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Simplicial d-Polytopic Numbers Defined on Lucas Sequences

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Abstract

This paper introduces the simplicial d-polytopic numbers defined on Lucas sequences. We establish basic identities and find q-identities. Furthermore, we find generating functions for the simplicial d-Lucas-polytopic numbers and for the squares of the Lucas-triangular numbers. Finally, we compute sums of reciprocals of Lucas sequences and Lucas-triangular numbers. We introduce an analogue of the zeta function defined on Lucas sequences.

1 Introduction

There is growing research on analogue sequences of numbers defined in terms of Lucas sequences, including Catalan numbers [1, 5], Bernoulli and Euler polynomials [7], Eulerian numbers [10], among others. In this paper, we define simplicial d-polytopic numbers on Lucas sequences. The advantage of doing so is that we obtain, for free, analogs of Fibonacci, Pell, Jacobsthal, and Mersenne sequences, among others.

The simplicial polytopic numbers [3] are a family of sequences of figurate numbers corresponding to the d-dimensional simplex for each dimension d, where d is a non-negative integer. For d ranging from 1 to 5, we have the following simplicial polytopic numbers, respectively: non-negative numbers $\mathbb{N}$, triangular numbers T_n, tetrahedral numbers Te_n, pentachoron numbers P_n and hexateron numbers H_n. A list of the above sets of numbers is as follows:

$$
\begin{aligned}
\mathbb{N} &= (0,1,2,3,4,5,6,7,8,9,\ldots),\\
T &= (0,1,3,6,10,15,21,28,36,45,55,66,\ldots),\\
\mathrm{Te} &= (0,1,4,10,20,35,56,84,120,165,\ldots),\\
P &= (0,1,5,15,35,70,126,210,330,495,715,\ldots),\\
H &= (0,1,6,21,56,126,252,462,792,1287,\ldots).
\end{aligned}
$$

The n^{th} simplicial d-polytopic numbers P_n^d are given by the formula

$$P_n^d = \binom{n+d-1}{d} = \frac{n^{(d)}}{d!},$$

where $x^{(d)} = x(x+1)(x+2)\cdots(x+d-1)$ is the rising factorial. The generating function of the simplicial d-polytopic numbers is

$$\sum_{n=1}^{\infty} \binom{n+d-1}{d} x^n = \frac{x}{(1-x)^{d+1}}.$$

In this paper, the n-th simplicial d-Lucas-polytopic number is defined by

$$\left\{ \begin{matrix} n+d-1 \\ d \end{matrix} \right\}_{s,t} = \frac{\{n\}_{s,t}\{n+1\}_{s,t}\cdots\{n+d-1\}_{s,t}}{\{d\}_{s,t}!},$$

where $\{n\}_{s,t}$ is the Lucas analogue of the positive integer n. We determine basic identities for simplicial Lucas polytopic numbers, especially for Lucas-triangular and Lucas-tetrahedral numbers. These sequences are part of the On-Line Encyclopedia of Integer Sequences [13]. Some known q-identities are found, [12, 14]. We establish generating functions for the simplicial d-Lucas-polytopic numbers $\left\{ \begin{smallmatrix} n+d-1 \\ d \end{smallmatrix} \right\}_{s,t}$ and for the sequence $\left\{ \begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right\}_{s,t}^2$. Finally, we introduce the Lucas-zeta function $\zeta_{s,t}(z)$ and find some values for $\zeta_{s,t}(1)$. In addition, we calculate reciprocal sums of Lucas sequences and Lucas-triangular numbers.

2 Preliminaries

The Lucas sequences [11] on the parameters s, t are defined by

$$\{n+2\}_{s,t} = s\{n+1\}_{s,t} + t\{n\}_{s,t}$$

with initial values $\{0\}_{s,t} = 0$ and $\{1\}_{s,t} = 1$, where $s \neq 0$ and $t \neq 0$. Below are some important specializations of Lucas sequences.

1. If $s = 2, t = -1$, then $\{n\}_{2,-1} = n$ are the positive integers.

2. If $s = 1, t = 1$, then $\{n\}_{1,1} = F_n$ are the Fibonacci numbers

$$F_n = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots).$$

3. If $s = 2, t = 1$, then $\{n\}_{2,1} = P_n$, where P_n are the Pell numbers

$$P_n = (0, 1, 2, 5, 12, 29, 70, 169, 408 \ldots).$$

4. If $s = 1, t = 2$, then $\{n\}_{1,2} = J_n$, where J_n are the Jacobsthal numbers

$$J_n = (0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots).$$

5. If $s = 3, t = -2$, then $\{n\}_{3,-2} = M_n$, where $M_n = 2^n - 1$ are the Mersenne numbers

$$M_n = (0, 1, 3, 7, 15, 31, 63, 127, 255, \ldots).$$

6. If $s = p + q, t = -pq$, then $\{n\}_{p+q,-pq} = [n]_{p,q}$, where $[n]_{p,q}$ are the (p, q)-numbers

$$[n]_{p,q} = (0, 1, [2]_{p,q}, [3]_{p,q}, [4]_{p,q}, [5]_{p,q}, [6]_{p,q}, [7]_{p,q}, [8]_{p,q} \ldots).$$

   If $p = 1$, we obtain the q-numbers $[n]_q = 1 + q + q^2 + q^3 + \cdots + q^{n-1}$.

7. If $s = 2t, t = -1$, then $\{n\}_{2t,-1} = U_{n-1}(t)$, where $U_n(t)$ are the Chebyshev polynomials of the second kind, with $U_{-1}(t) = 0$.

