = T_{n-i,i}$. Then the sum takes the form

$$p_n = \sum_{i=0}^{n} T_{n-i,i}.$$

Next we prove the following theorem.

Theorem 2. *For the sequences of numbers F_n, L_n and p_n, $k, m \in \mathbb{N}$ the following identity holds*

$$p_n F_{kn+2m} = \sum_{i=0}^{n} T_{n-i,i} L_{ki+m} F_{k(n-i)+m}. \tag{7}$$

Proof. We substitute $n = 0$ in the obtained identity (7) to get

$$p_0 F_{0+2m} = p_0 L_m F_m.$$

Based on the identity (6), we obtain the equality. Now we consider multiple pairs of members on the right side (7) such as

$$T_{n-i,i} = T_{i,n-i}.$$

Then for the sum

$$T_{i,n-i} L_{ki+m} F_{k(n-i)+m} + T_{n-i,i} L_{k(n-i)+m} F_{ki+m}$$

there holds true

$$T_{n-i,i}\big(L_{ki+m} F_{k(n-i)+m} + L_{k(n-i)+m} F_{ki+m}\big).$$

Based on the identity, we write

$$L_{ki+m} F_{k(n-i)+m} + L_{k(n-i)+m} F_{ki+m} = 2F_{ki+m+k(n-i)+m} = 2F_{kn+2m}.$$

For odd values of n, multiple pairs of members on the right side (7) are even and the resulting identity applies to each of these pairs. For even values of n multiple pairs of members on the right side (7) are odd. We consider $n = i + i$, then the sum of members takes the expression

$$T_{i,i} L_{ki+m} F_{ki+m}$$

Based on the identity (6), we obtain

$$L_{ki+m} F_{ki+m} = F_{2ki+2m} = F_{kn+2m}.$$

Then the right side of the identity (8) can be written as

$$\sum_{i=0}^{n} T_{n-i,i} L_{ki+m} F_{k(n-i)+m} = F_{kn+2m} \sum_{i=0}^{n} T_{i,n-i}.$$

Knowing that

$$p_n = \sum_{i=0}^{n} T_{n-i,i},$$

we obtain the desired identity

$$p_n F_{kn+2m} = \sum_{i=0}^{n} T_{n-i,i} L_{ki+m} F_{k(n-i)+m}.$$

□

Let us consider some examples of applying the identity (7). If we let

$$T_{n,i} = r_n r_i,$$

where $r_n \in \mathbb{R}$ and

$$p_n = \sum_{i=0}^{n} r_i r_{n-i},$$

then the identity (7) is

$$p_n F_{kn+2m} = \sum_{i=0}^{n} r_i L_{ki+m} r_{n-i} F_{k(n-i)+m}. \tag{8}$$

If $r_n = 1$, then the identity (8) is

$$(n+1) F_{kn+2m} = \sum_{i=0}^{n} L_{ki+m} F_{k(n-i)+m}.$$

For $m = n$ we obtain

$$(n+1)F_{(k+1)n} = \sum_{i=0}^{n} L_{ki+n}F_{k(n-i)+n}.$$

Let us consider the case of $r_n = F_n$. We substitute this into the identity (8)

$$p_n F_{n+2m} = \sum_{i=0}^{n} F_i L_{i+m} F_{n-i} F_{n-i+m}.$$

Here

$$p_n = \sum_{i=0}^{n} F_i F_{n-i}.$$

Hoggatt [13] obtained the following explicit expression for the convolution of Fibonacci numbers $\mathcal{R}(1,1)$

$$\sum_{i=0}^{n} F_i F_{n-i} = \frac{1}{5}\left((n-1)F_n + (n+1)F_{n-1}\right).$$

Using his method, we can obtain the convolution formula for the generalized Fibonacci numbers $\mathcal{R}(a,b)$

$$\sum_{i=0}^{n} F_i F_{n-i} = \frac{(n-1)F_{n+1} + b(n+1)F_{n-1}}{a^2+4b}. \tag{9}$$

When $m = 0$, $k = 1$, $\mathcal{R}(1,1)$, we obtain the identity

$$\sum_{i=0}^{n} F_i^2 L_{n-i} F_{n-i} = F_n \frac{(n-1)F_{n+1} + (n+1)F_{n-1}}{5}.$$

For Pell numbers we obtain $m = 0$, $k = 1$, $\mathcal{R}(2,1)$

$$\sum_{i=0}^{n} F_i^2 L_{n-i} F_{n-i} = F_n \frac{(n-1)F_{n+1} + (n+1)F_{n-1}}{8}.$$

