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On Reduced Unicellular Hypermonopoles

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Abstract

The problem of counting unicellular hypermonopoles by the number of their hyperedges is equivalent to describing the cycle length distribution of a product of two cyclic permutations, first solved by Zagier. The solution of this problem has also been used in the study of the cycle graph model of Bafna and Pevzner, and of related models in mathematical biology. In this paper we develop a method to compute the finite number of reduced unicellular hypermonopoles of a given genus. The problem of representing any hypermap as a drawing is known to be simplifiable to solving the same problem for reduced unicellular hypermonopoles. We also outline a correspondence between our hypermap model, the cycle graph model of Bafna and Pevzner, and the polygon gluing model of Alexeev and Zograf. Reduced unicellular hypermonopoles correspond to reduced objects in the other models as well, and the notion of genus is the same.

1 Introduction

In the study of the combinatorics of the symmetric group many authors have been interested in the statistics of cycle lengths of products of pairs of permutations. In this note we revisit the particular case of counting the cycles of the product of two cyclic permutations, or dually finding the number of decompositions of a given permutation as a product of two cyclic permutations.

Our renewed interest in this topic comes from the study of *hypermaps*. In a recent paper [7] we showed that the problem of drawing a hypermap may be reduced to considering the same problem for a *unicellular hypermonopole* of the same genus. A hypermap $H = (\sigma, \alpha)$ is a pair of permutations generating a transitive permutation group. A hypermonopole is a hypermap with a single vertex; that is, σ is a cyclic permutation. It is called *unicellular* if it has only one face, meaning that $\alpha^{-1}\sigma$ is also a cyclic permutation. The main result of this short paper concerns the enumeration of unicellular hypermonopoles without *buds*, meaning that α has no fixed point, we call such a unicellular hypermonopole *reduced*.

Our main tool is the enumeration formula obtained by Zagier [18] for the number of unicellular hypermonopoles having a given number of cycles. Notice that the number of cycles k and the genus g of a unicellular hypermonopole of S_n satisfy $k = n - 2g$. A combinatorial bijective proof of Zagier's formula was given by Cori, Marcus, and Schaeffer [8]. In the field of genome rearrangements, Bafna and Pevzner [2] introduced a cycle graph model that is essentially equivalent to studying the product of a pair of cyclic permutations. Their aim was to determine the minimum number of "transpositions"[1] needed to reduce a permutation (considered as a word) to the identity permutation. This allows to model the evolution of the DNA of some viruses. These questions have motivated several researchers to focus on the products of cyclic permutations. This is the case of the work of A. Hultman in his thesis [11], and more recently that of Alexeev and Zograf [1] who introduced gluings of polygons to represent these products.

Our paper is organized as follows. In the Preliminaries we recall the notions of the theory of hypermaps and state Zagier's formula and some of its reformulations. Section 3 is devoted to the description of the relationship between Zagier's factorization problem, counting unicellular hypermonopoles, and the models of Bafna and Pevzner [2] and of Alexeev and Zograf [1]. In Section 4 we give our main result expressing the number of reduced unicellular hypermonopoles on n points with a given number of hyperedges. We already observed in [7] that the number of all reduced unicellular hypermonopoles of a given genus is finite. Our main result allows to count these finite numbers explicitly, and we give the first values of these numbers.

It is worth noting that reduced unicellular hypermonopoles correspond to cycle graphs having breakpoints everywhere in the model of Bafna and Pevzner [2]. Furthermore, the genus of the corresponding polygon gluing diagram in the work of Alexeev and Zograf [1] is the same as the genus of the corresponding unicellular hypermonopole. The three models are intimately related, hence the counting problem we solved has also some significance in the related models in mathematical biology.

[1]The transpositions in biology are different from those considered in algebra; they displace an interval of the word representing the permutation.

