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On q-Analogs of the $3x+1$ Dynamical System

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Abstract

We construct an infinite family of maps on $F_2[[q]]$ that are conjugate to the famous $3x+1$ map on the 2-adic integers. We identify one such map that has the further property that the positive integers whose $3x+1$-orbit contains 1 correspond to polynomials via a conjugacy, and whose polynomial orbits all eventually enter the 2-cycle or one of the two fixed points. Proving that every positive integer corresponds to a polynomial via this conjugacy would settle the original $3x+1$ conjecture.

1 Introduction and background

A *discrete dynamical system* on a set X is obtained by iterating a map $f\colon X \to X$. For any function $f\colon X \to X$ and any $x \in X$, the f*-orbit* of x is the set $\left\{f^k(x) : k \in \mathbb{N}\right\}$ where f^0 is the identity map and $f^k = f \circ f^{k-1}$ for all positive integers k. Two dynamical systems $f\colon X \to X$ and $g\colon Y \to Y$ are *conjugate* (or *isomorphic* as dynamical systems) if and only if there exists a bijection $h\colon X \to Y$ such that $h \circ f = g \circ h$. Such a map h is called a *conjugacy*, and it is easy to see that conjugacies preserve the sizes of orbits as well as other properties of the dynamical system.

The famous $3x+1$ conjecture states that the T-orbit of every positive integer contains 1, where

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if 2 divides } x; \\ \frac{3x+1}{2}, & \text{otherwise.} \end{cases}$$

It has been an open problem in mathematics since the 1930s (see Lagarias [3] and Wirsching [8]).

More recently, several authors have studied a similar dynamical system, $T_q\colon F_2[q] \to F_2[q]$ where $F_2[q]$ is the ring of univariate polynomials in q with coefficients in the field with two elements, and for all $x \in F_2[q]$,

$$T_q(x) = \begin{cases} \frac{x}{q}, & \text{if } q \text{ divides } x; \\ \frac{(1+q)x+1}{q}, & \text{otherwise.} \end{cases}$$

There is a natural bijection $\xi\colon F_2[q] \to \mathbb{Z}^+$ that associates a q-polynomial $f(q)$ with the positive integer $f(2)$. Its inverse map sends a positive integer n to the q-polynomial obtained by replacing 2 with q in the base two representation of n. Note that the maps T and T_q have the same form if we replace q with 2 in the definition of T_q.

It was shown concurrently by Hicks, Mullen, Yucas, and Zavislak [2] and by Snapp and Tracy [7] that the T_q-orbit of any polynomial contains 1. The latter concluded that the analog of the $3x + 1$ conjecture holds for T_q [7, Theorem 2.1]. While the result is analogous if we identify q-polynomials p with their positive integer $\xi(p)$, this does not prove the $3x+1$ conjecture itself, since iterating T_q on any polynomial eventually reaches the fixed point 1, but iterating T on infinitely many positive integers eventually reaches the two-cycle $\{1, 2\}$. Thus, these two dynamical systems are not conjugate via any correspondence between the positive integers and polynomials.

In this paper, we show that the dynamical systems T and T_q actually are conjugate if we extend their domains to the ring of 2-adic integers, $\mathbb{Z}_2$, and the ring of formal power series in q, $F_2[[q]]$, respectively. Both dynamical systems have a unique two-cycle. Thus, to prove the $3x+1$ conjecture, it suffices to show for some conjugacy h between T and T_q that for any positive integer n, iterating T_q on $h(n)$ eventually reaches the unique two-cycle for T_q. Using h instead of ξ produces an analogous conjecture whose resolution would settle the original problem.

A similar situation occurs for any function that is conjugate to T. In particular, we will also show that if A, B are elements of $F_2[[q]]$ with constant term 1 (i.e., not divisible by q) then the map $T_{A,B}\colon F_2[[q]] \to F_2[[q]]$ given by

$$T_{A,B}(x) = \begin{cases} \frac{x}{q}, & \text{if } q \text{ divides } x; \\ \frac{Ax+B}{q}, & \text{otherwise} \end{cases}$$

is conjugate to T and the $3x + 1$ conjecture is true if and only if for any conjugacy $h\colon \mathbb{Z}_2 \to F_2[[q]]$ between T and $T_{A,B}$, and any positive integer n, the $T_{A,B}$-orbit of $h(n)$ eventually reaches the unique two-cycle for $T_{A,B}$. Note that T_q is $T_{1+q,1}$ in this notation.

