Journal of Integer Sequences, Vol. 28 (2025),
Article 25.4.8

Multiplicative Functions Additive on the Sums of Two Positive Triangular Numbers

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Abstract

Let T_2 be the set of sums of two positive triangular numbers. If a multiplicative function f satisfies $f(a+b) = f(a) + f(b)$ for all $a, b \in T_2$, then f is the identity function provided that $f(n) \neq 0$ for some $n \neq 1, 3, 5$.

1 Introduction

A function $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m) f(n)$ for all relatively prime integers m and n.

In 1992 Spiro showed that if a multiplicative function f satisfies the condition

$$f(p+q) = f(p) + f(q)$$

for all primes p, q and $f(p_0) \neq 0$ for some prime p_0, then f is the identity function [11]. She called this property *additive uniqueness*.

Since her paper, many mathematicians have studied the additive uniqueness of various sets. Fang, Dubickas, and Šarka [4, 3] obtained the same result with the extended condition

$$f(p_1 + p_2 + \cdots + p_k) = f(p_1) + f(p_2) + \cdots + f(p_k)$$

with $k \geq 3$.

Chung [1] found all multiplicative functions satisfying

$$f(m^2+n^2)=f(m^2)+f(n^2)$$

for all $m,n\in\mathbb{N}$. In this case, the function f is not determined uniquely. Later, the author [7] showed that if the condition is changed to

$$f(a_1^2+a_2^2+\cdots+a_k^2)=f(a_1^2)+f(a_2^2)+\cdots+f(a_k^2)$$

with $k\geq 3$, then f is the identity function.

Let T be the set of positive triangular numbers. That is,

$$T=\left\{\frac{n(n+1)}{2}\,|\,n=1,2,\dots\right\}=\{1,3,6,10,15,21,28,36,45,\dots\}.$$

Chung and Phong [2] showed that T is an additive uniqueness set for multiplicative functions. The author and his colleagues [6] showed that the set

$$P=\left\{\frac{n(3n-1)}{2}\,|\,n\in\mathbb{Z},n\neq 0\right\}=\{1,2,5,7,12,15,22,26,35,40,\dots\}$$

of positive generalized pentagonal numbers is an additive uniqueness set for multiplicative functions. Also, the condition for the additive uniqueness of the sets T and P can be extended as was done for the set of primes and the set of squares [8, 5, 10].

The author [9] found all multiplicative functions f satisfying the condition

$$f(a^2+b^2+c^2+d^2)=f(a^2+b^2)+f(c^2+d^2)$$

for all positive integers a, b, c, and d.

Now we consider the problem of replacing squares with triangular numbers in the previous condition. Let T_2 be the set of sums of two elements of T. That is,

$$T_2=\{2,4,6,7,9,11,12,13,16,18,20,21,22,24,25,27,29,30,\dots\}.$$

Then the following holds.

Theorem 1. *If a multiplicative function f satisfies the condition*

$$f(a+b)=f(a)+f(b)$$

for all $a,b\in T_2$, then f is one of the following:

1. *$f(n)=n$, the identity function,*
2. *$f(n)=0$ for all $n\neq 1,3,5$ and $f(3)f(5)=0$.*

2 Proof

We use induction. First, we compute some values of $f(n)$. Second, we represent $f(n)$ as a sum of $f(a)+f(b)$ with $a, b \in T_2$ for sufficiently large n.

Lemma 2. *If $f(2) \neq 0$, then $f(n) = n$ for $n \leq 21$.*

Proof. It is enough to consider powers of primes. Note that $f(4) = f(2) + f(2) = 2f(2)$. Since

$$\begin{aligned} f(2)\, f(3) &= f(6) \\ &= f(2+4) = f(2) + f(4) = 3f(2), \end{aligned}$$

we obtain that $f(3) = 3$.

Then we can obtain $f(7) = 7$, since

$$\begin{aligned} f(2)\, f(7) &= f(14) \\ &= f(2+12) = f(2) + f(3)\, f(4) = 7\, f(2). \end{aligned}$$

Also, we obtain $f(2) = 2$ and $f(4) = 4$, since

$$\begin{aligned} f(2)\, f(7) &= f(14) \\ &= f(7) + f(7) = 2\, f(7). \end{aligned}$$

Then $f(5) = 5$ from $f(2)\, f(5) = f(2)\, f(3) + f(4)$ and

$$\begin{aligned} f(8) &= f(4) + f(4) = 8, & f(9) &= f(2) + f(7) = 9, \\ f(11) &= f(4) + f(7) = 11, & f(13) &= f(4) + f(9) = 13, \\ f(16) &= f(7) + f(9) = 16, & f(17) &= f(4) + f(13) = 17, \\ f(19) &= f(7) + f(3)\, f(4) = 19. \end{aligned}$$