For Jacobsthal numbers, we obtain $m = 0$, $k = 1$, $\mathcal{R}(1,2)$

$$\sum_{i=0}^{n} F_i^2 L_{n-i} F_{n-i} = F_n \frac{(n-1)F_{n+1} + (n+1)F_{n-1}}{9}.$$

Note that the property $T_{i,n-i} = T_{n-i,i}$ can be replaced by the property $T_{n,i} = T_{n,n-i}$ for some triangle $T_{n,k}$ (for $n < k$ $T_{n,k} = 0$), provided that the triangle is symmetrical with respect to the main diagonal. In this case, the identity (7) takes the form

$$F_{kn+2m} \sum_{i=0}^{n} T_{n,i} = \sum_{i=0}^{n} T_{n,i} F_{ki+m} L_{k(n-i)+m}.$$

We write the identity for Pascal's triangle

$$2^n F_{kn+2m} = \sum_{k=0}^{n} \binom{n}{i} F_{ki+m} L_{k(n-i)+m}.$$

Similarly, for the entries of Pascal's triangle (see A008459)

$$\binom{2n}{n} F_{kn+2m} = \sum_{k=0}^{n} \binom{n}{i}^2 F_{ki+m} L_{k(n-i)+m}.$$

For Eulerian numbers $E(n,k)$ [11, par. 6.2, pp. 267–272] the following identity can be written

$$(n+1)! F_{kn+2m} = \sum_{k=0}^{n} E(n,k) F_{ki+m} L_{k(n-i)+m}.$$

Leibniz's harmonic triangle [22], Lozanic's triangle [24], and others [32] have such properties. The resulting identity (7) allows us to write a functional equation for the Hadamard product in the form

$$[F(x) * R(x)^2] = [F(x) * R(x)][L(x) * R(x)], \tag{10}$$

where $F(x)$ is the generating function for the generalized Fibonacci numbers, $L(x)$ is the generating function for the generalized Lucas numbers and

$$R(x) = \sum_{n=0}^{\infty} r_n x^n.$$

Knowing two of these three functions, one can find the third function. Further we consider obtaining the generating functions based on the equation (10).

3 Hadamard product of the generalized Fibonacci and harmonic numbers

Harmonic numbers are important in various fields of number theory and algorithms. The harmonic number H_n is defined as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

The generating function has a form

$$H(x) = \frac{1}{1-x} \log\left(\frac{1}{1-x}\right).$$

We know the identity the for harmonic numbers

$$\sum_{i=1}^{n-1} \frac{1}{i(n-i)} = \frac{2}{n} H\left(n-1\right).$$

Let us find the generating function for the Hadamard product of the form

$$\sum_{n>0} F_n H_n x^n.$$

To do this, we prove the following theorem.

Theorem 3. *The Hadamard product of the generating function of the generalized Fibonacci numbers and harmonic numbers has the form*

$$\frac{x}{1-ax-bx^2} * \frac{\log(x-1)}{1-x} = \frac{\frac{(2-ax)}{\sqrt{4b+a^2}} \log\left(\frac{2-\sqrt{a^2+4b}x-ax}{2+\sqrt{a^2+4b}x-ax}\right) + x\log\left(1-ax-bx^2\right)}{2\left(1-ax-bx^2\right)}. \tag{11}$$

Proof. We consider the case $r(n) = \frac{1}{n}$ with $r(0) = 0$, $m = 0$, $k = 1$. Then the identity (8) is

$$p_n F_n = \sum_{i=1}^{n-1} \frac{L_i}{i} \frac{F_{n-i}}{n-i},$$

where F_n are the generalized Fibonacci numbers, L_n are the generalized Lucas numbers.

Then

$$p_n = \sum_{i=1}^{n-1} \frac{1}{i(n-i)} = \frac{2}{n} H_{n-1}.$$

Next, we find the generating function $U_1(x,a,b)$ for $p(n)F(n)$. We note that

$$p_n F_n = [x^n]U_1(x) = [x^n]U_2(x)U_3(x),$$

where

$$U_2(x) = \sum_{n=1}^{\infty} \frac{F_n}{n} x^n,$$

$$U_3(x) = \sum_{n=1}^{\infty} \frac{L_n}{n} x^n.$$

Then

$$U_2(x,a,b) = \int \frac{1}{(1-ax-bx^2)} dx = \frac{1}{\sqrt{4b+a^2}} \log\left(\frac{-\sqrt{4b+a^2}x+ax-2}{\sqrt{4b+a^2}x+ax-2}\right),$$

$$U_3(x,a,b) = \int \left(\frac{2-ax}{(1-ax-bx^2)x} - \frac{2}{x}\right) dx = -\log\left(1-ax-bx^2\right).$$