Each element of $\mathbb{Z}_2$ is a formal sum $\sum_{i=0}^{\infty} x_i 2^i$ where $x_i \in \{0, 1\}$ for all $i \in \mathbb{N}$. Thus, we can extend ξ to a bijective correspondence between $F_2[[q]]$ and $\mathbb{Z}_2$ by associating a formal power series $f(q) \in F_2[[q]]$ to the 2-adic integer $f(2)$. We will refer to the elements of $F_2[[q]]$ as *q-series* in what follows for brevity.

The *binary representation* of either a 2-adic integer $x = \sum_{i=0}^{\infty} x_i 2^i$ or a q-series $x = \sum_{i=0}^{\infty} x_i q^i$ is the infinite sequence of zeroes and ones $x_0 x_1 x_2 \cdots$. By identifying 2-adic integers and q-series with their binary representations, we can consider both T and T_q to be dynamical systems from $\Sigma_2 \to \Sigma_2$, where Σ_2 is the set of all infinite binary sequences. We will say that a 2-adic integer or q-series is *even* if and only if the leading digit of its binary representation is 0 (which implies it is divisible by 2 or q, respectively), and *odd* otherwise. Similarly, the *parity* of an element is either 0 or 1 depending on whether the element is even or odd, respectively.

The ordinary rational numbers with odd denominators (when written as reduced fractions) are isomorphic to a subring of $\mathbb{Z}_2$ and correspond to the 2-adic integers with eventually repeating binary representations via the geometric series formula. We can use this as a compact way to represent a q-series whose binary representation is eventually repeating (keeping in mind that ordinary addition and multiplication of rational numbers correspond to addition and multiplication in $\mathbb{Z}_2$, not in $F_2[[q]]$). Also note that we can express such power series as rational expressions in q with coefficients in F_2.

Example 1. Let $P_1 = 1+q+q^3+q^4+q^6+q^7+\cdots = \frac{1+q}{1+q^3} \in F_2[[q]]$. The binary expansion of P_1 is $110110110\cdots$ which we can abbreviate as $\overline{110}$. But $\overline{110}$ is also the binary representation of the 2-adic $1 + 2 + 2^3 + 2^4 + 2^6 + 2^7 + \cdots$ which equals $-3/7$ by the geometric series formula. Thus, we have the correspondence $\xi(P_1) = -3/7$. Notice that $\xi(f)$ is a positive integer whenever $f \in F_2[[q]]$ is a polynomial, which is the correspondence assumed by Snapp and Tracy [7, Theorem 2.1] when referring to the analogous conjecture.

Notice that in this situation f is odd if and only if the numerators of both f and $\xi(f)$ are odd when each is written as a reduced fraction. So the parity of f is always the same as the parity of the numerators of the reduced fractions for f and $\xi(f)$.

2 Main results

While the correspondence ξ is a natural one to consider when comparing the dynamical systems T and T_q, it is not a conjugacy even when their domains are extended to $\mathbb{Z}_2$ and $F_2[[q]]$, respectively. That such conjugacies do exist, however, is the point of our first main result. For the remainder of this paper, we assume that $T\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ and $T_{A,B}\colon F_2[[q]] \to F_2[[q]]$.

Theorem 2. *For any odd $A, B \in F_2[[q]]$, the map $T_{A,B}$ is conjugate to T.*

Corollary 3. *The map T_q is conjugate to T.*

Thus, for any conjugacy h between T and $T_{A,B}$, the power series that correspond to the positive natural numbers are those in $h(\mathbb{N}^+)$. For T_q, this set does not include any polynomials, as all nonzero polynomial orbits end in the fixed point 1 of T_q, but no positive integer has a T-orbit that contains the fixed points 0 or -1 of T.

Similarly, any conjugacy between T and T_q must map 1 to an element of the unique 2-cycle of T_q, namely $\{P_1, qP_1\}$ (or equivalently $\{-3/7, -6/7\}$). If the conjugacy preserves parity, then for T_q it is $-3/7$ that corresponds to 1 for T, and so the analogous conjecture can be stated as follows:

Conjecture 4. For any conjugacy $h\colon \mathbb{Z}_2 \to F_2[[q]]$ between T and T_q and any positive integer n, the T_q-orbit of $h(n)$ contains $-3/7$.