Hence $f(n) = n$ for $n \leq 21$. □

Lemma 3. *If $f(2) = 0$ and $f(3) \neq 0$, then $f(n) = 0$ for all $4 \leq n \leq 21$.*

Proof. Cleary, $f(4) = f(2) + f(2) = 0$ and $f(8) = f(4) + f(4) = 0$. We obtain $f(7) = 0$, since

$$f(7) + f(7) = f(14) = f(2)\, f(7) = 0.$$

Then $f(11) = f(9) = 0$, since

$$f(4) + f(7) = f(11) = f(2) + f(9).$$

Also, $f(5) = f(13) = 0$, since

$$f(3)\, f(5) = f(15) = f(2)\, f(3) + f(9) = f(2) + f(13).$$

We obtain $f(16) = 0$, since
$$f(16) = f(7) + f(9).$$
Then $f(17) = f(19) = 0$, since
$$f(17) = f(4) + f(13) \quad \text{and} \quad f(19) = f(7) + f(3)\,f(4).$$
Hence $f(n) = 0$ for all $4 \leq n \leq 21$. □

Lemma 4. *If $f(2) = 0$ and $f(5) \neq 0$, then $f(n) = 0$ for all $2 \leq n \leq 21$ except $n = 5$.*

Proof. The proof is almost identical to the proof of Lemma 3.

We obtain $f(2) = f(4) = f(8) = 0$, since
$$f(4) = f(2) + f(2) \quad \text{and} \quad f(8) = f(4) + f(4).$$
Also, $f(7) = f(9) = f(11) = 0$, since
$$f(7) + f(7) = f(2)\,f(7) \quad \text{and} \quad f(4) + f(7) = f(11) = f(2) + f(9).$$
Then $f(3) = f(13) = 0$, since
$$f(3)\,f(5) = f(15) = f(2)\,f(3) + f(9) = f(2) + f(13).$$
We obtain $f(16) = f(17) = f(19) = 0$, since
$$f(16) = f(7) + f(9), \quad f(17) = f(4) + f(13) \quad \text{and} \quad f(19) = f(7) + f(3)\,f(4).$$
Hence $f(n) = 0$ for all $2 \leq n \leq 21$ and $n \neq 5$. □

Lemma 5. *If $f(2) = f(3) = f(5) = 0$, then $f(n) = 0$ for all $2 \leq n \leq 21$.*

Proof. We can derive $f(13) = 0$ from $f(3)\,f(5) = f(2) + f(13)$.

It is obvious that $f(n) = 0$ for other $4 \leq n \leq 21$ by the same methods as the proofs of Lemmas 3 and 4. □

Now we prove Theorem 1. Note that $21 \in T$ and
$$21 = 6 + 15 = 1 + 10 + 10.$$
We know that every positive number can be represented as a sum of three triangular numbers including 0 by Fermat's polygonal number theorem or Gauss's Eureka theorem. So, if $n > 21$, then n is a sum of four *positive* triangular numbers. Because, if $n - 21 = a + b + c$ for some triangular numbers $a \geq b \geq c \geq 0$, then
$$n = \begin{cases} a + b + c + 21 & \text{if } a \geq b \geq c \geq 1, \\ a + b + 10 + 11 & \text{if } a \geq b \geq 1 \text{ and } c = 0, \\ a + 1 + 10 + 10 & \text{if } a \geq 1 \text{ and } b = c = 0. \end{cases}$$
Hence, Theorem 1 can be proved by induction according to the values of $f(2)$, $f(3)$, and $f(5)$.

Since $3 \notin T_2$, information about $f(3)$ is always from $f(3)\,f(n)$ with $3 \nmid n$. Thus, when $f(2) = 0$ and $f(3) \neq 0$, we can set $f(3)$ to an arbitrary number. So can $f(5)$, if $f(2) = 0$ and $f(5) \neq 0$.

3 Acknowledgment

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (NRF-2021R1A2C1092930).

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2020 *Mathematics Subject Classification*: Primary 11N64.
Keywords: additive uniqueness, multiplicative function, sum of two positive triangular numbers.

(Concerned with sequence A051533.)

Received December 31 2024; revised versions received August 14 2025; September 1 2025. Published in *Journal of Integer Sequences*, September 8 2025.

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