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Identities for Permutations with Fixed Points

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Abstract

We present identities for permutations with fixed points. The formulas are based on successive derivations or integrations of the determinant of a particular matrix.

1 Introduction

The number of permutations with fixed points can be obtained by induction [1]. It is of interest for combinatorial interpretations of relations for the number of derangements [3]. For instance, the number of derangements can be interpreted as the sum of the values of the largest fixed points of all non-derangements of length $n-1$ [2]. In the same paper, Deutsch and Elizalde [2] showed that the analogous sum for the smallest fixed points equals the number of permutations of length n with at least two fixed points. In this short article, we derive identities for the fixed-point statistics over the symmetric group.

Definition 1. Let us consider, for $x \in \mathbb{R}$ and a natural number $n \geq 2$, the $n \times n$ matrix

$$M_x = \begin{bmatrix} x & 1 & \cdots & 1 & 1 \\ 1 & x & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & x & 1 \\ 1 & 1 & \cdots & 1 & x \end{bmatrix}.$$

The characteristic polynomial of the matrix $-M_0$ is

$$\chi_{-M_0}(X) = \det(XI_n + M_0).$$

Hence, since $\chi_{-M_0}(1) = 0$, it follows that 1 is an eigenvalue, the corresponding eigenspace being

$$E_1 = \ker(-M_0 - I_n) = \ker(M_0 + I_n).$$

We have

$$x = (x_1, \cdots, x_n) \in E_1 \iff x_1 + x_2 + \cdots + x_n = 0,$$

which is a hyperplane of dimension $n-1$. This implies that the multiplicity of 1 is $n-1$. In addition, the sum of all eigenvalues is equal to the trace of the matrix $-M_0$, i.e., 0, and therefore $1-n$ is also an eigenvalue. We have

$$\begin{aligned} x = (x_1, \cdots, x_n) \in E_{1-n} \quad &\iff \quad \forall i \in [1,n], \quad (n-1)x_i = \sum_{k=1, k\neq i}^{n} x_k \\ &\iff \quad x_1 = x_2 = \cdots = x_n \\ &\iff \quad x = \mathrm{span}\{(1,1,\cdots,1)\}, \end{aligned}$$

and the multiplicity of $1-n$ is 1. Thus

$$\begin{aligned} \det\, M_x &= \det(xI_n + M_0) \\ &= (x-1+n)(x-1)^{n-1}. \end{aligned} \tag{1}$$

By the Leibniz formula [4], the determinant of the matrix M_x gives

$$\det\, M_x = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \prod_{i=1}^{n} (m(x))_{\sigma(i),i},$$

where $\epsilon(\sigma)$ is the signature of the permutation $\sigma \in \mathfrak{S}_n$ and

$$(m(x))_{\sigma(i),i} = \begin{cases} 1, & \text{if } \sigma(i) \neq i; \\ x, & \text{if } \sigma(i) = i, \end{cases}$$

yielding

$$\det M_x = \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) \prod_{i\in \mathrm{fix}(\sigma)} x$$
$$= \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) x^{\mathrm{fix}(\sigma)}, \tag{2}$$

where $\mathrm{fix}(\sigma)$ is the number of fixed points of σ.

2 First relation by successive derivatives of the determinant

Theorem 2. *Let $x \in \mathbb{R}$ and n be a natural number. We have*

$$\sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\mathrm{fix}(\sigma)\left(\mathrm{fix}(\sigma)-1\right)\cdots\left(\mathrm{fix}(\sigma)-k+1\right) x^{\mathrm{fix}(\sigma)-k} = (x-1)^{n-k-1}\frac{n!(x+n-k-1)}{(n-k)!}. \tag{3}$$

Proof. By differentiating Eq. (2) k times, for $k \geq 1$, one gets

$$(\det M_x)^{(k)} = \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\mathrm{fix}(\sigma)\left(\mathrm{fix}(\sigma)-1\right)\cdots\left(\mathrm{fix}(\sigma)-k+1\right) x^{\mathrm{fix}(\sigma)-k}. \tag{4}$$