It follows from the results of Monks and Yazinski [4] that there exists a parity-preserving conjugacy h such that $h(1) = -3/7$. For any such h and any positive integer n, we can calculate the corresponding power series $h(n)$ for T_q. If we can show that all such power series have $-3/7$ in their T_q orbit, then it will prove the $3x+1$ conjecture. We list the first few such values in Table 1.

Given that polynomials do not correspond to positive integers for any conjugacy between T and T_q, a natural question to ask is whether such conjugacies might exist for some other maps $T_{A,B}$. Our second main result answers this in the affirmative.

Theorem 5. *The $T_{1,1+q^2}$-orbit of any polynomial contains the 2-cycle $\{1, q\}$ or one of the two fixed points 0 or $1+q$. Furthermore, $T_{1,1+q^2}(x)$ is a polynomial if and only if x is a polynomial.*[1]

Since the only 2-cycle for T is $\{1, 2\}$ and no positive integer has a T-orbit that contains the fixed points 0 or -1, it follows that for any conjugacy between T and $T_{1,1+q^2}$, the positive integers whose T-orbit contains 1 correspond to polynomials. Thus, to prove the $3x+1$ conjecture, it suffices to prove the following.

Conjecture 6. For any conjugacy $H\colon \mathbb{Z}_2 \to F_2[[q]]$ between T and $T_{1,1+q^2}$ and any positive integer n, the power series $H(n)$ is a polynomial.

Equivalently, the $3x+1$ conjecture is true if and only if $\xi(H(n))$ is an integer for every positive integer n. We provide the first few such values in Table 2 for any parity-preserving conjugacy, H. Given that Conjecture 6 is equivalent to the original $3x+1$ conjecture, it is not surprising that no integer sequence starting with the first twenty values of $\xi(H(n))$ appears in the OEIS [6].

Notice that arithmetic in $F_2[[q]]$ is somewhat simpler than it is in $\mathbb{Z}_2$, because addition and multiplication of their binary representations are binary arithmetic without carrying. For example, in $F_2[[q]]$ we have $3+6=5$, $3\cdot 6 = 3\cdot 4 + 3\cdot 2 = 10$. In this notation, since $\xi(1+q) = 3$, $\xi(1) = 1$, and $\xi(q) = 2$, the formula for T_q becomes

$$T_q(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even;} \\ \frac{3x+1}{2}, & \text{otherwise} \end{cases}$$

which looks exactly like the formula for T except that the addition and multiplication are the corresponding binary operations without carrying. Similarly, the formula for $T_{1,1+q^2}$ becomes

$$T_{1,1+q^2}(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even;} \\ \frac{x+5}{2}, & \text{otherwise} \end{cases}$$

with the same simpler addition and multiplication.

As the results of Hicks, Mullen, Yucas, and Zavislak [2], Snapp and Tracy [7], and Theorem 5 show, working with these simpler binary operations can allow us to more easily determine the behavior of the orbits. Thus, determining a closed formula for the correspondence shown in either Table 1 or Table 2 could very well lead to a solution to the original conjecture itself.

[1] A similar theorem holds more generally for any $T_{A,B}$, where B is a polynomial multiple of $A + q^2$.

2.1 Proof of Theorem 2

It is well known (cf. Lagarias [3]) that T is conjugate to the *shift map* $\sigma\colon \mathbb{Z}_2 \to \mathbb{Z}_2$, given by

$$\sigma(x) = \begin{cases} \frac{x}{2}, & x \text{ is even;} \\ \frac{x-1}{2}, & x \text{ is odd.} \end{cases}$$

Notice that if $x_0x_1x_2\cdots$ is the binary representation of x, then $\sigma(x_0x_1x_2\cdots) = x_1x_2x_3\cdots$, i.e., σ "shifts" the binary digits by deleting the first one. This makes it very easy to determine, for example, the number of cycles with n elements for a fixed n. In particular, σ has two fixed points $\{\overline{0}\}$ and $\{\overline{1}\}$, a unique cycle of size two $\{\overline{01}, \overline{10}\}$, and two cycles of size three $\{\overline{110}, \overline{011}, \overline{101}\}$ and $\{\overline{001}, \overline{100}, \overline{010}\}$.

Since conjugacy is an isomorphism in the category of discrete dynamical systems, to prove Theorem 2, it suffices to show that $T_{A,B}$ is conjugate to σ. We begin with a few easy lemmas.

Lemma 7. *Let $n \in \mathbb{N}$ and $U, V \in F_2[[q]]$ with V odd. Then*

$$T_{A,B}(U + q^{n+1}V) = T_{A,B}(U) + q^nW$$

for some odd $W \in F_2[[q]]$.

