Journal of Integer Sequences, Vol. 28 (2025),
Article 25.6.5

Counting Self-Dual Monotone Boolean Functions

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Abstract

Let D_n denote the set of monotone Boolean functions with n variables. The cardinality of D_n, denoted by d_n, is known as the n-th Dedekind number. Elements of D_n can be represented as strings of bits of length 2^n. For each $f \in D_n$, we have the *dual* function $f^* \in D_n$ which is obtained by reversing and negating all bits. An element $f \in D_n$ is *self-dual* if $f = f^*$. Let Λ_n denote the set of all self-dual monotone Boolean functions of n variables and let λ_n denote $|\Lambda_n|$. There is a natural action of the permutation group S_n on the set of Boolean functions by permutation of variables. The sets D_n and Λ_n are closed under this action. We let R_n and Q_n denote the sets of all equivalence classes in D_n and Λ_n, respectively. In this paper, we derive several algorithms for counting self-dual monotone Boolean functions and confirm the known result that λ_9 equals 423,295,099,074,735,261,880. Furthermore, we calculate $|Q_8|$ to be 6,001,501.

1 Introduction

Let B denote the set $\{0, 1\}$ and B^n the set of n-element sequences of B. A Boolean function with n variables is a function from B^n into B. There is the order relation in B (namely: $0 \le 0,\ 0 \le 1,\ 1 \le 1$) and the partial order in B^n: for any two elements: $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in B^n, $x \le y$ if and only if $x_i \le y_i$ for all $1 \le i \le n$. A function $h : B^n \to B$ is monotone if $x \le y$ implies $h(x) \le h(y)$. Let D_n denote the set of monotone functions with n variables. The cardinality of D_n, denoted by d_n, is known as the n-th Dedekind number. We have the partial order in D_n defined by

$$g \le h \quad \text{if and only if} \quad g(x) \le h(x) \quad \text{for all } x \in B^n.$$

We represent an element $f \in D_n$ as a string of bits of length 2^n where the values of $f(x)$ are listed in lexicographical order of x. The two elements of D_0 are represented as 0 and 1. For every $n \geq 1$, the set D_n can be represented as the set of all concatenations $g_0 \cdot g_1$, where $g_0, g_1 \in D_{n-1}$ and $g_0 \leq g_1$. For example, $D_1 = \{00, 01, 11\}$ and $D_2 = \{0000, 0001, 0011, 0101, 0111, 1111\}$.

For each $f \in D_n$, we have the *dual* $f^* \in D_n$, which is obtained by reversing and negating all bits. Formally, the dual of $f \in D_n$ is the function $f^* : B^n \to B$ defined by

$$f^*(x) = (f(x^c))^c,$$

where x^c is the negation of $x \in B^n$ and $(f(x^c))^c$ is the negation of $f(x^c) \in B$. For example, $1111^* = 0000$ and $0001^* = 0111$. An element $f \in D_n$ is *self-dual* if $f = f^*$. For example, 0101 and 0011 are self-dual in D_2.

Let Λ_n be the set of all self-dual monotone Boolean functions of n variables, and let λ_n denote the cardinality of this set. The value λ_n is also known as the n-th Hosten-Morris number (A001206 in the *On-Line Encyclopedia of Integer Sequences*).

The first attempt to determine the values of λ_n was made in 1968 by Riviere [6], who found all values up to λ_5. In 1972, Brouwer and Verbeek provided the values up to λ_7 [3]. The value of λ_8 was determined by Mills and Mills [5] in 1978.

The most recent known term, λ_9, was obtained by Brouwer, Mills, Mills, and Verbeek [2] in 2013. The value of λ_n corresponds to the number of non-dominated coteries on n members [1, Section 1], and also corresponds to the number of maximal linked systems (see Section 2.1 and [2, Section 1]).

Any two Boolean functions are said to be *equivalent* if one can be transformed into the other by a permutation of the input variables (see Section 2.2). Let Q_n denote the set of all equivalence classes in Λ_n and let q_n denote $|Q_n|$. The values of q_n are described by the A008840 OEIS sequence.

In this paper, we derive several algorithms for counting self-dual monotone Boolean functions and we confirm the result of [2] that λ_9 equals 423,295,099,074,735,261,880. Furthermore, employing Burnside's lemma and techniques discussed in [8, 10], we calculate q_8 to be 6,001,501, which is the first-ever calculation of the value q_8. We also show that the two sequences d_n and λ_n intertwine, or more precisely, we have

$$\lambda_1 \leq d_0 \leq \lambda_2 \leq d_1 \leq \cdots \leq \lambda_{n+1} \leq d_n \leq \lambda_{n+2} \leq \cdots$$

and we have the following estimates for the unknown value of λ_{10}

$$d_8 \leq \lambda_{10} \leq d_9.$$

n	λ_n	q_n
0	0	0
1	1	1
2	2	1
3	4	2
4	12	3
5	81	7
6	2,646	30
7	1,422,564	716
8	229,809,982,112	6,001,501
9	423,295,099,074,735,261,880	–

Table 1: Known values of λ_n (A001206) and q_n (A008840).