The Leibniz formula for the multiple derivative of a product gives, using Eq. (1):

$$(\det M_x)^{(k)} = \sum_{p=0}^{k} \binom{k}{p} (x-1+n)^{(p)} \left((x-1)^{n-1}\right)^{(k-p)}$$

and

$$(x+1-n)^{(p)} = \begin{cases} x-1+n, & \text{if } p=0; \\ 1, & \text{if } p=1; \\ 0, & \text{otherwise,} \end{cases}$$

and thus

$$\begin{aligned}(\det M_x)^{(k)} &= (x-1+n)\left((x-1)^{n-1}\right)^{(k)} + k\left((x-1)^{n-1}\right)^{(k-1)} \\ &= (x-1)^{n-k-1}\frac{(n-1)!}{(n-k-1)!}\left(x-1+n+\frac{k(x-1)}{n-k}\right) \\ &= (x-1)^{n-k-1}\frac{n!(x+n-k-1)}{(n-k)!}, \end{aligned} \tag{5}$$

which, combined with Eq. (4), completes the proof. □

For $x=1$, according to Eq. (5), one has

$$(\det\ M_1)^{(k)} = n!\,\delta_{k,n-1},$$

where $\delta_{i,j}$ is the usual Kronecker delta symbol. Setting $k=n-1$ in Eq. (4) yields

$$\sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\mathrm{fix}(\sigma)\left(\mathrm{fix}(\sigma)-1\right)\left(\mathrm{fix}(\sigma)-2\right)\cdots\left(\mathrm{fix}(\sigma)-n+2\right) = n!.$$

As another example, setting $x=2$ in Eq. (3) gives

$$\sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\mathrm{fix}(\sigma)\left(\mathrm{fix}(\sigma)-1\right)\left(\mathrm{fix}(\sigma)-2\right)\cdots\left(\mathrm{fix}(\sigma)-k+1\right)2^{\mathrm{fix}(\sigma)-k} = \frac{n!(n-k+1)}{(n-k)!}.$$

3 Second identity by successive integrations of the determinant

Theorem 3. *For $k\in\mathbb{N}^*$ we have*

$$\sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\frac{(\mathrm{fix}(\sigma))!}{(\mathrm{fix}(\sigma)+k)!} = (-1)^{n+1}\frac{(n^2+(k-1)n-(k-1))}{(k-1)!(n+k-1)(n+k)}. \tag{6}$$

Proof. The left-hand side of Eq. (6) is obtained by successive integrations of

$$\sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)x^{\mathrm{fix}(\sigma)}.$$

According to Eq. (2), it is therefore equal to $P_{n,k}(1)$ with, for $k\geq 2$:

$$P_{n,k}(x) = \int_0^x\left(\int_0^{x_{k-1}}\left(\int_0^{x_{k-2}}\cdots\left(\int_0^{x_2}\left(\int_0^{x_1}\det(M_u)\mathrm{d}u\right)\mathrm{d}x_1\right)\cdots\right)\mathrm{d}x_{k-2}\right)\mathrm{d}x_{k-1}, \tag{7}$$

or also

$$P_{n,k}(x) = \int_0^x P_{n,k-1}(u)\mathrm{d}u.$$

Since $\det(M_x) = (x-1)^n + n(x-1)^{n-1}$, one has in particular

$$P_{n,1}(x) = \int_0^x \det(M_u)\mathrm{d}u = (-1)^{n+1}\frac{n}{n+1} + \frac{(x-1)^{n+1}}{n+1} + (x-1)^n, \tag{8}$$

as well as

$$P_{n,2}(x) = \int_0^x\left(\int_0^{x_1}\det(M_u)\mathrm{d}u\right)\mathrm{d}x_1 = \int_0^x P_{n,1}(u)\mathrm{d}u$$

yielding

$$P_{n,2}(x) = (-1)^{n+1}\frac{nx}{(n+1)} + \frac{(-1)^{n+1}}{(n+1)(n+2)} + \frac{(-1)^n}{(n+1)} + \frac{(x-1)^{n+2}}{(n+1)(n+2)} + \frac{(x-1)^{n+1}}{(n+1)}. \quad (9)$$