The Lucas constant is the ratio toward which adjacent terms in a Lucas sequence tend. This is the only positive zero of $x^2 - sx - t = 0$. We let $\varphi_{s,t}$ denote this constant, where

$$\varphi_{s,t} = \frac{s + \sqrt{s^2 + 4t}}{2}$$

and

$$\varphi'_{s,t} = s - \varphi_{s,t} = -\frac{t}{\varphi_{s,t}} = \frac{s - \sqrt{s^2 + 4t}}{2}$$

denote the reciprocal of $\varphi_{s,t}$. Some specializations of the constants $\varphi_{s,t}$ and $\varphi'_{s,t}$ are

1. If $s = 2$ and $t = -1$, then $\varphi_{2,-1} = 1$ and $\varphi'_{2,-1} = 1$.
2. If $s = 1$ and $t = 1$, then $\varphi_{1,1} = \varphi = \frac{1+\sqrt{5}}{2}$ and $\varphi'_{1,1} = \varphi' = \frac{1-\sqrt{5}}{2}$.
3. If $s = 2$ and $t = 1$, then $\varphi_{2,1} = 1 + \sqrt{2}$ and $\varphi'_{2,1} = 1 - \sqrt{2}$.
4. If $s = 1$ and $t = 2$, then $\varphi_{1,2} = 2$ and $\varphi'_{1,2} = -1$.
5. If $s = 3$ and $t = -2$, then $\varphi_{3,-2} = 2$ and $\varphi'_{3,-2} = 1$.

6. If $s = p + q$ and $t = -pq$, then $\varphi_{p+q,-pq} = p$ and $\varphi'_{p+q,-pq} = q$.

7. If $s = 2t$ and $t = -1$, then $\varphi_{2t,-1} = \frac{t+\sqrt{t^2-1}}{2}$ and $\varphi'_{2t,-1} = \frac{t-\sqrt{t^2-1}}{2}$.

The Lucasnomial coefficients are defined by

$$\left\{ {n \atop k} \right\}_{s,t} = \frac{\{n\}_{s,t}!}{\{k\}_{s,t}!\{n-k\}_{s,t}!},$$

where $\{n\}_{s,t}! = \{1\}_{s,t}\{2\}_{s,t} \cdots \{n\}_{s,t}$. The Lucasnomial coefficients satisfy the following Pascal recurrence relationships. For $1 \le k \le n-1$ we have

$$\left\{ {n+1 \atop k} \right\}_{s,t} = \varphi_{s,t}^{k} \left\{ {n \atop k} \right\}_{s,t} + \varphi_{s,t}^{\prime(n+1-k)} \left\{ {n \atop k-1} \right\}_{s,t}, \tag{1}$$

$$= \varphi_{s,t}^{\prime(k)} \left\{ {n \atop k} \right\}_{s,t} + \varphi_{s,t}^{n+1-k} \left\{ {n \atop k-1} \right\}_{s,t}. \tag{2}$$

A proof of the above identities was provided by Corcino [2].

Set $s, t \in \mathbb{R}$, with $s \neq 0$ and $t \neq 0$. If $s^2 + 4t \neq 0$, we define the Lucas-derivative $D_{s,t}$ of the function $f(x)$ as

$$(D_{s,t}f)(x) = \begin{cases} \frac{f(\varphi_{s,t}x) - f(\varphi'_{s,t}x)}{(\varphi_{s,t} - \varphi'_{s,t})x}, & \text{if } x \neq 0; \\ f'(0), & \text{if } x = 0, \end{cases}$$

provided $f(x)$ is differentiable at $x = 0$. If $s^2 + 4t = 0$, with $t < 0$, we define the Lucas-derivative of the function $f(x)$ as

$$(D_{\pm 2i\sqrt{t},t}f)(x) = f'(\pm i\sqrt{t}x).$$

The Lucas-derivative $D_{s,t}$ fulfills the following properties.

- Linearity:

$$D_{s,t}(\alpha f + \beta g) = \alpha D_{s,t} f + \beta D_{s,t} g.$$

- Product rules:

$$D_{s,t}(f(x)g(x)) = f(\varphi_{s,t}x) D_{s,t} g(x) + g(\varphi'_{s,t}x) D_{s,t} f(x),$$

and

$$D_{s,t}(f(x)g(x)) = f(\varphi'_{s,t}x) D_{s,t} g(x) + g(\varphi_{s,t}x) D_{s,t} f(x).$$

- Quotient rules:

$$D_{s,t}\left(\frac{f(x)}{g(x)}\right) = \frac{g(\varphi_{s,t}x) D_{s,t} f(x) - f(\varphi_{s,t}x) D_{s,t} g(x)}{g(\varphi_{s,t}x) g(\varphi'_{s,t}x)},$$

and

$$D_{s,t}\left(\frac{f(x)}{g(x)}\right) = \frac{g(\varphi'_{s,t}x) D_{s,t} f(x) - f(\varphi'_{s,t}x) D_{s,t} g(x)}{g(\varphi_{s,t}x) g(\varphi'_{s,t}x)}.$$

Define the n-th Lucas-derivative of the function $f(x)$ recursively as

$$D^n_{s,t} f(x) = D_{s,t}(D^n_{s,t} f(x)).$$

3 d-Lucas-polytopic numbers

3.1 Definition and basic properties

Definition 1. The n-th simplicial d-Lucas-polytopic number is defined by

$$\left\{ \begin{matrix} n+d-1 \\ d \end{matrix} \right\}_{s,t} = \frac{\{n\}_{s,t}\{n+1\}_{s,t}\cdots\{n+d-1\}_{s,t}}{\{d\}_{s,t}!}.$$

The Lucas analogues of the triangular numbers, tetrahedral numbers, pentachoron numbers, and hexateron numbers are

$$T_{s,t} = \left\{ T_n(s,t) = \left\{ \begin{matrix} n+1 \\ 2 \end{matrix} \right\}_{s,t} : n \geq 0 \right\},$$

$$\mathrm{Te}_{s,t} = \left\{ \mathrm{Te}_n(s,t) = \left\{ \begin{matrix} n+2 \\ 3 \end{matrix} \right\}_{s,t} : n \geq 0 \right\},$$

$$P_{s,t} = \left\{ P_n(s,t) = \left\{ \begin{matrix} n+3 \\ 4 \end{matrix} \right\}_{s,t} : n \geq 0 \right\},$$

$$H_{s,t} = \left\{ H_n(s,t) = \left\{ \begin{matrix} n+4 \\ 5 \end{matrix} \right\}_{s,t} : n \geq 0 \right\}.$$