2 Preliminaries

2.1 Hypermaps and their two disk diagrams

A *hypermap* (σ, α) is a pair of permutations of a set $\{1, 2, \ldots, n\}$ generating a transitive permutation group. It is used to represent a (connected) hypergraph on an oriented surface. The cycles of σ are the vertices, the cycles of α are the hyperedges, and the cycles of $\alpha^{-1}\sigma$

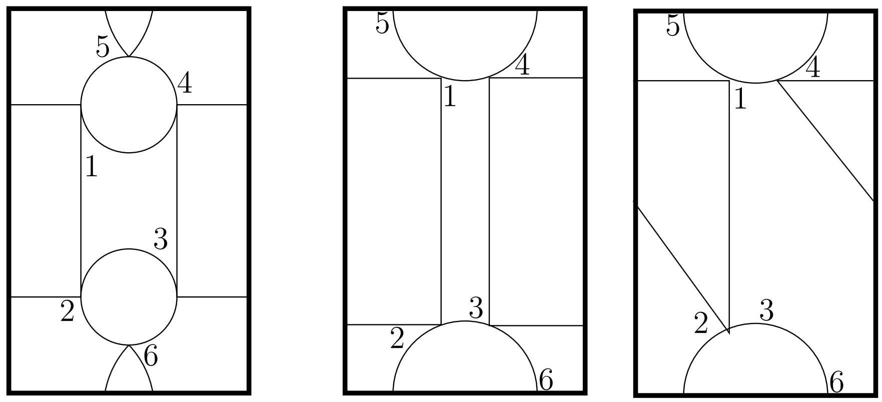

Figure 1: Three hypermaps drawn on the torus.

are the faces. The *points* $1, 2, \ldots, n$ are the points of incidence between the vertices and the hyperedges. Three hypermaps drawn on the torus are shown in Figure 1. The leftmost drawing represents the hypermap (σ, α) with $\sigma = (1, 4, 5)(2, 6, 3)$ and $\alpha = (1, 2, 3, 4)(5, 6)$. The description of the other two drawings will follow below. To obtain three tori, the parallel sides of the rectangles shown in bold should be identified. In our drawings, points follow in counterclockwise order along the vertices and in the clockwise order along the hyperedges. The *genus* $g(\sigma, \alpha)$ of the oriented surface on which a hypermap may be drawn with noncrossing lines is given by the following formula due to Jacques [12]:

$$n + 2 - 2g(\sigma, \alpha) = z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma), \tag{1}$$

where $z(\pi)$ denotes the number of cycles of the permutation π. A detailed introduction to the theory of hypermaps may be found in the survey papers of Cori and Machì[6].

This paper is motivated by the following observation in [7]: using a sequence of *topological hyperdeletions* $(\sigma, \alpha) \mapsto (\sigma, \alpha\delta)$ and of *topological hypercontractions* $(\sigma, \alpha) \mapsto (\gamma\sigma, \gamma\alpha)$, each hypermap may be reduced to a hypermap (σ', α') of the same genus, such that $z(\sigma') = 1$; that is, (σ', α') is a *hypermonopole*, and $z(\alpha'^{-1}\sigma') = 1$; that is, (σ', α') is *unicellular*. By definition $(\sigma, \alpha) \mapsto (\sigma, \alpha\delta)$ is a topological hyperdeletion when $\delta = (i, j)$ is a transposition such that i and j belong to the same cycle of α but to different cycles of $\alpha^{-1}\sigma$. Similarly $(\sigma, \alpha) \mapsto (\gamma\sigma, \gamma\alpha)$ is a topological hypercontraction when $\gamma = (i, j)$ is a transposition such that i and j belong to the same cycle of α and to different cycles of σ. We refer the interested reader for the detailed description of the process to [7]. Here we illustrate it in Figure 1. First we apply the topological hypercontraction $(\sigma,\alpha)\mapsto(\gamma\sigma,\gamma\alpha)$ with $\gamma=(5,6)$ thus obtaining the hypermap $(\gamma\sigma,\gamma\alpha)$ with $\gamma\sigma=(1,4,6,3,2,5)$ and $\gamma\alpha=(1,2,3,4)(5)(6)$, drawn in the center of Figure 1. Finally we apply the topological hyperdeletion $(\gamma\sigma,\gamma\alpha)\mapsto(\gamma\sigma,\gamma\alpha\delta)$ with $\delta=(2,3)$ and we obtain the unicellular hypermonopole (σ',α') with $\sigma'=\gamma\sigma=(1,4,6,3,2,5)$ and $\alpha'=\gamma\alpha\delta=(1,2,4)(3)(5)(6)$, shown on the right hand side of Figure 1.