Proof. Let $n \in \mathbb{N}$ and $U, V \in F_2[[q]]$ with V odd. Either U is even or U is odd.

Case 1: Assume U is even. Then $U + Vq^{n+1}$ is even as well. So

$$\begin{aligned} T_{A,B}(U + q^{n+1}V) &= \frac{U + q^{n+1}V}{q} \\ &= \frac{U}{q} + q^nV \\ &= T_{A,B}(U) + q^nV \end{aligned}$$

and V is odd.

Case 2: Assume U is odd. Then $U + Vq^{n+1}$ is odd as well. So

$$\begin{aligned} T_{A,B}(U + q^{n+1}V) &= \frac{A(U + q^{n+1}V) + B}{q} \\ &= \frac{AU + B}{q} + q^nAV \\ &= T_{A,B}(U) + q^nAV, \end{aligned}$$

and AV is odd since both A and V are.

So in both cases, the result follows. □

Corollary 8. *Let $n \in \mathbb{N}$ and $U, V \in F_2[[q]]$ with V odd. Then for all $k \leq n+1$*

$$T^k_{A,B}(U + q^{n+1}V) = T^k_{A,B}(U) + q^{n+1-k}W_k$$

for some odd $W_k \in F_2[[q]]$.

Proof. We proceed by induction on k for arbitrary n. For the base case, we have

$$\begin{aligned} T^0_{A,B}(U + q^{n+1}V) &= U + q^{n+1}V \\ &= T^0_{A,B}(U) + q^{n+1-0}V \end{aligned}$$

where V is odd.

For the inductive step, let $k \le n$ and assume

$$T^k_{A,B}(U + q^{n+1}V) = T^k_{A,B}(U) + q^{n+1-k}W_k$$

for some odd W_k. Then

$$\begin{aligned} T^{k+1}_{A,B}(U + q^{n+1}V) &= T_{A,B}(T^k_{A,B}(U + q^{n+1}V)) \\ &= T_{A,B}(T^k_{A,B}(U) + q^{n+1-k}W_k) \\ &= T^{k+1}_{A,B}(U) + q^{n-k}W_{k+1} \\ &= T^{k+1}_{A,B}(U) + q^{n+1-(k+1)}W_{k+1} \end{aligned}$$

for some odd W_{k+1} by Lemma 7, which completes the induction. □

This gives us the next interesting corollary, which essentially says that changing the coefficient of q^n in a q-series changes the parity of its n^{th} iterate. We write $\underset{m}{\equiv}$ to indicate congruence with modulus m.

Corollary 9. *Let $n \in \mathbb{N}$ and $U, V \in F_2[[q]]$ with V odd. Then $T^n_{A,B}(U + q^nV)$ and $T^n_{A,B}(U)$ have opposite parity, i.e.,*

$$T^n_{A,B}(U + q^nV) \underset{q}{\not\equiv} T^n_{A,B}(U).$$

Proof. We consider two cases, either $n = 0$ or $n > 0$.

If $n = 0$ then since V is odd, $U + V \underset{q}{\not\equiv} U$, and so

$$\begin{aligned} T^n_{A,B}(U + q^nV) &= T^0_{A,B}(U + q^0V) \\ &= U + V \\ &\underset{q}{\not\equiv} U \\ &= T^0_{A,B}(U). \end{aligned}$$

If $n > 0$, then by the previous corollary, there exists an odd $W \in F_2[[q]]$ such that $T^n_{A,B}(U + q^nV) = T^n_{A,B}(U) + q^{n-n}W$, so

$$\begin{aligned} T^n_{A,B}(U + q^nV) &= T^n_{A,B}(U) + q^{n-n}W \\ &= T^n_{A,B}(U) + W \\ &\underset{q}{\not\equiv} T^n_{A,B}(U). \end{aligned}$$

□

Definition 10. For $x \in F_2[[q]]$, define *the parity vector* $\mathrm{PV}_{A,B}(x)$ to be the 2-adic integer whose binary representation is $x_0x_1x_2\cdots$ where for each $k \in \mathbb{N}$, $x_k \in \{0,1\}$ is the parity of $T^k_{A,B}(x)$.

We abbreviate $\mathrm{PV}_{A,B}$ as PV when the subscripts are clear from the context.