Therefore the sought-for generating function is

$$U_1(x,a,b)=\frac{1}{\sqrt{a^2+4b}}\log\left(\frac{2-\sqrt{a^2+4b}x-ax}{2+\sqrt{a^2+4b}x-ax}\right)\log\left(1-ax-bx^2\right)$$

and

$$\frac{2}{n}H_{n-1}F_n=[x^n]U_1(x,a,b).$$

Therefore we can write

$$H_{n-1}F_n=[x^n]\frac{xdU_1(x,a,b)}{2dx}.$$

Then the generating function for the Hadamard product H_nF_{n+1} is

$$U_4(x,a,b)=\frac{(a+2bx)\log\left(\frac{2-\sqrt{a^2+4b}x-ax}{2+\sqrt{a^2+4b}x-ax}\right)+\sqrt{a^2+4b}\log\left(1-ax-bx^2\right)}{2\sqrt{a^2+4b}\left(1-ax-bx^2\right)}.$$

Next, we find the generating function $U_5(x,a,b)$ for H_nF_n. We write

$$H_nF_n=H_{n-1}F_n+\frac{1}{n}F_n,$$

where

$$H_nF_n=[x^n]\left(\frac{xdU_1(x,a,b)}{2dx}+U_2(x,a,b)\right).$$

Then the generating function for the Hadamard product H_nF_n is

$$U_5(x,a,b)=\frac{\frac{(2-ax)}{\sqrt{4b+a^2}}\log\left(\frac{2-\sqrt{a^2+4b}x-ax}{2+\sqrt{a^2+4b}x-ax}\right)+x\log\left(1-ax-bx^2\right)}{2\left(1-ax-bx^2\right)}.$$

$\square$

Next we consider some examples. If we let $a=1$, $b=1$, and $\mathcal{R}(1,1)$, then the Hadamard product for the Fibonacci numbers and the harmonic numbers is

$$H_nF_n=[x^n]U_5(x,1,1)=[x^n]\left(\frac{\sqrt{5}x\log\left(1-x-x^2\right)+(2-x)\log\left(\frac{\left(2-\sqrt{5}+1\right)x}{\left(2+\sqrt{5}-1\right)x}\right)}{2\sqrt{5}(x^2+x-1)}\right).$$

The presented series converges at $x=\frac{1}{2}$. Therefore it follows that

$$\sum_{n=1}^{\infty}\frac{H_nF_n}{2^n}=\frac{3}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)+\log 4=\frac{6\log(1+\phi)}{\sqrt{5}}+2\log 2,$$

where ϕ is the golden ratio.

If we let $a = 1$, $b = 2$, and $\mathcal{R}(1, 2)$, then the Hadamard product for the Jacobsthal numbers J_n and the harmonic numbers is

$$H_n J_n = [x^n]U_5(x, 1, 2) = [x^n] \left(\frac{(2 - x)\log(\frac{1+x}{1-2x}) - 3x\log(1 - x - 2\,x^2)}{6(1 - x - 2x^2)} \right).$$

The presented series converges at $x = \frac{1}{3}$. Therefore it follows that

$$\sum_{n=1}^{\infty} \frac{1}{3^n} H_n J_n = \frac{3\log 3 + 2\log 2}{4}$$

.

The obtained generating functions $U_4(x, a, b)$ and $U_5(x, a, b)$ allow us to write the generating function for $H(n)F(n + j + 1)$ with $j \geq 0$. For this purpose we use the identity (4) and obtain

$$H(n)F(n + j + 1) = F(j + 1)H(n)F(n + 1) + bF(j)H(n)F(n).$$

Therefore the generating function has the following expression

$$U_6(x, a, b) = F(j + 1)U_5(x, a, b) + bF(j)U_4(x, a, b). \tag{12}$$

4 Hadamard product of Fibonacci and Catalan numbers

Catalan numbers, along with Fibonacci numbers and harmonic numbers are the subject of extensive research by many mathematicians. For instance, Stanley [33] gives over 300 combinatorial objects that have been described by Catalan numbers. We now provide an explicit expression for the Catalan numbers

$$C_n = \frac{1}{n + 1}\binom{2n}{n},$$

a generating function

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

and the well-known recurrence relation

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i},$$

where $C_0 = 1$.

Let us find the Hadamard product

$$[C(x) * F(x)] = \sum_{n>0} C_n F_n x^n$$

For that we consider the application of the obtained identity (8) for $r(n) = C_n$, where $k = 1$ and C_n are the Catalan numbers.