A process of reducing the original hypermap (σ,α) to a unicellular hypermonopole (σ',α') via hyperdeletions and hypercontractions may be found without drawings: as long as there is more than one face, we can always find a hyperdeletion reducing the number of faces; and as long as there is more than one vertex, we can always reduce the number of vertices using a hypercontraction. If (σ,α) is drawn on an oriented surface, hyperdeletions and hypercontractions may be visualized as in Figure 1. Conversely, if we are able to draw the unicellular hypermonopole (σ',α') by some means on an oriented surface then we may easily extend this drawing to a drawing of (σ,α) as follows. Consider the representation of (σ',α')

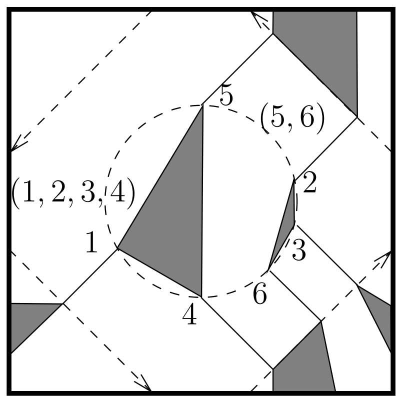

Figure 2: A 2-disk diagram of (σ,α).

on the torus shown in Figure 2. The dashed circle in the center represents

$$\sigma'=(1,4,6,3,2,5),$$

and the dashed square whose sides are in the corners represents

$$\alpha'^{-1}(\sigma')=(1,2,5,4,6,3).$$

We may complete the drawing of (σ',α') by replacing the only cycle σ' with a noncrossing partition having $z(\sigma)$ parts and the face $\alpha'^{-1}(\sigma')$ with a noncrossing partition having $z(\alpha^{-1}\sigma)$ parts. Noncrossing partitions, introduced by Kreweras [13], are partitions of the set $\{1,2,\ldots,n\}$ such that the polygons representing the parts do not cross if we place the points in the cyclic order of $(1,2,\ldots,n)$ on a circle. The cycles of σ and $\alpha^{-1}(\sigma)$ are shown as shaded polygons in Figure 2, together with the line segments connecting the corresponding points of σ' and $\alpha'^{-1}(\sigma')$ they divide the rest of the torus into two parts, which naturally correspond to the cycles of α. The resulting representation is called a *two-disk representation of* (σ,α)

in [7]. Hence the problem of drawing an arbitrary hypermap on some oriented surface is reduced to the same question for unicellular hypermonopoles only.

There are infinitely many unicellular hypermonopoles of a fixed genus: trivially, for each n the hypermap (σ, α) with $\sigma = (1, 2, \ldots, n)$ and $\alpha = (1)(2) \cdots (n)$ is a unicellular hypermonopole of genus zero. Fortunately, in the case of unicellular hypermonopoles it is easy to remove or reinsert *buds*: a bud is a fixed point i of α. For $n \geq 2$, the removal of i from the cycles representing σ and α results in a hypermap (σ', α') on the set of points $\{1, 2, \ldots, n\} - \{i\}$, which is still a hypermonopole (as $z(\sigma') = z(\sigma) = 1$), and it is also still unicellular: the action of $\alpha'^{-1}\sigma'$ on a $j \notin \{i, \sigma^{-1}(i)\}$ is the same as that of $\alpha^{-1}\sigma$, whereas $\alpha'^{-1}\sigma'$ takes $\sigma^{-1}(i)$ into $\alpha'^{-1}(\sigma(i))$. (Note that neither $\sigma^{-1}(i)$ nor $\sigma(i)$ is equal to i, as σ is cyclic and has at least 2 points.) Hence the cycle representing $\alpha'^{-1}\sigma'$ is obtained from the cycle representing $\alpha^{-1}\sigma$ by simply removing the point i from the cyclic list. It is also worth noting that the numbering of the set of points $\{1, 2, \ldots, n\} - \{i\}$ may be made consecutive by decreasing each $j > i$ in the point set by 1. Clearly if we are able to draw (σ', α') on an oriented surface with noncrossing lines, adding a bud to the figure amounts to adding a single point.