From the Pascal recurrence in Eqs. (1) and (2) we have

$$\left\{ \begin{matrix} n+d \\ d \end{matrix} \right\}_{s,t} = \varphi^d_{s,t} \left\{ \begin{matrix} n+d-1 \\ d \end{matrix} \right\}_{s,t} + \varphi'^n_{s,t} \left\{ \begin{matrix} n+d-1 \\ d-1 \end{matrix} \right\}_{s,t}, \tag{3}$$

$$= \varphi'^d_{s,t} \left\{ \begin{matrix} n+d-1 \\ d \end{matrix} \right\}_{s,t} + \varphi^n_{s,t} \left\{ \begin{matrix} n-d-1 \\ d-1 \end{matrix} \right\}_{s,t}. \tag{4}$$

It is a well-known fact that the sum of the first n terms of a sequence of d-polytopic numbers is the n-th term of a sequence of $(d+1)$-polytopic numbers, i.e,

$$\sum_{k=1}^{n} P^d_n = P^{d+1}_n.$$

We then obtain the Lucas analogue of the above formula.

Theorem 2. *For all $n \geq 1$, the sum of simplicial d-Lucas-polytopic numbers is*

$$\left\{ \begin{matrix} n+d \\ d+1 \end{matrix} \right\}_{s,t} = \sum_{k=1}^{n} \varphi_{s,t}^{(d+1)(n-k)} \varphi_{s,t}'^{(k-1)} \left\{ \begin{matrix} k+d-1 \\ d \end{matrix} \right\}_{s,t},$$

$$\left\{ \begin{matrix} n+d \\ d+1 \end{matrix} \right\}_{s,t} = \sum_{k=1}^{n} \varphi_{s,t}'^{(d+1)(n-k)} \varphi_{s,t}^{k-1} \left\{ \begin{matrix} k+d-1 \\ d \end{matrix} \right\}_{s,t}.$$

Proof. The proof is by induction on n. When $n = 1$, then $\left\{ \begin{smallmatrix} d+1 \\ d+1 \end{smallmatrix} \right\}_{s,t} = \left\{ \begin{smallmatrix} d \\ d \end{smallmatrix} \right\}_{s,t}$. For $n = 2$, it follows that

$$\left\{ \begin{matrix} d+2 \\ d+1 \end{matrix} \right\}_{s,t} = \varphi_{s,t}^{d+1} \left\{ \begin{matrix} d \\ d \end{matrix} \right\}_{s,t} + \varphi'_{s,t} \left\{ \begin{matrix} d+1 \\ d \end{matrix} \right\}_{s,t}.$$

Suppose the statement is true for n and we prove the statement for $n+1$. We have

$$\begin{aligned}
\left\{ \begin{matrix} n+d+1 \\ d+1 \end{matrix} \right\}_{s,t} &= \varphi_{s,t}^{d+1} \left\{ \begin{matrix} n+d \\ d+1 \end{matrix} \right\}_{s,t} + \varphi_{s,t}'^{n} \left\{ \begin{matrix} n+d \\ d \end{matrix} \right\}_{s,t} \\
&= \varphi_{s,t}^{d+1} \sum_{k=1}^{n} \varphi_{s,t}^{(d+1)(n-k)} \varphi_{s,t}'^{(k-1)} \left\{ \begin{matrix} k+d-1 \\ d \end{matrix} \right\}_{s,t} + \varphi_{s,t}'^{n} \left\{ \begin{matrix} n+d \\ d \end{matrix} \right\}_{s,t} \\
&= \sum_{k=1}^{n} \varphi_{s,t}^{(d+1)(n+1-k)} \varphi_{s,t}'^{(k-1)} \left\{ \begin{matrix} k+d-1 \\ d \end{matrix} \right\}_{s,t} + \varphi_{s,t}'^{n} \left\{ \begin{matrix} n+d \\ d \end{matrix} \right\}_{s,t} \\
&= \sum_{k=1}^{n+1} \varphi_{s,t}^{(d+1)(n+1-k)} \varphi_{s,t}'^{(k-1)} \left\{ \begin{matrix} k+d-1 \\ d \end{matrix} \right\}_{s,t}.
\end{aligned}$$

The proof is reached. □

From Theorem 2 we obtain the following known result about q-binomial coefficients:

$$\begin{bmatrix} n+d \\ d+1 \end{bmatrix}_q = \sum_{k=1}^{n} q^{k-1} \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q = \sum_{k=1}^{n} q^{(d+1)(n-k)} \begin{bmatrix} k+d-1 \\ d \end{bmatrix}_q.$$

4 Some specializations

4.1 Lucas-triangular numbers

Some specializations of Lucas-triangular numbers are

$$\left\{ \begin{matrix} n+1 \\ 2 \end{matrix} \right\}_{1,1} = F_n F_{n+1} = (0, 1, 2, 6, 15, 40, 104, 273, \ldots),$$

$$\begin{aligned}
\left\{ {n+1 \atop 2} \right\}_{2,1} &= \frac{1}{2} P_n P_{n+1} = (0, 1, 5, 30, 174, 1015, 5915, \ldots), \\
\left\{ {n+1 \atop 2} \right\}_{1,2} &= J_n J_{n+1} = (0, 1, 2, 6, 15, 55, 231, 903, 3655, \ldots), \\
\left\{ {n+1 \atop 2} \right\}_{3,-2} &= \frac{1}{3} (2^n - 1)(2^{n+1} - 1) \\
&= (0, 1, 7, 35, 155, 651, 2667, 10795, 43435, 174251, \ldots).
\end{aligned}$$

The $\left\{ {n+1 \atop 2} \right\}_{1,1}$-numbers are known as golden rectangle numbers, (A001654 in [13]). The $\left\{ {n+1 \atop 2} \right\}_{2,1}$-numbers may be called Pell triangles, (A084158 in [13]). The $\left\{ {n+1 \atop 2} \right\}_{1,2}$-numbers are known as Jacobsthal oblong numbers, (A084175 in [13]). The $\left\{ {n+1 \atop 2} \right\}_{3,-2}$-numbers are the Gaussian binomial coefficients $\left[{n \atop 2} \right]_q$ for $q = 2$, (A006095 in [13]).