Then the identity is

$$C_{n+1}F_{n+m} = \sum_{i=0}^{n} C_i L_{i+m} C_{n-i} F_{n-i+m}. \tag{13}$$

Applying the result obtained by Barry and Mwafise [2, p. 22]

$$C_n p_{n+1} = [x^n]V_1(x,a,b) = [x^n]\left(\frac{1}{x}\sqrt{\frac{1-2ax-\sqrt{1-4ax-16bx^2}}{(2(a^2+4b))}}\right), \tag{14}$$

where

$$p_n = [x^n]\left(\frac{1}{1-ax-bx^2}\right),$$

we obtain the following generating function

$$V_2(x,a,b) = \sum_{n=0}^{\infty} C_n\, L_{n+1} x^n.$$

Corollary 4. *The generating function for the product of the generalized Lucas and the Catalan numbers $C_n L_{n+1}$ has the following expression*

$$V_2(x,a,b) = \frac{1}{x}\left(1 - \sqrt{\frac{1-2ax+\sqrt{1-4ax-16bx^2}}{2}}\right). \tag{15}$$

Proof. If we let

$$V_1(x) = \sum_{n=0} C_n F_{n+1} x^n,$$

then we use the recurrence equation based on the identity (13) for $m = 1$

$$C_{n+1}F_{n+2} = \sum_{i=0}^{n} C_i L_{i+1} C_{n-i} F_{n-i+1}.$$

Then

$$C_n F_{n+1} = \begin{cases} 1, & \text{if } n = 0; \\ \sum\limits_{i=0}^{n-1} C_i L_{i+1} C_{n-i-1} F_{n-i}, & \text{if } n > 0. \end{cases}$$

Then we write the following functional equation related to the generating functions $V_1(x,a,b)$ and $V_2(x,a,b)$

$$V_1(x,a,b) = 1 + xV_1(x,a,b)V_2(x,a,b).$$

Therefore we find the explicit expression for the sought-for generating function

$$V_2(x,a,b) = \frac{1}{x}\left(1 - \frac{1}{V_1(x,a,b)}\right).$$

$\square$

Corollary 5. *The generating function for the product of the generalized Fibonacci and the Catalan numbers C_nF_n has the following expression*

$$V_3(x,a,b) = \frac{1}{2bx}\left(1 - \sqrt{\frac{2b\sqrt{-16bx^2 - 4ax + 1} + 4abx + 2b + a^2}{4b + a^2}}\right). \tag{16}$$

Proof. By definition we obtain

$$L_{n+1} = F_{n+2} + bF_n = aF_{n+1} + 2bF_n.$$

Then

$$C_nL_{n+1} = aC_nF_{n+1} + 2bF_nC_n,$$
$$V_2(x,a,b) = aV_1(x,a,b) + 2bV_3(x,a,b),$$
$$V_3(x,a,b) = \frac{1}{2b}\left(V_2(x,a,b) - aV_1(x,a,b)\right).$$

Knowing the generating functions for the right side terms of the expression, we obtain the sought-for expression. $\square$

Let us consider some examples. If we let $a = 1$, $b = 1$, and $\mathcal{R}(1,1)$, then the Hadamard product for the Fibonacci and the Catalan numbers will have the expression

$$F_nC_n = [x^n]\left(\frac{1}{2x}\left(1 - \frac{\sqrt{2\sqrt{1 - 4x - 16x^2} + 4x + 3}}{\sqrt{5}}\right)\right).$$

We add this generating function to the sequence A119694. The presented series converges at $x = \frac{1}{8}$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{2^{3n}} C_nF_n = 4\left(1 - \frac{3}{\sqrt{10}}\right).$$

If we let $a = 2$, $b = 1$, and $\mathcal{R}(2,1)$, then the Hadamard product for the Pell numbers P_n and the Catalan numbers will have the expression

$$P_nC_n = [x^n]\left(\frac{1}{2x}\left(1 - \frac{\sqrt{2\sqrt{1 - 8x - 16x^2} + 8x + 6}}{2\sqrt{2}}\right)\right).$$

We add this generating function to the sequence A372216. The presented series converges at $x = \frac{1}{16}$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{2^{4n}} C_n F_n = 8 - 2\sqrt{13 + \sqrt{7}}.$$

Based on the identity (4) and using the generating functions $V_1(x, a, b)$ and $V_3(x, a, b)$ we obtain a generating function for the product $C_n F_{n+j+1}$, where $j \geq 0$

$$V_4(x, a, b) = F_{j+1} V_1(x, a, b) + bF_j V_3(x, a, b). \tag{17}$$

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