Definition 1. We call a unicellular hypermonopole *reduced* if it contains no bud.

It has been first observed in [7] that there are only finitely many reduced unicellular hypermonopoles of a fixed genus.

Lemma 2. *If (σ, α) is a genus g reduced unicellular hypermonopole on n points then*

$$2g + 1 \leq n \leq 4g \quad \textit{holds.}$$

Proof. Substituting $z(\sigma) = 1$ and $z(\alpha^{-1}\sigma) = 1$ into (1) we obtain

$$n = z(\alpha) + 2g(\sigma, \alpha). \tag{2}$$

Since each cycle of α has at least 2 elements, we get $z(\alpha) \leq n/2$, yielding the upper bound for n. The lower bound is a direct consequence of $z(\alpha) \geq 1$. □

Example 3. By the end of this paper we will see that there are two reduced unicellular hypermonopoles (σ, α) of genus 1. Assuming that the points are numbered in such a way that $\sigma = (1, 2, \ldots, n)$ holds, it is not difficult to verify that one of them satisfies $\sigma = (1, 2, 3)$ and $\alpha = (1, 3, 2)$, and the other one satisfies $\sigma = (1, 2, 3, 4)$ and $\alpha = (1, 3)(2, 4)$. The unicellular hypermonopole (σ', α') shown in Figure 2 has three buds: 3, 5 and 6. After removing these and renumbering the points in such a way that the unique vertex becomes the cycle $(1, 2, 3)$, we obtain the reduced unicellular hypermonopole (σ'', α'') with $\sigma'' = (1, 2, 3)$ and $\alpha'' = (1, 3, 2)$.

2.2 Products of two cyclic permutations

A permutation is *cyclic* if it has exactly one cycle. In his paper we will need the number of pairs of cyclic permutations of $\{1, 2, \ldots, n\}$ whose product has exactly k cycles. The answer to this question was first given by Zagier [18, application 3 of Theorem 1].

Theorem 4 (Zagier)**.** *The probability that the product of two cyclic permutations of the set* $\{1, 2, \ldots, n\}$ *has* k *cycles is*

$$P(n,k) = \frac{1+(-1)^{n-k}}{(n+1)!} c(n+1,k).$$

Here $c(n+1,k) = |s(n+1,k)|$ *is the number of permutations of* $\{1, 2, \ldots, n+1\}$ *with* k *cycles, and* $s(n+1,k)$ *is a Stirling number of the first kind.*

The first purely combinatorial proof of this result was provided by Cori, Marcus, and Schaeffer [8, Corollary 1]. Note that $P(n,k) = 0$ if $n-k$ is odd. This is obvious: all cyclic permutations have the same parity, hence the product of two cyclic permutations must be an even permutation. The parity of a permutation is the parity of the number of its cycles of even length in its cycle decomposition. This number cannot be even if $n-k$ is odd. After noting that one of the two cyclic permutations may be fixed to be $(1, 2, \ldots, n)$, Zagier's result may be restated in combinatorial terms as follows.