2 Preliminaries

A *poset (partially ordered set)* $(S, \leq)$ consists of a set S (called the carrier) together with a binary relation (partial order) $\leq$ which is reflexive, transitive, and antisymmetric. For example, B, B^n, and D_n are posets. Given two posets $(S, \leq)$ and $(T, \leq)$, a function $f : S \to T$ is *monotone*, if $x \leq y$ implies $f(x) \leq f(y)$. We let T^S denote the poset of all monotone functions from S to T with the partial order defined by

$$f \leq g \quad \text{if and only if} \quad f(x) \leq g(x) \text{ for all } x \in S.$$

Notice that $D_n = B^{B^n}$ and $B^n = B^{A_n}$, where A_n is the antichain with the carrier $\{1, \ldots, n\}$. A set A is an antichain if any two distinct elements in A are incomparable. In this paper we use the following well-known lemma [10]:

Lemma 1. *The poset D_{n+k} is isomorphic to the poset $D_n^{B^k}$—the poset of monotone functions from B^k to D_n.*

By $\top$ we denote the maximal element in D_n, that is $\top = (1 \ldots 1)$, and by $\perp$ the minimal element in D_n, that is $\perp = (0 \ldots 0)$. For two elements $f, g \in D_n$, by $f|g$ we denote the *bitwise or*; and by $f\&g$ the *bitwise and*. Furthermore, let $\mathrm{re}(f, g)$ denote the number of elements in the interval $[f, g]$, that is $|\{h \in D_n : f \leq h \leq g\}|$. Note that $\mathrm{re}(f, \top) = |\{h \in D_n : f \leq h\}|$ and $\mathrm{re}(\perp, g) = |\{h \in D_n : h \leq g\}|$. For $f \in D_n$, by $\ell(f)$ we denote the number of ones in f, also known as its Hamming weight. For example, $\ell(0000) = 0$ and $\ell(0101) = 2$. The next result is straightforward.

Lemma 2. *For each $f, g \in D_n$, we have*

1. $f^{**} = f$

2. if $f \leq g$ then $g^* \leq f^*$

3. $(f|g)^* = f^* \& g^*$

4. $(f \& g)^* = f^* | g^*$

2.1 Maximal linked system

Let $X = \{1, \ldots, n\}$ and $\mathcal{P}(X)$ be the power set of X. A family $\mathcal{W} \subseteq \mathcal{P}(X)$ is *linked* if for all A and B in $\mathcal{W}$, $A \cap B$ is not empty. A family $\mathcal{U} \subseteq \mathcal{P}(X)$ is a *maximal linked system (mls)* on X if $\mathcal{U}$ is linked and for all $\mathcal{W}$ with $\mathcal{U} \subseteq \mathcal{W} \subseteq \mathcal{P}(X)$, either $\mathcal{W} = \mathcal{U}$ or $\mathcal{W}$ is not linked.

If $\mathcal{U}$ is linked, then $\emptyset \notin \mathcal{U}$ and for each set $A \in \mathcal{P}(X)$, it is not possible that both A and its complement $A^c = X - A$ belong to $\mathcal{U}$.

For $n = 0$, $X = \emptyset$, $\mathcal{P}(X) = \{\emptyset\}$ and we have one mls; namely, the empty family. Notice that the set of self-dual functions Λ_0 is empty and $\lambda_0 = 0$.

Lemma 3. *If $n \geq 1$ and a family $\mathcal{U} \subset \mathcal{P}(X)$ is a mls, then*

(L1) $\mathcal{U}$ is an upset, i.e., if $A \subseteq B$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$.

(L2) For every subset $A \in \mathcal{P}(X)$ exactly one of the two subsets A or A^c is in $\mathcal{U}$.

Proof. (L1) For each $C \in \mathcal{U}$, $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. Hence, either $B \in \mathcal{U}$ or $\mathcal{U}$ is not maximal.

(L2) We have two cases:

Case 1. For every $B \in \mathcal{U}$, $A \cap B \neq \emptyset$. Then either $A \in \mathcal{U}$ or $\mathcal{U}$ is not maximal.

Case 2. There is $B \in \mathcal{U}$ such that $B \cap A = \emptyset$. In this case we have that $B \subseteq A^c$ and, by (L1), $A^c \in \mathcal{U}$. □

Lemma 4. *If $n \geq 1$ and a family $\mathcal{U} \subseteq \mathcal{P}(X)$ satisfies conditions (L1) and (L2), then $\mathcal{U}$ is an mls.*

Proof. First, we prove that $\mathcal{U}$ is linked. Suppose, for a contradiction, that there are two subsets $A, B \in \mathcal{U}$ with $A \cap B = \emptyset$. Then $B \subseteq A^c$ and $A^c \in \mathcal{U}$, a contradiction.

If $\mathcal{U}$ satisfies condition (L2), then exactly half of the elements in $\mathcal{P}(X)$ belong to $\mathcal{U}$. On the other hand, if $|\mathcal{U}| > 2^{n-1}$ then $\mathcal{U}$ contains a pair A, A^c and $\mathcal{U}$ is not linked. □

For $n = 1$, the family $\{\{1\}\}$ is the only mls on $X = \{1\}$.

For $n = 2$, we have two mls: $\{\{1\}, \{1, 2\}\}$ and $\{\{2\}, \{1, 2\}\}$.

Recall that $\mathcal{P}(X)$ is isomorphic to B^n by identifying a subset of X with its characteristic vector and any subset of $\mathcal{P}(X)$ can be represented by a function from B^n to B. For $n = 1$, the mls $\{\{1\}\}$ can be represented as the string 01, which is the only self-dual function in Λ_1. For $n = 2$, the two mls can be represented as 0101 and 0011, and they form the set Λ_2. For $n \geq 1$, the set of mls on $\{1, \ldots, n\}$ can be represented as the set of self-dual functions Λ_n [2, Section 1.1].