From Eqs. (3) and (4), and Theorem 2, we obtain

$$\begin{aligned}
\left\{ {n+2 \atop 2} \right\}_{s,t} &= \varphi_{s,t}^{2} \left\{ {n+1 \atop 2} \right\} + \varphi_{s,t}^{\prime n} \{n+1\}_{s,t}, \\
\left\{ {n+2 \atop 2} \right\}_{s,t} &= \varphi_{s,t}^{\prime 2} \left\{ {n+1 \atop 2} \right\}_{s,t} + \varphi_{s,t}^{n} \{n+1\}_{s,t}, \\
\left\{ {n+1 \atop 2} \right\}_{s,t} &= \sum_{k=1}^{n} \varphi_{s,t}^{2(n-k)} \varphi_{s,t}^{\prime (k-1)} \{k\}_{s,t}, \\
\left\{ {n+1 \atop 2} \right\}_{s,t} &= \sum_{k=1}^{n} \varphi_{s,t}^{\prime 2(n-k)} \varphi_{s,t}^{k-1} \{k\}_{s,t}.
\end{aligned}$$

From the above identities, we obtain the identity of Warnaar [14]

$$\left[{n+1 \atop 2} \right]_q = \sum_{k=1}^{n} \frac{1-q^k}{1-q} q^{2(n-k)} = \sum_{k=1}^{n} q^{k-1} \frac{1-q^k}{1-q}.$$

Theorem 3. *For all $n \in \mathbb{N}$,*

1. *The Lucas analogue of the identity $T_n + T_{n-1} = n^2$ is*

$$\left\{ {n+1 \atop 2} \right\}_{s,t} = t \left\{ {n \atop 2} \right\}_{s,t} + \{n\}^2.$$

2. *Another Lucas analogue of the identity $T_n + T_{n-1} = n^2$ is*

$$\varphi_{s,t}^{\prime (n-1)} \left\{ {n+1 \atop 2} \right\}_{s,t} + \varphi_{s,t}^{n+1} \left\{ {n \atop 2} \right\}_{s,t} = \frac{\langle n \rangle_{s,t}}{\{2\}_{s,t}} \{n\}_{s,t}^2,$$

where $\langle n \rangle_{s,t} = \varphi_{s,t}^{n} + \varphi_{s,t}^{\prime n}$.

3. The Lucas analogue of the alternating sum squares $T_n = \sum_{k=1}^{n}(-1)^{n-k}k^2$ is

$$\left\{ {n+1 \atop 2} \right\}_{s,t} = \sum_{k=1}^{n} t^{n-k}\{k\}_{s,t}^2.$$

Proof. We have

$$\begin{aligned}
\left\{ {n+1 \atop 2} \right\}_{s,t} &= \frac{\{n\}_{s,t}\{n+1\}_{s,t}}{\{2\}_{s,t}} \\
&= \frac{\{n\}_{s,t}(s\{n\}_{s,t} + t\{n-1\}_{s,t})}{\{2\}_{s,t}} \\
&= s\frac{\{n\}_{s,t}^2}{s} + t\frac{\{n\}_{s,t}\{n-1\}_{s,t}}{\{2\}_{s,t}} \\
&= \{n\}_{s,t}^2 + t\left\{ {n \atop 2} \right\}_{s,t}.
\end{aligned}$$

Statement 2 is proved as follows:

$$\begin{aligned}
\varphi_{s,t}^{n+1}\left\{ {n \atop 2} \right\}_{s,t} + \varphi_{s,t}^{\prime(n-1)}\left\{ {n+1 \atop 2} \right\}_{s,t} &= \varphi_{s,t}^{n+1}\frac{\{n\}_{s,t}\{n-1\}_{s,t}}{\{2\}_{s,t}} + \varphi_{s,t}^{\prime(n-1)}\frac{\{n\}_{s,t}\{n+1\}_{s,t}}{\{2\}_{s,t}} \\
&= \frac{\{n\}_{s,t}}{\{2\}_{s,t}}(\varphi_{s,t}^{n+1}\{n-1\}_{s,t} + \varphi_{s,t}^{\prime(n-1)})\{n+1\}_{s,t}) \\
&= \frac{\{n\}_{s,t}}{\{2\}_{s,t}}\{2n\}_{s,t} \\
&= \frac{\langle n\rangle_{s,t}}{\{2\}_{s,t}}\{n\}_{s,t}^2.
\end{aligned}$$

By iterating statement (1), we obtain statement (3). □

If we set $s = 1 + q$, and $t = -q$ in the previous theorem, then

$$\begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q = -q\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q + \left(\frac{1-q^{n+1}}{1-q}\right)^2 \tag{5}$$

and

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^{n-1}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q = \frac{1+q^n}{1+q}[n]_q^2.$$

By iterating Eq. (5), we obtain a result due to Schlosser [12]:

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q = \sum_{k=1}^{n}(-q)^{n-k}\left(\frac{1-q^k}{1-q}\right)^2.$$

Theorem 4. *For all $n \in \mathbb{N}$,*

$$\left\{ {n+2 \atop 2} \right\}_{s,t}^2 - t^2 \left\{ {n+1 \atop 2} \right\}_{s,t}^2 = \left(\frac{\{n+2\}_{s,t} + t\{n\}_{s,t}}{s} \right) \{n+1\}_{s,t}^3. \tag{6}$$

Proof. From Theorem 3, we have that

$$\begin{aligned}
\left\{ {n+2 \atop 2} \right\}_{s,t}^2 &= \left(t \left\{ {n+1 \atop 2} \right\}_{s,t} + \{n+1\}_{s,t}^2 \right)^2 \\
&= t^2 \left\{ {n+1 \atop 2} \right\}_{s,t}^2 + 2t \left\{ {n+1 \atop 2} \right\}_{s,t}^2 \{n+1\}_{s,t}^2 + \{n+1\}_{s,t}^4 \\
&= t^2 \left\{ {n+1 \atop 2} \right\}_{s,t}^2 + 2t \frac{\{n\}_{s,t}\{n+1\}_{s,t}^3}{\{2\}_{s,t}} + \{n+1\}_{s,t}^4 \\
&= t^2 \left\{ {n+1 \atop 2} \right\}_{s,t}^2 + \left(\frac{2t}{s} \{n\}_{s,t} + \{n+1\}_{s,t} \right) \{n+1\}_{s,t}^3 \\
&= t^2 \left\{ {n+1 \atop 2} \right\}_{s,t}^2 + \left(\frac{t\{n\}_{s,t} + \{n+2\}_{s,t}}{s} \right) \{n+1\}_{s,t}^3.
\end{aligned}$$