Theorem 5. *The number of cyclic permutations* ψ *of* $\{1, \ldots, n\}$ *such that the product* $(1, \ldots, n)\psi$ *has exactly* k *cycles is*

$$H(n,k) = \begin{cases} c(n+1,k)/\binom{n+1}{2}, & \text{if } n-k \text{ is even;} \\ 0, & \text{if } n-k \text{ is odd.} \end{cases} \tag{3}$$

Table 1 shows the values of $H(n,k)$ for $n \le 9$. It is worth noting that when $n-k$ is even, the sign of $s(n+1,k)$ is negative, hence we may also replace $c(n+1,k)$ with $-s(n+1,k)$ in (3) above. The numbers $H(n,k)$ were later rediscovered by A. Hultman in his MS Thesis [11] who defined them in terms of counting alternating cycles in the cycle graph of a permutation. The numbers $H(n+1,k)$ were named *Hultman numbers* in the work of Doignon and Labarre [9], and they are listed as A164652 in the On-Line Encyclopedia of Integer Sequences [15]. The equivalence of the two definitions is made apparent in [4, Corollary 1], which is based on a result of Doignon and Labarre [9]. Citing Stanley [16], Bóna and Flynn [4, p. 931] also published (3) for these numbers. A simple proof of (3) (relying on Hultman's definition) was also found by Grusea and Labarre [10, Section 7]. We will use the following lemma of Grusea and Labarre [10, Lemma 8.1].

Lemma 6 (Grusea-Labarre)**.** *The numbers* $H(n,k)$ *satisfy*

$$\sum_{k=0}^{n} H(n,k) x^k = \frac{(x)^{(n+1)} - (x)_{n+1}}{(n+1)n}.$$

Here $(x)^{(n+1)} = x \cdot (x+1) \cdots (x+n)$ *and* $(x)_{n+1} = x \cdot (x-1) \cdots (x-n)$ *are falling, respectively rising factorials (Pochhammer symbols).*

n \ k	1	2	3	4	5	6	7	8	9	10
0	1									
1	0	1								
2	1	0	1							
3	0	5	0	1						
4	8	0	15	0	1					
5	0	84	0	35	0	1				
6	180	0	469	0	70	0	1			
7	0	3 044	0	1 869	0	126	0	1		
8	8 064	0	26 060	0	5 985	0	210	0	1	
9	0	193 248	0	152 900	0	16 401	0	330	0	1

Table 1: The values of $H(n,k)$ for $n \leq 9$.

3 Basic facts about unicellular hypermonopoles

As a direct consequence of the definitions we may observe the following.

Remark 7. For a unicellular hypermonopole (σ, α) the permutations σ and $\pi = \alpha^{-1}\sigma$, representing the unique vertex, respectively, the unique face of the hypermap, are cyclic permutations. Conversely, given a pair (σ, π) of cyclic permutations of the same set, there is a unique unicellular hypermonopole (σ, α) satisfying $\pi = \alpha^{-1}\sigma$.

Indeed, the unique α satisfying $\pi = \alpha^{-1}\sigma$ is

$$\alpha = \sigma\pi^{-1}. \tag{4}$$

Regardless of α, the pair (σ, α) is always a hypermap because σ is a cyclic permutation, and any permutation group containing a cyclic permutation is transitive. Theorem 5 and Remark 7 have the following consequence.

Corollary 8. *The number of unicellular hypermonopoles (σ, α) satisfying $\sigma = (1, \ldots, n)$ and $z(\alpha) = k$ is the number $H(n,k)$ given in (3).*

In the rest of this section we show that unicellular hypermonopoles are bijectively equivalent to two other combinatorial models for the Hultman numbers $H(n+1,k)$.