2.2 Permutations and equivalence relation

Let S_n denote the set of permutations on $\{1,\ldots,n\}$. Every permutation $\pi \in S_n$ defines a permutation on B^n as follows: for every $x=(x_1,\ldots,x_n) \in B^n$ let $\pi(x)=(x_{\pi(1)},\ldots,x_{\pi(n)})$.

The permutation π also defines the function $f\circ\pi : B^n \to B$ for every function $f : B^n \to B$ by

$$(f\circ\pi)(x) = f(\pi(x)) \text{ for } x \in B^n.$$

By $\sim$ we denote an equivalence relation on D_n. Namely, two functions $f, g \in D_n$ are *equivalent*, $f \sim g$, if there is a permutation $\pi \in S_n$ such that $f = g\circ\pi$. For a function $f \in D_n$, its *equivalence class* is the set $[f] = \{g \in D_n : g \sim f\}$. By R_n we denote the set of equivalence classes in D_n and by r_n we denote $|R_n|$. By $\gamma(f)$ we denote $|[f]|$. For the class $[f]$, its *representative* is its minimal element (according to the lexicographical order in D_n). Sometimes, we identify the class $[f] \in R_n$ with its representative and treat $[f]$ as an element in D_n. Let Q_n denote the set of all equivalence classes in Λ_n, and let q_n denote $|Q_n|$.

The following lemma is straightforward.

Lemma 5. *For every function $f : B^n \to B$ and for every permutation $\pi \in S_n$*

1. *The function $f\circ\pi$ is monotone if f is monotone.*
2. *If $f \in D_n$, then $f^*\circ\pi = (f\circ\pi)^*$.*
3. *If $f \in \Lambda_n$, then for every equivalent $g \in [f]$ we have $g \in \Lambda_n$.*

For $n = 2$, we have $d_2 = 6$ monotone functions,

$$D_2 = \{0000, 0001, 0011, 0101, 0111, 1111\}$$

and $r_2 = 5$ equivalence classes in D_2; namely

$$R_2 = \{\{0000\}, \{0001\}, \{0011, 0101\}, \{0111\}, \{1111\}\}.$$

Furthermore, there are two self-dual functions $\Lambda_2 = \{0011, 0101\}$ and they form one equivalence class. Hence $\lambda_2 = 2$ and $q_2 = 1$.

For $n = 3$, we have $d_3 = 20$ monotone functions and $r_3 = 10$ equivalence classes in D_3. There are four self-dual functions

$$\Lambda_3 = \{01010101, 00110011, 00001111, 00010111\}$$

and they form two equivalence classes:

$$Q_3 = \{\{01010101, 00110011, 00001111\}, \{00010111\}\}.$$

Hence, $\lambda_3 = 4$ and $q_3 = 2$.

3 Counting functions from B to D_n

Let $n \geq 0$. By Lemma 1, the poset D_{n+1} is isomorphic to the poset D_n^B—the poset of monotone functions from $B = \{0, 1\}$ to D_n. Consider a monotone function $H : B \to D_n$. It can be represented as the concatenation

$$H = H(0) \cdot H(1)$$

with $H(0), H(1) \in D_n$ and $H(0) \leq H(1)$. The dual of H is

$$H^* = H(1)^* \cdot H(0)^*.$$

Recall that we identify each function in D_n with the sequence of bits. If $H \in D_{n+1}$ is self-dual then it is of the form $b \cdot b^*$ with $b \in D_n$ and $b \leq b^*$. And vice versa, if $b \in D_n$ and $b \leq b^*$, then the concatenation $b \cdot b^*$ is self-dual in D_{n+1}. Therefore, we have proved the following theorem.

Theorem 6. *For every $n \geq 0$, the number of self-dual functions λ_{n+1} is equal to the number of elements $b \in D_n$ that satisfy $b \leq b^*$. In other words*

$$\lambda_{n+1} = \sum_{\substack{b \in D_n \\ b \leq b^*}} 1.$$

Furthermore,

$$\lambda_{n+1} = \sum_{\substack{b \in R_n \\ b \leq b^*}} \gamma(b).$$

Here we identify each class $b \in R_n$ with its representative.

The following corollary is presented, in a different form, in [11]

Corollary 7. *For every $n \geq 0$, we have $\lambda_{n+1} \leq d_n$.*

The next result is straightforward.

Lemma 8. *Let b be a function in D_n. Then*

- $b \leq b^*$*, only if* $\ell(b) \leq 2^{n-1}$.
- *if* $\ell(b) = 2^{n-1}$ *and* $b \leq b^*$*, then* $b = b^*$*, and* b *is self-dual in* D_n.
- *if* b *is self-dual, then* $\ell(b) = 2^{n-1}$.
- $\ell(b^*) = 2^n - \ell(b)$*, hence,*

$$|\{f \in D_n : \ell(f) < 2^{n-1}\}| = |\{f \in D_n : \ell(f) > 2^{n-1}\}|.$$

As a corollary we have the following theorem.