The remainder of the proof follows straightforwardly. □

Eq. (6) is the Lucas analogue of the identity

$$T_{n+1}^2 - T_n^2 = n^3.$$

From Warnaar [14] we have

$$\begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q^2 - q^2 \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 = \left(\frac{1-q^{2(n+1)}}{1-q^2} \right) \left(\frac{1-q^{n+1}}{1-q} \right)^2.$$

Theorem 5. *For all $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} t^{2(n-k)} \left(\frac{\{k+1\}_{s,t} + t\{k-1\}_{s,t}}{s} \right) \{k\}_{s,t}^3 = \left\{ {n+1 \atop 2} \right\}_{s,t}^2. \tag{7}$$

Proof.

$$\begin{aligned}
\left\{ {n+1 \atop 2} \right\}_{s,t}^2 = &\left(\left\{ {n+1 \atop 2} \right\}_{s,t}^2 - t^2 \left\{ {n \atop 2} \right\}_{s,t}^2 \right) + \left(t^2 \left\{ {n \atop 2} \right\}_{s,t}^2 - t^4 \left\{ {n-1 \atop 2} \right\}_{s,t}^2 \right) \\
&+ \cdots + \left(t^{2n-2} \left\{ {2 \atop 2} \right\}_{s,t}^2 - t^{2n} \left\{ {1 \atop 2} \right\}_{s,t}^2 \right)
\end{aligned}$$

$$= \left(\frac{\{n+1\}_{s,t} + t\{n-1\}_{s,t}}{s}\right)\{n\}_{s,t}^3 + t^2\left(\frac{\{n\}_{s,t} + t\{n-2\}_{s,t}}{s}\right)\{n-1\}_{s,t}^3 +$$
$$\cdots + t^{2n-2}\left(\frac{\{2\}_{s,t}}{s}\right)\{1\}_{s,t}^3$$
$$= \sum_{k=1}^{n} t^{2(n-k)}\left(\frac{\{k+1\}_{s,t} + t\{k-1\}_{s,t}}{s}\right)\{k\}_{s,t}^3.$$

□

Eq. (7) can be written as

$$\sum_{k=1}^{n} t^{2(n-k)}\left(\frac{\varphi_{s,t}^{2k} - \varphi_{s,t}'^{2k}}{\varphi_{s,t}^{2} - \varphi_{s,t}'^{2}}\right)\left(\frac{\varphi_{s,t}^{k} - \varphi_{s,t}'^{k}}{\varphi_{s,t} - \varphi_{s,t}'}\right)^2 = \sum_{k=1}^{n} t^{2(n-k)}[k]_{\varphi^2,\varphi'^2}[k]_{\varphi,\varphi'}^2 = \left\{ n+1 \atop 2 \right\}_{s,t}^2. \quad (8)$$

Eq. (7) is the Lucas analogue of the identity

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

Corollary 6. *For all $n \in \mathbb{N}$, we have*

1. *The Fibonacci analogue of the sum of cubes:*
$$\sum_{k=1}^{n} (F_{k+1} + F_{k-1})F_k^3 = F_n^2 F_{n+1}^2.$$
2. *The Pell analogue of the sum of cubes:*
$$\sum_{k=1}^{n} (P_{k+1} + P_{k-1})P_k^3 = \frac{1}{2}P_n^2 P_{n+1}^2.$$
3. *The Jacobsthal analogue of the sum of cubes:*
$$\sum_{k=1}^{n} 4^{n-k}(J_{k+1} + 2J_{k-1})J_k^3 = J_n^2 J_{n+1}^2.$$
4. *The Mersenne analogue of the sum of cubes:*
$$\sum_{k=1}^{n} 4^{n-k}(2^k + 1)(2^k - 1)^3 = \frac{1}{3}(2^n - 1)^2(2^{n+1} - 1)^2.$$
5. *The q-identity of Warnaar [14]:*
$$\sum_{k=1}^{n} q^{2(n-k)}\left(\frac{1-q^{2k}}{1-q^2}\right)\left(\frac{1-q^k}{1-q}\right)^2 = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2.$$

4.2 Lucas-tetrahedral numbers

Some specializations of Lucas-tetrahedral numbers are

$$\begin{aligned}
\left\{ {n+2 \atop 3} \right\}_{1,1} &= \frac{1}{2} F_n F_{n+1} F_{n+2} \\
&= (0, 1, 3, 15, 60, 260, 1092, 4641, 19635, \ldots), \\
\left\{ {n+2 \atop 3} \right\}_{2,1} &= \frac{1}{10} P_n P_{n+1} P_{n+2} \\
&= (1, 12, 174, 2436, 34307, 482664, \ldots), \\
\left\{ {n+2 \atop 3} \right\}_{1,2} &= \frac{1}{3} J_n J_{n+1} J_{n+2} \\
&= (0, 1, 5, 55, 385, 3311, 25585, 208335, \ldots), \\
\left\{ {n+2 \atop 3} \right\}_{3,-2} &= \frac{1}{21} (2^n - 1)(2^{n+1} - 1)(2^{n+2} - 1) \\
&= (0, 1, 15, 155, 1395, 11811, 97155, \ldots).
\end{aligned}$$