Theorem 11. *For $n \in \mathbb{N}$ and $a, b \in F_2[[q]]$,*

$$\mathrm{PV}(a) \underset{2^{n+1}}{\equiv} \mathrm{PV}(b) \Leftrightarrow a \underset{q^{n+1}}{\equiv} b.$$

Proof. We proceed by induction on n.

For the base case, if $n = 0$, then for any $a, b \in F_2[[q]]$, $T^0_{A,B}(a) = a$ and $T^0_{A,B}(b) = b$. So $\mathrm{PV}(a) \underset{2^{0+1}}{\equiv} \mathrm{PV}(b)$ if and only if $a \underset{q^{0+1}}{\equiv} b$.

For the inductive hypothesis, let $k \in \mathbb{N}$ and assume for all $a, b \in F_2[[q]]$,

$$\mathrm{PV}(a) \underset{2^{k+1}}{\equiv} \mathrm{PV}(b) \Leftrightarrow a \underset{q^{k+1}}{\equiv} b.$$

Let $a, b \in F_2[[q]]$.

($\Leftarrow$) Assume $a \underset{q^{k+2}}{\equiv} b$. Then $a = p + q^{k+1}c$ and $b = p + q^{k+1}d$ for some $p, c, d \in F_2[[q]]$ such that p is a polynomial of degree less than $k+1$ and $c \underset{q}{\equiv} d$ (i.e., they have the same parity). Clearly,

$$\begin{aligned} a &= p + q^{k+1}c \\ &\underset{q^{k+1}}{\equiv} p \\ &\underset{q^{k+1}}{\equiv} p + q^{k+1}d \\ &= b, \end{aligned}$$

so by the inductive hypothesis $\mathrm{PV}(a) \underset{2^{k+1}}{\equiv} \mathrm{PV}(b)$.

Additionally, by Corollary 8,

$$\begin{aligned} T^{k+1}_{A,B}(a) &= T^{k+1}_{A,B}(p + q^{k+1}c) \\ &= T^{k+1}_{A,B}(p) + q^0c \\ &= T^{k+1}_{A,B}(p) + c \\ &\underset{q}{\equiv} T^{k+1}_{A,B}(p) + d \\ &= T^{k+1}_{A,B}(p) + q^0d \\ &= T^{k+1}_{A,B}(p + q^{k+1}d) \\ &= T^{k+1}_{A,B}(b). \end{aligned}$$

Combining these two results shows $\mathrm{PV}(a) \underset{2^{k+2}}{\equiv} \mathrm{PV}(b)$ as desired.

($\Rightarrow$) Now assume $\mathrm{PV}(a) \underset{2^{k+2}}{\equiv} \mathrm{PV}(b)$. Then $\mathrm{PV}(a) \underset{2^{k+1}}{\equiv} \mathrm{PV}(b)$, so by the induction hypothesis $a \underset{q^{k+1}}{\equiv} b$.

Assume $a \underset{q^{k+2}}{\not\equiv} b$. Since $a \underset{q^{k+1}}{\equiv} b$, it must be that $a = p + q^{k+1}c$ and $b = p + q^{k+1}d$ for some $p, c, d \in F_2[[q]]$ such that p is a polynomial of degree less than $k+1$ and $c \underset{q}{\not\equiv} d$ (i.e., c, d have opposite parity). Thus, again by Corollary 8,

$$\begin{aligned}
T^{k+1}_{A,B}(a) &= T^{k+1}_{A,B}(p + q^{k+1}c) \\
&= T^{k+1}_{A,B}(p) + q^0 c \\
&= T^{k+1}_{A,B}(p) + c \\
&\underset{q}{\not\equiv} T^{k+1}_{A,B}(p) + d \\
&= T^{k+1}_{A,B}(p) + q^0 d \\
&= T^{k+1}_{A,B}(p + q^{k+1}d) \\
&= T^{k+1}_{A,B}(b)
\end{aligned}$$

so $\mathrm{PV}(a) \underset{2^{k+2}}{\not\equiv} \mathrm{PV}(b)$, which is a contradiction. So $a \underset{q^{k+2}}{\equiv} b$ as desired. □

It follows that PV induces a bijection from $F_2[[q]]/q^n$ to $\mathbb{Z}_2/2^n$ for any positive n.