Theorem 9.

$$\lambda_{n+1} = \lambda_n + \sum_{\substack{b \in R_n \\ \ell(b) < 2^{n-1} \\ b \leq b^*}} \gamma(b).$$

$$\lambda_{n+1} \leq \lambda_n + \frac{1}{2}(d_n - \lambda_n) = \frac{1}{2}(d_n + \lambda_n).$$

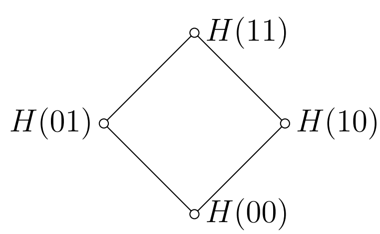
The second part of the theorem is presented, in a different form, in [11]. Notice that $\lambda_{n+1} = \frac{1}{2}(d_n + \lambda_n)$ for each $n \leq 3$.

4 Counting functions from B^2 to D_n

Let $n \geq 0$. By Lemma 1, the poset D_{n+2} is isomorphic to the poset $D_n^{B^2}$—the poset of monotone functions from $B^2 = \{00, 01, 10, 11\}$ to D_n. Consider a monotone function $H : B^2 \to D_n$. It can be represented as the concatenation

$$H(00) \cdot H(01) \cdot H(10) \cdot H(11)$$

and as the graph below.



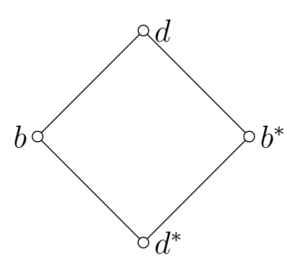
where the vertices represent the set of values $H(B^2)$ and the edges represent the partial order:

$$H(00) \leq H(01) \leq H(11) \text{ and } H(00) \leq H(10) \leq H(11).$$

Recall that we identify each function in D_n with the sequence of bits. The dual of H is represented by

$$H(11)^* \cdot H(10)^* \cdot H(01)^* \cdot H(00)^*.$$

If H is self-dual then $H(00) = H(11)^*$, $H(01) = H(10)^*$, $H(10) = H(01)^*$, and $H(11) = H(00)^*$ and we can represent H as the graph



where d denotes $H(11)$ and b denotes $H(01)$. Furthermore, d^* denotes $H(00)$ and b^* denotes $H(10)$. If H is self-dual then it is of the form $d^* \cdot b \cdot b^* \cdot d$ with $b, d \in D_n$ and $d \geq b|b^*$. And vice versa, if $b, d \in D_n$, and $d \geq b|b^*$, then, by Lemma 2, $d^* \leq b\&b^*$ and the concatenation $d^* \cdot b \cdot b^* \cdot d$ is self-dual in D_{n+2}. Hence, we have proved the following theorem.

Theorem 10. *For every $n \geq 0$, the number of self-dual functions λ_{n+2} is equal to the number of pairs $b, d \in D_n$ which satisfy condition $d \geq b|b^*$. In other words*

$$\lambda_{n+2} = \sum_{b \in D_n} \operatorname{re}(b|b^*, \top).$$

Furthermore, Lemma 5 implies

$$\lambda_{n+2} = \sum_{b \in R_n} \gamma(b) \cdot \operatorname{re}(b|b^*, \top).$$

Here we identify each class $b \in R_n$ with its representative.

Corollary 11. *For every $n \geq 0$, we have $\lambda_{n+2} \geq d_n$.*

By Colloraries 7 and 11, we have that

$$\lambda_1 \leq d_0 \leq \lambda_2 \leq d_1 \leq \cdots \leq \lambda_{n+1} \leq d_n \leq \lambda_{n+2} \leq \cdots$$

and we have the following estimates for the unknown value of λ_{10}

$$d_8 \leq \lambda_{10} \leq d_9.$$

5 Counting functions from B^4 to D_n

Let $n \geq 0$. By Lemma 1, the poset D_{n+4} is isomorphic to the poset $D_n^{B^4}$—the set of monotone functions from $B^4 = \{0000, 0001, \ldots, 1111\}$ to D_n. Consider a monotone function $H\colon B^4 \to D_n$. It can be represented as the concatenation

$$\begin{aligned} &H(0000) \cdot H(0001) \cdot H(0010) \cdot H(0011) \cdot H(0100) \cdot H(0101) \cdot H(0110) \cdot H(0111) \cdot \\ &H(1000) \cdot H(1001) \cdot H(1010) \cdot H(1011) \cdot H(1100) \cdot H(1101) \cdot H(1110) \cdot H(1111). \end{aligned}$$

and its dual as

$$\begin{aligned} &H(1111)^* \cdot H(1110)^* \cdot H(1101)^* \cdot H(1100)^* \cdot H(1011)^* \cdot H(1010)^* \cdot H(1001)^* \cdot H(1000)^* \cdot \\ &H(0111)^* \cdot H(0110)^* \cdot H(0101)^* \cdot H(0100)^* \cdot H(0011)^* \cdot H(0010)^* \cdot H(0001)^* \cdot H(0000)^*. \end{aligned}$$

If H is self-dual, then

$$\begin{aligned} H(0000) &= H(1111)^*, & H(0001) &= H(1110)^*, \\ H(0010) &= H(1101)^*, & H(0011) &= H(1100)^*, \\ H(0100) &= H(1011)^*, & H(0101) &= H(1010)^*, \\ H(0110) &= H(1001)^*, & H(0111) &= H(1000)^*. \end{aligned}$$

Recall that we identify each function in D_n with the sequence of bits. Similarly, as in Section 4 we can represent H as a graph. See Figure 1 where h represents $H(1111)$, d represents $H(0111)$, e represents $H(1011)$, f represents $H(1101)$, g represents $H(1110)$, $\ldots$, and h^* represents $H(0000)$.