The sequences $\left\{ {n+2 \atop 3} \right\}_{1,1}$, $\left\{ {n+2 \atop 3} \right\}_{2,1}$, and $\left\{ {n+2 \atop 3} \right\}_{3,-2}$ correspond to A001655, A099930, and A006096, respectively, in [13]. The sequence $\left\{ {n+2 \atop 3} \right\}_{1,2}$ has not been previously investigated. From Eqs. (3) and (4), and Theorem 2,

$$\begin{aligned}
\left\{ {n+3 \atop 3} \right\}_{s,t} &= \varphi_{s,t}^{3} \left\{ {n+2 \atop 3} \right\}_{s,t} + \varphi_{s,t}^{\prime n} \left\{ {n+2 \atop 2} \right\}_{s,t}, \\
\left\{ {n+3 \atop 3} \right\}_{s,t} &= \varphi_{s,t}^{\prime 3} \left\{ {n+2 \atop 3} \right\}_{s,t} + \varphi_{s,t}^{n} \left\{ {n+2 \atop 2} \right\}_{s,t}, \\
\left\{ {n+2 \atop 3} \right\}_{s,t} &= \sum_{k=1}^{n} \varphi_{s,t}^{3(n-k)} \varphi_{s,t}^{\prime (k-1)} \left\{ {k+1 \atop 2} \right\}_{s,t}, \\
\left\{ {n+3 \atop 3} \right\}_{s,t} &= \sum_{k=1}^{n} \varphi_{s,t}^{\prime 3(n-k)} \varphi_{s,t}^{k-1} \left\{ {k+1 \atop 2} \right\}_{s,t}.
\end{aligned}$$

Theorem 7. *For all $n \geq 0$,*

$$\left\{ {n+3 \atop 3} \right\}_{s,t} = st \left\{ {n+2 \atop 3} \right\}_{s,t} + \{n+1\}_{s,t} \left\{ {n+2 \atop 2} \right\}_{s,t}. \tag{9}$$

Proof. We have

$$\begin{aligned}\left\{ {n+3 \atop 3} \right\}_{s,t} &= \frac{\{n+1\}_{s,t}\{n+2\}_{s,t}\{n+3\}_{s,t}}{\{3\}_{s,t}!} \\ &= \frac{\{n+1\}_{s,t}\{n+2\}_{s,t}}{\{3\}_{s,t}!}(s\{n+2\}_{s,t} + t\{n+1\}_{s,t}) \\ &= \frac{\{n+1\}_{s,t}\{n+2\}_{s,t}}{\{3\}_{s,t}!}((s^2+t)\{n+1\}_{s,t} + st\{n\}_{s,t}) \\ &= \{n+1\}_{s,t}\left\{ {n+2 \atop 2} \right\}_{s,t} + st\left\{ {n+2 \atop 3} \right\}_{s,t}.\end{aligned}$$

□

The above theorem is Prop. 2.2 given by Sagan and Savage [1]. Eq. (9) is the Lucas analogue of the identity

$$\mathrm{Te}_{n+1} + 2\mathrm{Te}_n = (n+1)T_{n+1}.$$

If we choose $s = 1+q$ and $t = -q$ in Eq. (9), we obtain

$$\begin{bmatrix} n+3 \\ 3 \end{bmatrix}_q = -(1+q)q\begin{bmatrix} n+2 \\ 3 \end{bmatrix}_q + \frac{1-q^{n+1}}{1-q}\begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q.$$

5 Generating functions

Theorem 8. *For all $n \geq 1$,*

$$D^n_{\varphi,\varphi'}\left(\frac{1}{1-x}\right) = \frac{\{n\}_{s,t}!}{(\varphi^n_{s,t}x;q)_{n+1}}, \tag{10}$$

where $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ is the q-shifted factorial.

Proof. Note that

$$D_{\varphi,\varphi'}\left(\frac{1}{1-x}\right) = \frac{1}{(1-\varphi_{s,t}x)(1-\varphi'_{s,t}x)} = \frac{\{1\}_{s,t}!}{(\varphi_{s,t}x;q)_2}.$$

Suppose that Eq. (10) is true for n and let us prove by induction for $n+1$. As

$$D_{\varphi,\varphi'}(\varphi^n_{s,t}x;q)_{n+1} = -\{n+1\}_{s,t}(\varphi^n_{s,t}\varphi'_{s,t}x;q)_n,$$

then

$$
\begin{aligned}
D^{n+1}_{\varphi,\varphi'}\left(\frac{1}{1-x}\right) &= D_{\varphi,\varphi'}D^{n}_{\varphi,\varphi'}\left(\frac{1}{1-x}\right)\\
&= D_{\varphi,\varphi'}\left(\frac{\{n\}_{s,t}!}{(\varphi^n_{s,t}x;q)_{n+1}}\right)\\
&= \frac{-\{n\}_{s,t}!D_{\varphi,\varphi'}(\varphi^n_{s,t}x;q)_{n+1}}{(\varphi^{n+1}_{s,t}x;q)_{n+1}(\varphi^n_{s,t}\varphi'_{s,t}x;q)_{n+1}}\\
&= \frac{\{n+1\}_{s,t}!(\varphi^n_{s,t}\varphi'_{s,t}x;q)_{n}}{(\varphi^{n+1}_{s,t}x;q)_{n+1}(\varphi^n_{s,t}\varphi'_{s,t}x;q)_{n+1}}\\
&= \frac{\{n+1\}_{s,t}!}{(1-(\varphi_{s,t}q)^{n+1}x)(\varphi^{n+1}_{s,t}x;q)_{n+1}}\\
&= \frac{\{n+1\}_{s,t}!}{(\varphi^{n+1}_{s,t}x;q)_{n+2}}.
\end{aligned}
$$

The proof is completed. □

Theorem 9. *For all $d \in \mathbb{N}$,*

$$
\sum_{n=1}^{\infty} \left\{ \begin{matrix} n+d-1 \\ d \end{matrix} \right\}_{s,t} x^n = \frac{x}{(\varphi^d_{s,t}x;q)_{d+1}}.
$$