Corollary 12. *Let $n \in \mathbb{N}^+$. For any $a \in F_2[[q]]$, define $\widehat{\mathrm{PV}}([a]) = [\mathrm{PV}(a)]$ where $[a]$ is the equivalence class of a in $F_2[[q]]/q^n$ and $[\mathrm{PV}(a)]$ is the equivalence class of $\mathrm{PV}(a)$ in $\mathbb{Z}_2/2^n$. Then $\widehat{\mathrm{PV}}\colon F_2[[q]]/q^n \to \mathbb{Z}_2/2^n$ is a bijection.*

Proof. By Theorem 11, $\widehat{\mathrm{PV}}$ maps to the same equivalence class for all elements in the same equivalence class, so $\widehat{\mathrm{PV}}$ is a well-defined function.

If $[a], [b] \in F_2[[q]]/q^n$ and $\widehat{\mathrm{PV}}([a]) = \widehat{\mathrm{PV}}([b])$, then

$$\begin{aligned}
[\mathrm{PV}(a)] &= \widehat{\mathrm{PV}}([a]) \\
&= \widehat{\mathrm{PV}}([b]) \\
&= [\mathrm{PV}(b)].
\end{aligned}$$

Thus, $\mathrm{PV}(a) \underset{2^n}{\equiv} \mathrm{PV}(b)$. Therefore, $a \underset{q^n}{\equiv} b$ by Theorem 11 and thus $[a] = [b]$. So $\widehat{\mathrm{PV}}$ is injective.

Finally, both $F_2[[q]]/q^n$ and $\mathbb{Z}_2/2^n$ are finite sets with 2^n elements, so any injective function between them must be bijective as well. □

Taking the inverse limit of these maps with respect to n produces the desired conjugacy needed to prove our main result.

Proof (of Theorem 2). We will show that $\mathrm{PV}\colon F_2[[q]] \to \mathbb{Z}_2$ is a conjugacy between $T_{A,B}$ and the shift map σ.

Let $a, b \in F_2[[q]]$ and assume $\mathrm{PV}(a) = \mathrm{PV}(b)$. Then for all positive n, $\mathrm{PV}(a) \underset{2^n}{\equiv} \mathrm{PV}(b)$ and thus $a \underset{q^n}{\equiv} b$ by Theorem 11. But two elements of $F_2[[q]]$ which are congruent mod q^n for all n must be equal. Therefore, $a = b$ and so PV is injective.

To see that PV is surjective, let $x \in \mathbb{Z}_2$. Then by Corollary 12, for each positive integer n there exists $a_n \in F_2[[q]]$ such that $\mathrm{PV}(a_n) \underset{2^n}{\equiv} x$. Since $s \underset{2^n}{\equiv} x$ implies $s \underset{2^m}{\equiv} x$ for all $m \leq n$, by Theorem 11, $a_n \underset{q^n}{\equiv} a_m$ for all $m \leq n$. So there is a unique $a \in F_2[[q]]$ such that $a \underset{q^n}{\equiv} a_n$ and $\mathrm{PV}(a) \underset{2^n}{\equiv} x$ for all n. Thus, $\mathrm{PV}(a) = x$ as desired.

Finally, to see that PV is a conjugacy, let $a \in F_2[[q]]$. Then by the definition of parity vector,

$$\begin{aligned}(\mathrm{PV} \circ T_{A,B})(a) &= \mathrm{PV}(T_{A,B}(a)) \\ &= \sigma(\mathrm{PV}(a)) \\ &= (\sigma \circ \mathrm{PV})(a).\end{aligned}$$

So $T_{A,B}$ is conjugate to the shift map via conjugacy PV. But it is well known (cf. Lagarias [3]) that the shift map is conjugate to T with conjugacy $\Phi \colon \mathbb{Z}_2 \to \mathbb{Z}_2$, and so $T_{A,B}$ is conjugate to T with conjugacy $h = \Phi \circ \mathrm{PV}$. ☐

This proves Theorem 2. Corollary 3 follows immediately as a special case since $T_q = T_{1+q,1}$.

2.2 Proof of Theorem 5

To determine which $T_{A,B}$ are conjugate to T via a conjugacy that maps polynomials to positive integers, we can first ask which $T_{A,B}$ have a polynomial 2-cycle, since such a 2-cycle will be unique and correspond to the unique 2-cycle $\{1, 2\}$ for T.