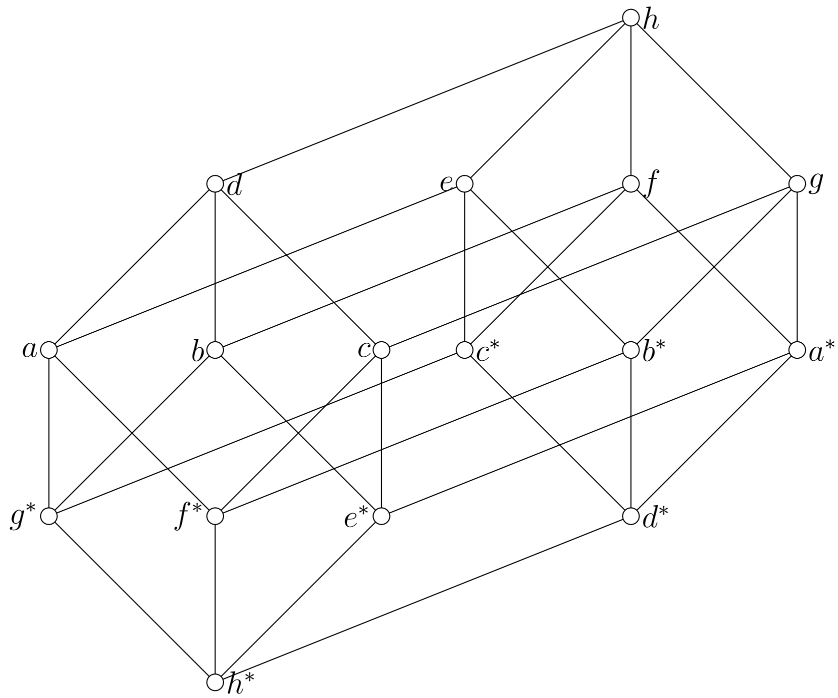


Figure 1: Structure of $H : B^4 \to D_n$ if H is self-dual. The graph represents the hypercube B^4. The label at a vertex x represents the value $H(x) \in D_n$.

Theorem 12. *For each $a, b, c \in D_n$,*
for each $h \in D_n$ such that $h \geq a|b|c|a^*|b^*|c^*$,
for each $d, e, f, g \in D_n$ such that

$$\begin{aligned} a|b|c \leq d \leq h, \\ a|b^*|c^* \leq e \leq h, \\ b|a^*|c^* \leq f \leq h, \\ c|a^*|b^* \leq g \leq h, \end{aligned}$$

the concatenation

$$h^* \cdot g^* \cdot f^* \cdot a \cdot e^* \cdot b \cdot c \cdot d \cdot d^* \cdot c^* \cdot b^* \cdot e \cdot a^* \cdot f \cdot g \cdot h$$

represents a self-dual function in D_{n+4}; see Figure 1. And vice versa, each self-dual function in D_{n+4} is of the above form.

The proof is straightforward.

Theorem 13. *The number of self-dual functions*

$$\lambda_{n+4} = \sum_{a,b,c \in D_n} \sum_{\substack{h \in D_n \\ h \geq (a|b|c|a^*|b^*|c^*)}} \mathrm{re}(a|b|c,h) \cdot \mathrm{re}(a|b^*|c^*,h) \cdot \mathrm{re}(b|a^*|c^*,h) \cdot \mathrm{re}(c|a^*|b^*,h).$$

Theorem 14. *For each $h \in D_n$ such that $h \geq h^*$,*
for each $a,b,c \in D_n$, $h^* \leq a,b,c \leq h$,
for each $d,e,f,g \in D_n$ such that

$$\begin{aligned} a|b|c &\leq d \leq h, \\ a|b^*|c^* &\leq e \leq h, \\ b|a^*|c^* &\leq f \leq h, \\ c|a^*|b^* &\leq g \leq h, \end{aligned}$$

the concatenation

$$h^* \cdot g^* \cdot f^* \cdot a \cdot e^* \cdot b \cdot c \cdot d \cdot d^* \cdot c^* \cdot b^* \cdot e \cdot a^* \cdot f \cdot g \cdot h$$

represents a self-dual function in D_{n+4}; see Figure 1. And vice versa, each self-dual function in D_{n+4} is of the above form.

Let

$$F(h) = \sum_{\substack{a,b,c \in D_n \\ h \geq a,b,c \geq h^*}} \mathrm{re}(a|b|c,h) \cdot \mathrm{re}(a|b^*|c^*,h) \cdot \mathrm{re}(b|a^*|c^*,h) \cdot \mathrm{re}(c|a^*|b^*,h).$$

Observe that $F(h)$ is the number of self-dual functions $H \in D_n^{B^4}$, with $H(0000) = h^*$ and $H(1111) = h$. Therefore,

$$\lambda_{n+4} = \sum_{\substack{h \in D_n \\ h^* \leq h}} F(h).$$

Furthermore, Lemma 5 implies that for any two elements $h_1 \sim h_2$ we have $F(h_1) = F(h_2)$. Hence, we have

$$\lambda_{n+4} = \sum_{\substack{h \in R_n \\ h \geq h^*}} \gamma(h) \cdot F(h).$$

Here again we identify the class $h \in R_n$ with its representative. Similarly as in Lemma 8 we can observe that $h \geq h^*$, only if $\ell(h) \geq 2^{n-1}$. Furthermore, if $\ell(h) = 2^{n-1}$ and $h \geq h^*$, then $h = h^*$, and we have only one self-dual function $H \in D_n^{B^4}$ with $H(0000) = H(1111) = h$. Hence,

$$\lambda_{n+4} = \lambda_n + \sum_{\substack{h \in R_n \\ h^* \leq h \\ \ell(h) > 2^{n-1}}} \gamma(h) \cdot F(h). \tag{1}$$

6 Implementation

In this section we present three algorithms based on results from the previous sections. We implemented the algorithms in Rust and ran them on a 32-thread Xeon CPU.