Proof. From Theorem 8,

$$
\begin{aligned}
\frac{x}{(\varphi^d_{s,t}x;q)_{d+1}} &= \frac{x}{\{d\}_{s,t}!}D^{d}_{\varphi,\varphi'}\left(\frac{1}{1-x}\right)\\
&= \frac{x}{\{d\}_{s,t}!}D^{d}_{\varphi,\varphi'}\left(\sum_{n=0}^{\infty} x^n\right)\\
&= \frac{x}{\{d\}_{s,t}!}\sum_{n=d}^{\infty}\{n\}_{s,t}\{n-1\}_{s,t}\cdots\{n-d+1\}_{s,t}x^{n-d}\\
&= \frac{x}{\{d\}_{s,t}!}\sum_{n=0}^{\infty}\frac{\{d+n\}_{s,t}!}{\{n\}_{s,t}!}x^n\\
&= \sum_{n=1}^{\infty} \left\{ \begin{matrix} n+d-1 \\ d \end{matrix} \right\}_{s,t} x^n.
\end{aligned}
$$

□

Theorem 10. *The generating function of the squares of the Lucas-triangular numbers is*

$$\sum_{n=1}^{\infty} \left\{ {n+1 \atop 2} \right\}_{s,t}^{2} x^n$$
$$= \frac{x + (\{4\}\varphi_{s,t} - \varphi_{s,t}^4)x^2 - \{3\}_{s,t}\varphi_{s,t}^3\varphi_{s,t}'^3 x^3 + (\{3\}_{s,t}\varphi_{s,t}^7\varphi_{s,t}'^3 - \{4\}_{s,t}\varphi_{s,t}^6\varphi_{s,t}'^3)x^4}{(1-t^2x)(1-\varphi_{s,t}'^4 x)(\varphi_{s,t}^4 x;q)_4}.$$

Proof. From Eq. (8),

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\{ {n+1 \atop 2} \right\}_{s,t}^{2} x^n \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n} t^{2(n-k)}[k]_{\varphi^2,\varphi'^2}[k]_{\varphi,\varphi'}^2 x^n \\
&= \sum_{k=1}^{\infty}[k]_{\varphi^2,\varphi'^2}[k]_{\varphi,\varphi'}^2 x^k \sum_{n=k}^{\infty} t^{2(n-k)}x^n \\
&= \sum_{k=1}^{\infty}[k]_{\varphi^2,\varphi'^2}[k]_{\varphi,\varphi'}^2 x^k \sum_{n=0}^{\infty} (t^2x)^n \\
&= \frac{1}{1-t^2x}(xD_{\varphi,\varphi'})^2(xD_{\varphi^2,\varphi'^2})\left\{\frac{x}{1-x}\right\} \\
&= \frac{x + (\{4\}_{s,t}\varphi_{s,t} - \varphi_{s,t}^4)x^2 - \{3\}_{s,t}\varphi_{s,t}^3\varphi_{s,t}'^3 x^3 + (\{3\}_{s,t}\varphi_{s,t}^7\varphi_{s,t}'^3 - \{4\}_{s,t}\varphi_{s,t}^6\varphi_{s,t}'^3)x^4}{(1-t^2x)(1-\varphi_{s,t}'^4 x)(\varphi_{s,t}^4 x;q)_4}.
\end{aligned}$$

□

The q-analogue of the previous theorem is

$$\sum_{n=1}^{\infty} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 x^n = \frac{x + ([4]_q - 1)x^2 - [3]_q q^3 x^3 + ([3]_q q^3 - [4]_q q^3)x^4}{(1-q^2x)(1-q^4x)(x;q)_4}.$$

6 Sum of reciprocals

The Lucas-zeta function, or (s,t)-zeta function, is the function defined by

$$\zeta_{s,t}(z) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_{s,t}^z} = 1 + \frac{1}{s^z} + \frac{1}{(s^2+t)^z} + \frac{1}{(s^3+2st)^z} + \cdots .$$

Egami [4] and Navas [9] independently studied the zeta function $\zeta_{1,1}(z)$. Landau [8] studied the problem of evaluating $\zeta_{1,1}(1)$. The function $\zeta_{s,t}(z)$ is convergent when $z > 0$ and when either $\varphi_{s,t} > 1$ and $0 < |q| < 1$ or $\varphi'_{s,t} > 1$ and $|q| > 1$. Take $z = \sigma + i\beta$. Then $\zeta_{s,t}(z)$ converges when $\Re(z) > 0$. Some specializations of $\zeta_{s,t}(1)$ are

$$\zeta_{1,1}(1) = \zeta_F(1) = \sum_{n=1}^{\infty} \frac{1}{F_n} = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots \doteq 3.359885666243\ldots,$$

$$\zeta_{2,1}(1) = \zeta_P(1) = \sum_{n=1}^{\infty} \frac{1}{P_n} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{12} + \frac{1}{29} + \cdots \doteq 1.81781609195402\ldots,$$

$$\zeta_{1,2}(1) = \zeta_J(1) = \sum_{n=1}^{\infty} \frac{1}{J_n} = 1 + 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{11} + \frac{1}{21} + \cdots \doteq 2.67186147\ldots,$$

$$\zeta_{3,-2}(1) = \zeta_M(1) = \sum_{n=1}^{\infty} \frac{1}{M_n} = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \cdots \doteq 1.57511520737327\ldots,$$

$$\zeta_{2t,-1}(1) = \zeta_U(1) = \sum_{n=1}^{\infty} \frac{1}{U_{n-1}(t)} = 1 + \frac{1}{2t} + \frac{1}{4t^2-1} + \frac{1}{8t^3-4t} + \frac{1}{16t^4-12t^2+1} + \cdots,$$

with $t \neq 0, \cos\frac{k\pi}{n+1}$, and $k = 1, 2, \ldots, n$.

Theorem 11.

$$\sum_{n=1}^{\infty} \frac{t^n}{\{2n\}_{s,t}} = \sqrt{s^2+4t}\left(L\left(\varphi'^2_{s,t}/t\right) - L\left(\varphi'^4_{s,t}/t^2\right)\right),$$

where $L(q) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$ *is the Lambert function.*

Proof. Take into account that

$$\frac{t^n}{\{2n\}_{s,t}} = \sqrt{s^2+4t}\left(\frac{(\varphi'^2_{s,t}/t)^n}{1-(\varphi'^2_{s,t}/t)^n} - \frac{(\varphi'^4_{s,t}/t^2)^n}{1-(\varphi'^4_{s,t}/t^2)^n}\right)$$

and sum for all $n \geq 1$. □

Theorem 12.

$$\sum_{n=1}^{\infty} \frac{1}{\{2n-1\}_{s,1}} = -\frac{\sqrt{s^2+4}}{4}\theta_2(\varphi'^2_{s,1})^2,$$

where $\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}$.