Lemma 13. *The unique 2-cycle for $T_{A,B}$ is $\left\{\frac{B}{A+q^2}, \frac{qB}{A+q^2}\right\}$, and its fixed points are 0 and $\frac{B}{A+q}$.*

Proof. Since $T_{A,B}$ is conjugate to the shift map, it must also have a unique 2-cycle and two fixed points. These elements must all satisfy $T_{A,B}(T_{A,B}(x)) = x$. If both x and $T_{A,B}(x)$ are even, then $x = T_{A,B}(T_{A,B}(x)) = (x/q)/q$, so $x = 0$. If both x and $T_{A,B}(x)$ are odd, then $x = T_{A,B}(T_{A,B}(x)) = (A((Ax+B)/q)+B)/q$, so $x = \frac{B}{A+q}$. Finally, if x is even and $T_{A,B}(x)$ is odd, then $x = T_{A,B}(T_{A,B}(x)) = (A((x/q)+B)/q$, so $x = \frac{qB}{A+q^2}$ and $T_{A,B}(x) = \frac{B}{A+q^2}$ (reversing the parities just swaps x and $T_{A,B}(x)$). ☐

Thus, the 2-cycle of $T_{A,B}$ will consist of polynomials whenever $\frac{B}{A+q^2}$ is a polynomial, i.e., whenever B is a polynomial multiple of $A + q^2$. The simplest example of this family of dynamical systems is when $A = 1$ and $B = 1 + q^2$. In that case, the 2-cycle of $T_{1,1+q^2}$ is $\{1, q\}$ and its fixed points are 0 and $1 + q$. Thus, to prove Theorem 5 it suffices to show that every polynomial eventually enters either the 2-cycle or one of the two fixed points, and that $T_{1,1+q^2}$ maps every polynomial, and only polynomials, to other polynomials.

Proof (of Theorem 5). Let $x \in F_2[[q]]$ be a polynomial of degree $n > 1$. If x is even then $T_{1,1+q^2}(x) = x/q$ is a polynomial of degree $n - 1$. If x is odd then $x = P + q^n$ for some odd polynomial P of degree less than n (i.e., $P = x + q^n$). So

$$\begin{aligned} T_{1,1+q^2}(x) &= T_{1,1+q^2}(P + q^n) \\ &= \frac{(P + q^n) + 1 + q^2}{q} \\ &= \frac{1 + P}{q} + q + q^{n-1}. \end{aligned}$$

But $(1 + P)/q + q + q^{n-1}$ is either 0 or is a polynomial of degree less than n because P is odd.

In both cases, the degree of $T_{1,1+q^2}(x)$ is strictly less than the degree of x. So the $T_{1,1+q^2}$-orbit of every polynomial of degree greater than 1 is a sequence of polynomials of decreasing degrees and thus must eventually reach 0, 1, q, or $1 + q$, which are the two fixed points and the two elements of the 2-cycle as claimed.

For the second part of the theorem, let $x \in F_2[[q]]$. If x is an even polynomial, then $x = qP$ for some polynomial P, so $T_{A,B}(x) = x/q = qP/q = P$ is a polynomial. If x is odd then $x = 1+qP$ for some polynomial P, so $T_{1,1+q^2}(x) = T_{1,1+q^2}(1+qP) = (1+qP+1+q^2)/q = P+q$ which is also a polynomial.

Conversely, let $x \in F_2[[q]]$ such that $T_{1,1+q^2}(x)$ is a polynomial. If x is even then $T_{1,1+q^2}(x) = x/q$, so $x = qT_{1,1+q^2}(x)$ is a polynomial. Similarly, if x is odd then $T_{1,1+q^2}(x) = (x + 1 + q^2)/q$ so $x = qT_{1,1+q^2}(x) + 1 + q^2$ which, once again, is a polynomial.

This completes the proof. □

In conclusion, recent authors have shown that the map T_q on $F_2[q]$ has the property that the T_q-orbit of every polynomial contains 1. This does not prove the original $3x + 1$ conjecture, however, because the correspondence ξ between nonzero polynomials and positive integers is not a conjugacy of discrete dynamical systems.

By extending the domains of T_q and T to $F_2[[q]]$ and $\mathbb{Z}_2$, respectively, we find that T_q and T are actually conjugate as discrete dynamical systems, but not via a conjugacy that maps polynomials to positive integers. This result extends to an infinite family of maps on $F_2[[q]]$ which are all conjugate to T, and some of these, such as $T_{1,1+q^2}$, do have the property that the positive integers whose T-orbit contains 1 correspond to polynomials.