Algorithm 1 Calculation of λ_{n+2}

Input: R_n (each class is represented by its minimal element) with $re(x, \top)$ for all $x \in R_n$
Output: $s = \lambda_{n+2}$

Initialize $s = 0$,
for all $b \in R_n$ **do**
    $s = s + \mathrm{re}(b|b^*, \top) \cdot \gamma(b)$
end for

Algorithm 1 is based on Theorem 10. After loading the preprocessed data into main memory, λ_9 was computed in 15 seconds. However, preprocessing (the calculation of R_7 and its intervals) took approximately 2.5 hours.

Algorithm 2 Calculation of λ_{n+4} based on Theorem 13

Input: D_n; R_n (each class is represented by its minimal element); $re(x, y)$ for all $(x, y) \in D_n \times D_n$
Output: $s = \lambda_{n+4}$

Initialize $s = 0$,
for all $a \in R_n$ **do**
    for all $b \in D_n$ **do**
        for all $c \in D_n$ **do**
            for all $h \in D_n, h \geq (a|b|c|a^*|b^*|c^*)$ **do**
                $s = s + \mathrm{re}(a|b|c, h) \cdot \mathrm{re}(a|b^*|c^*, h) \cdot \mathrm{re}(b|a^*|c^*, h) \cdot \mathrm{re}(c|a^*|b^*, h) \cdot \gamma(a)$
            end for
        end for
    end for
end for

Using our implementation of the algorithm, we calculated λ_9 in 76 seconds, and the preprocessing was almost instantaneous. The calculation of λ_9 using our implementation of the algorithm lasted approximately 25 minutes. In all cases, we have obtained the following value:

$$\lambda_9 = 423295099074735261880,$$

which confirms the result of Brouwer et al. [2].

Algorithm 3 Calculation of λ_{n+4} based on Equation 1

Input: D_n; R_n (each class is represented by its minimal element); $\mathrm{re}(x,y)$ for all $(x,y) \in D_n \times D_n$

Output: $s = \lambda_{n+4}$

Initialize $s = \lambda_n$,
for all $h \in R_n, h^* \leq h, \ell(h) > 2^{n-1}$ **do**
    for all $a \in D_n, h^* \leq a \leq h$ **do**
        for all $b \in D_n, h^* \leq b \leq h$ **do**
            for all $c \in D_n, h^* \leq c \leq h$ **do**
                $s = s + \mathrm{re}(a|b|c, h) \cdot \mathrm{re}(a|b^*|c^*, h) \cdot \mathrm{re}(b|a^*|c^*, h) \cdot \mathrm{re}(c|a^*|b^*, h) \cdot \gamma(h)$
            end for
        end for
    end for
end for

Algorithms 2 and 3 differ in the order in which values of the functions are chosen. It should be noted that Algorithm 2 works much faster than Algorithm 3 due to a significantly smaller number of iterations, requiring only 76 seconds compared to 25 minutes of computation time.

7 Calculation of q_n

The number of inequivalent self-dual monotone Boolean functions (q_n) is listed on the OEIS A008840 sequence. In order to calculate q_n for $n \leq 7$, we can use the following simple approach:

$$q_n = \sum_{\substack{a \in R_n \\ a = a^*}} 1.$$

For $n = 8$, this direct approach becomes computationally infeasible. Instead, we apply Burnside's lemma to our specific problem. Recall that each permutation $\pi \in S_n$ can be represented as a product of disjoint cycles. The *cycle type* of π is defined as the tuple of lengths of its disjoint cycles arranged in increasing order. For example, the type of permutation $\pi = (12)(34)(567)$ is $(2,2,3)$, and its total length is 7. In the sequel, we use a list of cycle types in S_n and we represent each cycle type by its index in the list, see the tables in Section 7.2.

To calculate q_8 we use the approach developed in [8, 9, 10].

$$q_n = \frac{1}{n!} \sum_{i=1}^{k} \mu_i \cdot |\Phi(\pi_i, \Lambda_n)| \tag{2}$$

where

- q_n is the number of equivalence classes in Λ_n,

- k is the number of different cycle types in S_n,
- π_i is a representative permutation of cycle type i,
- $\Phi(\pi_i, \Lambda_n)$ is the set of all elements in Λ_n which are fixed under π_i,
- μ_i number of permutations $\pi \in S_n$ with cycle type i.