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\{2n-1\}_{s,1}} &= -\frac{\sqrt{s^2+4}}{2}\sum_{n=1}^{\infty} \frac{2(\varphi'^2_{s,1})^{n-1/2}}{1+(\varphi'^2_{s,1})^{2n-1}} \\ &= -\frac{\sqrt{s^2+4}}{4}\sum_{a=-\infty}^{\infty} \frac{2(\varphi'^2_{s,1})^{a-1/2}}{1+(\varphi'^2_{s,1})^{2a-1}} \\ &= -\frac{\sqrt{s^2+4}}{4}\theta_2(\varphi'^2_{s,1})^2.\end{aligned}$$

□

Theorem 13.

$$\sum_{n=1}^{\infty} \frac{1}{\{n\}_{s,t}\{n+1\}_{s,t}} = \varphi_{s,t} - (1+t)\ln_{s,t}\left(1 + \frac{\varphi_{s,t}}{t}\right), \tag{11}$$

where

$$\ln_{s,t}(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{\{n\}_{s,t}}$$

is a Lucas analogue of the logarithm function.

Proof. Suppose that $0 < |q| < 1$, and $q = \varphi'_{s,t}/\varphi_{s,t}$, and set $a_n = 1/\{n\}_{s,t}\{n+1\}_{s,t}$. As

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\{n\}_{s,t}}{\{n+2\}_{s,t}} = \lim_{n\to\infty} \frac{1-q^n}{\varphi_{s,t}^2 - \varphi'^2_{s,t} q^n} = \frac{1}{\varphi_{s,t}^2},$$

then the series in the left-hand of Eq. (11) is convergent only if $|\varphi_{s,t}| > 1$. If $|q| > 1$, then the series in the left-hand of Eq. (11) is convergent only if $|\varphi'_{s,t}| > 1$. By partial fractions

$$\frac{1}{\{n\}_{s,t}\{n+1\}_{s,t}} = \frac{A}{\{n\}_{s,t}} + \frac{B}{\{n+1\}_{s,t}} = \frac{A}{\{n\}_{s,t}} + \frac{B}{\varphi_{s,t}\{n\}_{s,t} + \varphi'^n_{s,t}},$$

where $A = (-\varphi_{s,t}/t)^n$ and $B = (-\varphi_{s,t}/t)^{n+1}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\{n\}_{s,t}\{n+1\}_{s,t}} &= \sum_{n=1}^{\infty}\left(\frac{(-\varphi_{s,t}/t)^n}{\{n\}_{s,t}} + \frac{t(-\varphi_{s,t}/t)^{n+1}}{\{n+1\}_{s,t}}\right) \\
&= -\frac{\varphi_{s,t}}{t} + (1+t)\sum_{n=2}^{\infty} \frac{(-\varphi_{s,t}/t)^n}{\{n\}_{s,t}} \\
&= -\frac{\varphi_{s,t}}{t} - (1+t)\left(\ln_{s,t}\left(1+\frac{\varphi_{s,t}}{t}\right) - \frac{\varphi_{s,t}}{t}\right) \\
&= \varphi_{s,t} - (1+t)\ln_{s,t}\left(1+\frac{\varphi_{s,t}}{t}\right).
\end{aligned}$$

□

The function $\ln_{s,t}(1-x)$ is convergent for all $x \in (-|\varphi_{s,t}|, |\varphi_{s,t}|)$.

Some specializations of Theorem 13 are

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = \frac{1+\sqrt{5}}{2} - 2\ln_F\left(\frac{3+\sqrt{5}}{2}\right),$$

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n+1}} = 1 + \sqrt{2} - 2\ln_P(2+\sqrt{2}),$$

$$\sum_{n=1}^{\infty} \frac{1}{J_n J_{n+1}} = 2 - 3 \ln_J(2),$$
$$\sum_{n=1}^{\infty} \frac{1}{M_n M_{n+1}} = 2 + \ln_M(0),$$

where $\ln_F \equiv \ln_{1,1}$, $\ln_P \equiv \ln_{2,1}$, $\ln_J \equiv \ln_{1,2}$, and $\ln_M \equiv \ln_{3,-2}$. A very important remark is that $\ln_M(0)$ is finite.

Corollary 14.

$$\sum_{n=1}^{\infty} \frac{(-t)^n}{\{n\}_{s,t}\{n+1\}_{s,t}} = \frac{s+\sqrt{s^2+4t}}{2}.$$

Proof. If $t=-1$ in the previous theorem, then

$$\sum_{n=1}^{\infty} \frac{1}{\{n\}_{s,-1}\{n+1\}_{s,-1}} = \varphi_{s,-1}.$$

Note that $\{n\}_{s,-1} = (i/\sqrt{t})^{n-1}\{n\}_{a,t}$, where $s = ia/\sqrt{t}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\{n\}_{s,-1}\{n+1\}_{s,-1}} &= \sum_{n=1}^{\infty} \frac{1}{(i/\sqrt{t})^{2n-1}\{n\}_{a,t}\{n+1\}_{a,t}} \\ &= \frac{i}{\sqrt{t}} \sum_{n=1}^{\infty} \frac{(-t)^n}{\{n\}_{a,t}\{n+1\}_{a,t}} = \frac{i}{\sqrt{t}}\varphi_{a,t}.\end{aligned}$$

The proof is completed. □

Some specializations of Corollary 14 are

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1+\sqrt{5}}{2}.$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{P_n P_{n+1}} = 1+\sqrt{2}.$$
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{J_n J_{n+1}} = 2.$$
$$\sum_{n=1}^{\infty} \frac{2^n}{M_n M_{n+1}} = 2.$$
$$\sum_{n=1}^{\infty} \frac{1}{U_{n-1}(t)U_n(t)} = t+\sqrt{t^2-1}.$$

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