By applying the correspondence ξ to the polynomials that correspond to the positive integers whose T-orbit contains 1, we obtain the sequence $\xi(H(n))$ that starts with

$$1, 2, 47, 4, 21, 94, 2763, 8, 11049, 42, 1383, 188, 173, 5526, 5427, 16, 689, 22098, 22143, 84, \ldots$$

Showing that every term in this sequence is an integer, or equivalently, proving that every positive integer corresponds to a polynomial via a conjugacy with $T_{1,1+q^2}$, would resolve the original $3x + 1$ conjecture.

n	$\xi(h(n))$	$h(n)$
1	$-3/7$	$(1+q)\cdot\frac{1}{(1+q^3)}$
2	$-6/7$	$(q+q^2)\cdot\frac{1}{(1+q^3)}$
3	$-13/7$	$1+(1+q^2)\cdot\frac{q^2}{(1+q^3)}$
4	$-12/7$	$(q+q^2)\cdot\frac{q}{(1+q^3)}$
5	$9/7$	$1+q+(1+q)\cdot\frac{q^2}{(1+q^3)}$
6	$-26/7$	$q+(1+q^2)\cdot\frac{q^3}{(1+q^3)}$
7	$-175/9$	$1+q^3+(q^3+q^4+q^5)\cdot\frac{q^5}{(1+q^6)}$
8	$-24/7$	$(q+q^2)\cdot\frac{q^2}{(1+q^3)}$
9	$1187/65$	$1+q+q^5+(1+q+q^2+q^3+q^5+q^{10})\cdot\frac{q^6}{(1+q^{12})}$
10	$18/7$	$q+q^2+(1+q)\cdot\frac{q^3}{(1+q^3)}$
11	$-37/7$	$1+q^2+q^3+q^2\cdot\frac{q^5}{(1+q^3)}$
12	$-52/7$	$q^2+(1+q^2)\cdot\frac{q^4}{(1+q^3)}$
13	$-31/7$	$1+q+q^2+(1+q^2)\cdot\frac{q^4}{(1+q^3)}$
14	$-350/9$	$q+q^4+(q^3+q^4+q^5)\cdot\frac{q^6}{(1+q^6)}$
15	$-359/9$	$1+q^4+(q^3+q^4+q^5)\cdot\frac{q^6}{(1+q^6)}$
16	$-48/7$	$(q+q^2)\cdot\frac{q^3}{(1+q^3)}$
17	$-115/7$	$1+q+q^3+(q+q^2)\cdot\frac{q^5}{(1+q^3)}$
18	$2374/65$	$q+q^2+q^6+(1+q+q^2+q^3+q^5+q^{10})\cdot\frac{q^7}{(1+q^{12})}$
19	$2309/65$	$1+q^2+q^6+(1+q+q^2+q^3+q^5+q^{10})\cdot\frac{q^7}{(1+q^{12})}$
20	$36/7$	$q^2+q^3+(1+q)\cdot\frac{q^4}{(1+q^3)}$

Table 1: The power series in $F_2[[q]]$ that correspond to the first twenty positive integers n via any parity-preserving conjugacy $h\colon \mathbb{Z}_2 \to F_2[[q]]$ between T and T_q.

n	$\xi(H(n))$	$H(n)$
1	1	1
2	2	q
3	47	$q^5+q^3+q^2+q+1$
4	4	q^2
5	21	q^4+q^2+1
6	94	$q^6+q^4+q^3+q^2+q$
7	2763	$q^{11}+q^9+q^7+q^6+q^3+q+1$
8	8	q^3
9	11049	$q^{13}+q^{11}+q^9+q^8+q^5+q^3+1$
10	42	q^5+q^3+q
11	1383	$q^{10}+q^8+q^6+q^5+q^2+q+1$
12	188	$q^7+q^5+q^4+q^3+q^2$
13	173	$q^7+q^5+q^3+q^2+1$
14	5526	$q^{12}+q^{10}+q^8+q^7+q^4+q^2+q$
15	5427	$q^{12}+q^{10}+q^8+q^5+q^4+q+1$
16	16	q^4
17	689	$q^9+q^7+q^5+q^4+1$
18	22098	$q^{14}+q^{12}+q^{10}+q^9+q^6+q^4+q$
19	22143	$q^{14}+q^{12}+q^{10}+q^9+q^6+q^5+q^4+q^3+q^2+q+1$
20	84	$q^6+q^4+q^2$

Table 2: The polynomials in $F_2[[q]]$ that correspond to the first twenty positive integers n via any parity-preserving conjugacy $H\colon \mathbb{Z}_2 \to F_2[[q]]$ between T and $T_{1,1+q^2}$.

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