For $n = 1$, we have $\lambda_1 = q_1 = 1$. For $n = 2$, we have two permutations: the identity e with $|\Phi(e, \Lambda_2)| = |\Lambda_2| = 2$, and the inversion $\pi = (12)$ with three cycles when acting on B^2, namely: $C_1 = (00)$, $C_2 = (01, 10)$, and $C_3 = (11)$. The two elements 01 and 10 form a cycle, hence, if a function $f \in D_n$ is a fixed point of π, then $f(01) = f(10)$. On the other hand, if f is self-dual, then it represents an mls on $\{1, 2\}$, and $f(01) \neq f(10)$, because 01 and 10 represent subsets $\{1\}$ and $\{2\}$ in $\{1, 2\}$ which are the complements of each other. Thus, the set of fixed points $\Phi((12), \Lambda_2) = \emptyset$. By Burnside's lemma, we have

$$q_2 = \frac{1}{2}(|\Phi(e, \Lambda_2)| + |\Phi((12), \Lambda_2)|) = \frac{1}{2}(2 + 0) = 1.$$

Indeed, there is one equivalence class in Λ_2; namely, $\{0101, 0011\}$. We have just shown that $\Phi((12), \Lambda_2) = \emptyset$. Similarly, we can show that $\Phi(\pi, \Lambda_8) = \emptyset$, if $\pi = (12)(34)(56)(78)$. Indeed, for π, the two elements $01010101,\ 10101010 \in B^8$ form a cycle. Hence, if $f \in D_8$ is a fixed point of π, then $f(01010101) = f(10101010)$. On the other side, if f is self-dual, then $f(01010101) \neq f(10101010)$, because 101010101 and 01010101 represent subsets of $\{1, \ldots, 8\}$ which are the complements of each other.

Lemma 15. *Suppose that n is even, and a permutation π, when acting on $\{1, \ldots, n\}$, is a product of disjoint cycles of even length. Then $\Phi(\pi, \Lambda_n) = \emptyset$.*

Proof. If the permutation π is a product of disjoint cycles of even length, then there exist two elements $x, y \in B^n$ such that

- x and y represent subsets of $\{1, \ldots, n\}$ which are the complements of each other.
- $\pi(x) = y$ and $\pi(y) = x$, so x, y form a cycle in B^n.

Hence, if $f \in D_n$ is a fixed point of π, then $f(x) = f(y)$. On the other side, if f is self-dual, then $f(x) \neq f(y)$. $\square$

Corollary 16. *We have $\Phi(\pi, \Lambda_8) = \emptyset$ for each of the following permutations:* (12345678)*,* $(12)(345678)$*,* $(1234)(5678)$*,* $(12)(34)(5678)$*, and* $(12)(34)(56)(78)$*.*

For $n = 3$, we have three cycle types:

- the identity e with $|\Phi(e, \Lambda_3)| = |\Lambda_3| = 4$;
- three inversions, with $|\Phi((12), \Lambda_3)| = 2$; and

- two cycles of length 3 with $|\Phi((123), \Lambda_3)| = 1$.

By Burnside's lemma, we have

$$q_3 = \frac{1}{6}(4 + 3 \cdot 2 + 2 \cdot 1) = 2.$$

Notice, that the element MAJ = 00010111 $\in D_3$ is self-dual, and is a fixed point for every permutation $\pi \in S_3$. Hence, for every $\pi \in S_3$, $\Phi(\pi, \Lambda_3) \neq \emptyset$. Similarly, we can show the following lemma:

Lemma 17. *For each odd n and each permutation $\pi \in S_n$, we have $\Phi(\pi, \Lambda_n) \neq \emptyset$.*

Proof. Consider the function MAJ $\in D_n$, which returns MAJ$(x) = 1$ if and only if $\ell(x) > n/2$. The function MAJ is self-dual, and is a fixed point for every permutation $\pi \in S_n$. $\square$

7.1 Algorithms counting fixed points in Λ_n

In order to count or generate fixed points of permutations in Λ_n we use Lemma 15 and two algorithms.

Algorithm 4 Generation of $\Phi(\pi, \Lambda_n)$

Input: $\Phi(\pi, D_n)$
Output: $S = \Phi(\pi, \Lambda_n)$

Initialize $S = \emptyset$
for all $b \in \Phi(\pi, D_n)$ **do**
  if $b = b^*$ **then**
    Add b to S
  end if
end for

Algorithm 4 simply runs through the set of fixed points $\Phi(\pi, D_n)$ and selects self-dual functions. For example, there are five fixed points in

$$\Phi((123), D_3) = \{00000000, 00000001, 00010111, 01111111, 11111111\}$$

and only one of them is self-dual; namely, 00010111, so $|\Phi((123), \Lambda_3)| = 1$.

Algorithm 5 Calculation of $|\Phi(\pi, \Lambda_{n+2})|$

Input: $\Phi(\pi, D_n)$
Output: $s = |\Phi(\pi, \Lambda_{n+2})|$

Initialize $s = 0$,
for all $b \in \Phi(\pi, D_n)$ **do**
  Calculate $up = |\{h \in \Phi(\pi, D_n) : h \geq (b|b^*)\}|$
  $s = s + up$
end for

Algorithm 5 is based on the following facts. Consider a permutation π acting on B^n and on D_n. We can say that π also acts on B^{n+2} and on D_{n+2}. By [10, Lemma 6], $\Phi(\pi, D_{n+2}) = \Phi(\pi, D_n)^{B^2}$. Every function $F \in \Phi(\pi, D_n)^{B^2}$ can be represented as the concatenation

$$F = F(00) \cdot F(01) \cdot F(10) \cdot F(11),$$

where $F(00)$, $F(01)$, $F(10)$, $F(11) \in \Phi(\pi, D_n)$, and

$$F(00) \leq F(01), F(10) \leq F(11).$$

The dual of F can be represented as

$$F^* = F(11)^* \cdot F(10)^* \cdot F(01)^* \cdot F(00)^*.$$

If F is self-dual, then it is of the form

$$dbb^*d^*,$$

where $b, d \in \Phi(\pi, D_n)$ and $d^* \geq b|b^*$. Notice that this implies that $d \leq b\&b^*$. On the other hand, if $b, d \in \Phi(\pi, D_n)$ and $d^* \geq b|b^*$, then $dbb^*d^* \in \Phi(\pi, \Lambda_{n+2})$.