Applying the same procedure, the polynomial for $k=3$ gives

$$P_{n,3}(x) = \int_0^x \left(\int_0^{x_1}\left(\int_0^{x_2} \det(M_u)\mathrm{d}u\right)\mathrm{d}x_1\right)\mathrm{d}x_2 = \int_0^x P_{n,2}(u)\mathrm{d}u$$

leading to

$$\begin{aligned} P_{n,3}(x) = {} & (-1)^{n+1}\frac{nx^2}{2(n+1)} \\ & + (-1)^{n+1}\frac{x}{(n+1)(n+2)} + (-1)^n\frac{x}{(n+1)} \\ & + \frac{(-1)^n}{(n+1)(n+2)(n+3)} + \frac{(-1)^{n+1}}{(n+1)(n+2)} \\ & + \frac{(x-1)^{n+3}}{(n+1)(n+2)(n+3)} + \frac{(x-1)^{n+2}}{(n+1)(n+3)}. \end{aligned} \quad (10)$$

Setting $k = 1$, 2 and 3 in Eqs. (8), (9) and (10) gives (all the terms proportional to powers of $x-1$ vanish):

$$P_{n,1}(1) = (-1)^{n+1}\frac{n}{(n+1)},$$

$$P_{n,2}(1) = (-1)^{n+1}\frac{n}{(n+1)} + \frac{(-1)^{n+1}}{(n+1)(n+2)} + \frac{(-1)^n}{(n+1)},$$

and

$$\begin{aligned} P_{n,3}(1) = {} & (-1)^{n+1}\frac{n}{2(n+1)} \\ & + \frac{(-1)^{n+1}}{(n+1)(n+2)} + \frac{(-1)^n}{(n+1)} \\ & + \frac{(-1)^n}{(n+1)(n+2)(n+3)} + \frac{(-1)^{n+1}}{(n+1)(n+2)}. \end{aligned}$$

Thus, for any value of $k \geq 2$, we have

$$P_{n,k}(1) = \frac{(-1)^{n+1}n}{(n+1)(k-1)!} - \sum_{j=2}^{k}\frac{(-1)^{n+j}}{(k-j)!}\left(\frac{1}{\prod_{l=n+1}^{n+j} l} - \frac{1}{\prod_{l=n+1}^{n+j-1} l}\right),$$

with

$$\sum_{j=2}^{k}\frac{1}{(k-j)!}\frac{(-1)^{n+j}}{\prod_{l=n+1}^{n+j} l} = \sum_{j=2}^{k}\frac{(-1)^{n+j}}{(k-j)!}\frac{n!}{(n+j)!}.$$

Let us set

$$S_n(k) = \sum_{j=2}^{k} \frac{(-1)^{n+j}}{(k-j)!(n+j)!},$$

so that we can write

$$P_{n,k}(1) = \frac{(-1)^{n+1} n}{(n+1)(k-1)!} - n!\, S_n(k) - n!\, S_{n-1}(k). \tag{11}$$

Let us now assume that $k \geq 2$. We have

$$\begin{aligned} S_n(k) &= \sum_{j=2}^{k} \frac{(-1)^{n+j}}{(k-j)!(n+j)!} \\ &= \frac{1}{(n+k)!} \sum_{j=2}^{k} \frac{(-1)^{n+j}(n+k)!}{(k-j)!(n+j)!}. \end{aligned}$$