7.2 Result tables

In this section we present three tables which contain the values of $|\Phi(\pi_i, \Lambda_n)|$, for $n \in \{6, 7, 8\}$ and all permutations.

i	π_i	μ_i	$\|\Phi(\pi_i, \Lambda_6)\|$
1	(1)	1	2646
2	(12)	15	372
3	(123)	40	54
4	(1234)	90	18
5	(12345)	144	6
6	(123456)	120	0
7	(12)(34)	45	130
8	(12)(345)	120	18
9	(12)(3456)	90	0
10	(123)(456)	40	18
11	(12)(34)(56)	15	0

$$q_6 = \frac{1}{720} \sum_{i=1}^{11} \mu_i \cdot |\Phi(\pi_i, \Lambda_6)| = \frac{21600}{720} = 30.$$

Table 2: Values of $|\Phi(\pi_i, \Lambda_6)|$ and calculation of q_6.

i	π_i	μ_i	$\lvert\Phi(\pi_i, \Lambda_7)\rvert$
1	(1)	1	1422564
2	(12)	21	43556
3	(123)	70	1332
4	(1234)	210	216
5	(12345)	504	34
6	(123456)	840	12
7	(1234567)	720	3
8	(12)(34)	105	7212
9	(12)(345)	420	218
10	(12)(3456)	630	76
11	(12)(34567)	504	6
12	(123)(456)	280	210
13	(123)(4567)	420	6
14	(12)(34)(56)	105	1284
15	(12)(34)(567)	210	36

$$q_7 = \frac{1}{5040} \sum_{i=1}^{15} \mu_i \cdot |\Phi(\pi_i, \Lambda_7)| = \frac{3608640}{5040} = 716.$$

Table 3: Values of $|\Phi(\pi_i, \Lambda_7)|$ and calculation of q_7.

i	π_i	μ_i	$\lvert\Phi(\pi_i, \Lambda_8)\rvert$
1	(1)	1	229809982112
2	(12)	28	300991356
3	(123)	112	476120
4	(1234)	420	18984
5	(12345)	1344	662
6	(123456)	3360	296
7	(1234567)	5760	46
8	(12345678)	5040	0
9	(12)(34)	210	12716048
10	(12)(345)	1120	18384
11	(12)(3456)	2520	7952
12	(12)(34567)	4032	116
13	(12)(345678)	3360	0
14	(123)(456)	1120	21020
15	(123)(4567)	3360	120
16	(123)(45678)	2688	20
17	(1234)(5678)	1260	0
18	(12)(34)(56)	420	2230724
19	(12)(34)(567)	1680	3152
20	(12)(34)(5678)	1260	0
21	(12)(345)(678)	1120	1488
22	(12)(34)(56)(78)	105	0

$$q_8 = \frac{1}{40320} \sum_{i=1}^{22} \mu_i \cdot |\Phi(\pi_i, \Lambda_8)| = \frac{241980137280}{40320} = 6001501.$$

Table 4: Values of $|\Phi(\pi_i, \Lambda_8)|$ and calculation of q_8.

References

[1] J. C. Bioch and T. Ibaraki, Generating and approximating nondominated coteries, *IEEE Trans. Parallel Distrib. Syst.* **6** (1995), 905–914.

[2] A. E. Brouwer, C. F. Mills, W. H. Mills, and A. Verbeek, Counting families of mutually intersecting sets, *Electron. J. Combin.* **20** (2013), Article P8.

[3] A. E. Brouwer and A. Verbeek, Counting families of mutually intersecting sets, Report ZN 41, Mathematical Centre, Amsterdam, 1972.

[4] D. E. Loeb and A. R. Conway, Voting fairly: transitive maximal intersecting families of sets, *J. Combin. Theory Ser. A* **91** (2000), 386–410.

[5] C. F. Mills and W. H. Mills, The calculation of $\lambda(8)$, preprint, 1979.

[6] N. M. Riviere, Recursive formulas on free distributive lattices, *J. Combin. Theory* **5** (1968), 229–234.

[7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Available at `https://oeis.org`.

[8] B. Pawelski, On the number of inequivalent monotone Boolean functions of 8 variables, *J. Integer Sequences* **25** (2022), Article 22.7.7.

[9] B. Pawelski, On the number of inequivalent monotone Boolean functions of 9 variables, *IEEE Trans. Inform. Theory* **70** (2024), 5358–5364.

[10] A. Szepietowski, Fixes of permutations acting on monotone Boolean functions, *J. Integer Sequences* **25** (2022), Article 22.9.6.

[11] M. Timotijević, Note on combinatorial structure of self-dual simplicial complexes, *Mat. Vesnik* **71** (2019), 104–122.

2020 *Mathematics Subject Classification*: Primary 06E30.

Keywords: monotone Boolean function, self-dual monotone Boolean function, Dedekind number, Hosten-Morris number.

(Concerned with sequences A001206 and A008840.)

Received February 26 2024; revised versions received December 11 2024; June 12 2025; October 22 2025. Published in *Journal of Integer Sequences*, October 31 2025.