The latter quantity is equal to

$$\begin{aligned} S_n(k) &= \frac{1}{(n+k)!} \sum_{j=2}^{k} \binom{n+k}{n+j} (-1)^{n+j} \\ &= \frac{1}{(n+k)!} \sum_{m=n+2}^{n+k} \binom{n+k}{m} (-1)^m \\ &= \frac{1}{(n+k)!} \left(\sum_{m=0}^{n+k} (-1)^m \binom{n+k}{m} - \sum_{m=0}^{n+1} (-1)^m \binom{n+k}{m} \right) \\ &= -\frac{1}{(n+k)!} \sum_{m=0}^{n+1} (-1)^m \binom{n+k}{m} \\ &= -\frac{T_n(k)}{(n+k)!}, \end{aligned}$$

with

$$T_n(k) = \sum_{m=0}^{n+1} \binom{n+k}{m} (-1)^m.$$

Since

$$\binom{n+k}{m} = \binom{n+k-1}{m} + \binom{n+k-1}{m-1},$$

we get

$$\begin{aligned}
T_n(k) &= \sum_{m=1}^{n+1}(-1)^m\binom{n+k-1}{m} + \sum_{m=1}^{n+1}(-1)^m\binom{n+k-1}{m-1} + 1\\
&= \sum_{m=1}^{n+1}(-1)^m\binom{n+k-1}{m} + \sum_{m=0}^{n}(-1)^{m+1}\binom{n+k-1}{m} + 1\\
&= (-1)^{n+1}\binom{n+k-1}{n+1} - 1 + 1\\
&= (-1)^{n-1}\binom{n+k-1}{n+1}.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
S_n(k) &= \frac{(-1)^n(n+k-1)!}{(n+1)!(k-2)!(n+k)!}\\
&= \frac{(-1)^n}{(n+1)!(k-2)!(n+k)},
\end{aligned}$$

which, combined with Eq. (11), yields

$$\begin{aligned}
P_{n,k}(1) &= \frac{(-1)^{n+1}n}{(n+1)(k-1)!} - n!\frac{(-1)^n}{(n+1)!(k-2)!(n+k)} - n!\frac{(-1)^{n+1}}{n!(k-2)!(n-1+k)}\\
&= (-1)^{n+1}\frac{(n^2+(k-1)n-(k-1))}{(k-1)!(n+k-1)(n+k)},
\end{aligned}$$

which completes the proof. □

For $k=1$, Eq. (6) becomes

$$\sum_{\sigma\in\mathfrak{S}_n} \frac{\epsilon(\sigma)}{(\mathrm{fix}(\sigma)+1)} = (-1)^{n+1}\frac{n}{(n+1)},$$

while for $k=2$ one has

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_n} \frac{\epsilon(\sigma)}{(\mathrm{fix}(\sigma)+1)(\mathrm{fix}(\sigma)+2)} &= (-1)^{n+1}\frac{(n+\phi)(n+1-\phi)}{(n+1)(n+2)}\\
&= (-1)^{n+1}\frac{(n+\phi)(n-1/\phi)}{(n+1)(n+2)},
\end{aligned}$$

where ϕ is the golden ratio, and in the $k=3$ case:

$$\sum_{\sigma\in\mathfrak{S}_n} \frac{\epsilon(\sigma)}{(\mathrm{fix}(\sigma)+1)(\mathrm{fix}(\sigma)+2)(\mathrm{fix}(\sigma)+3)} = (-1)^{n+1}\frac{((n+1)^2-3)}{2(n+2)(n+3)}.$$

Here also, as in the first section, we have set $x = 1$, but of course, further sum rules can be obtained by taking different values of x in Eq. (7) and using

$$P_{n,k}(x) = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \frac{(\mathrm{fix}(\sigma))!}{(\mathrm{fix}(\sigma) + k)!} x^{\mathrm{fix}(\sigma)+k}.$$

4 Conclusion

We introduced two families of identities concerning permutations with fixed points, obtained respectively through successive differentiation and repeated integration of the determinant of a specific matrix. Beyond these results, the methods presented here may lead to the derivation of further sum rules.

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