The first model we consider is the *cycle graph model* introduced by Bafna and Pevzner [2]. We present it using the same simplification that was introduced by Doignon and Labarre in [9] where the vertex $n+1$ is identified with 0, and we adjust that model even further, by replacing each permutation

$$\begin{pmatrix} 0 & 1 & \cdots & n \\ 0 & \pi_1 & \cdots & \pi_n \end{pmatrix}$$

of $\{0, 1, \ldots, n\}$, taking 0 into 0, with the cyclic permutation $\pi = (0, \pi_1, \ldots, \pi_n)$. This correspondence is a bijection between all permutations of $\{0, 1, \ldots, n\}$ that have 0 as a fixed point and all cyclic permutations of $\{0,1,\ldots,n\}$. Let us fix the cyclic permutation $\sigma=(0,1,\ldots,n)$, and let π be any cyclic permutation of the set $\{0,1,\ldots,n\}$. The *cycle graph* $G(\pi)$ of the cyclic permutation π is a digraph on the vertex set $\{0,1,\ldots,n\}$ whose edges are colored with two colors:

1. the black edges go from i to $\pi^{-1}(i)$ (mod $n+1$) for $0\leq i\leq n$;

2. the gray edges go from i to $\sigma(i)$; in other words they go from i to $i+1$ (mod $n+1$) for $0\leq i\leq n$.

Note that in the work of Doignon and Labarre [9] the black edges go from π_i to π_{i-1} for $1\leq i\leq n$ and from π_0 to π_n. Our rule (1) is the same if we read the array $(0,\pi_1,\cdots,\pi_n)$ as the code for a cyclic permutation.

Each vertex is the head, respectively the tail of one edge of each color, hence the cycle graph may be uniquely decomposed into disjoint color-alternating cycles. Note that even though edges do not repeat in such cycles, vertices may occur twice. To remedy this slight confusion, we introduce two copies of each vertex i: a negative copy i^- and a positive copy i^+. Each negative vertex i^- will be the head of a black edge whose tail is $\pi(i)^+$, and it will be the tail of a gray edge whose head is $(i+1)^+$. Equivalently, each positive vertex i^+ will be the head of a gray edge whose tail is $(i-1)^-$, and it will be the tail of a black edge whose head is $\pi^{-1}(i)^+$. Instead of using colors we will label the black edges with π^{-1} and the gray edges with σ. For example, for $n=7$ and $\pi=(0,4,1,6,2,5,7,3)$ (the same array, read as a permutation word, gives rise to [9, Figure 1]) we obtain the following two cycles:

$$0^-\xrightarrow{\sigma}1^+\xrightarrow{\pi^{-1}}4^-\xrightarrow{\sigma}5^+\xrightarrow{\pi^{-1}}2^-\xrightarrow{\sigma}3^+\xrightarrow{\pi^{-1}}7^-\xrightarrow{\sigma}0^+\xrightarrow{\pi^{-1}}3^-\xrightarrow{\sigma}4^+\xrightarrow{\pi^{-1}}0^-\quad\text{and}$$

$$6^-\xrightarrow{\sigma}7^+\xrightarrow{\pi^{-1}}5^-\xrightarrow{\sigma}6^+\xrightarrow{\pi^{-1}}1^-\xrightarrow{\sigma}2^+\xrightarrow{\pi^{-1}}6^-$$

Using this notation one may notice immediately that these cycles may be uniquely reconstructed from the positive vertices only: we may identify the first cycle with $(1,5,3,0,4)$ and the second cycle with $(7,6,2)$. Observe next that

$$\alpha=(1,5,3,0,4)(7,6,2)=\sigma\pi^{-1};$$

that is, $\pi=\alpha^{-1}\sigma$. The cycle graph of π may be identified with the unicellular hypermonopole (σ,α) whose only vertex is σ and only face is $\pi=\alpha^{-1}\sigma$. The number of alternating cycles in $G(\pi)$ is $z(\alpha)$, the number of hyperedges.

Bafna and Pevzner [2] call a pair $(i,\pi(i))$ a *breakpoint* if $\pi(i)\neq\sigma(i)$. In our setting $(i,\pi(i))$ is a breakpoint if and only if $\alpha^{-1}\sigma(i)\neq\sigma(i)$, equivalently $\sigma(i)$ is not a bud of (σ,α). Reduced unicellular hypermonopoles bijectively correspond to cyclic permutations for which every pair $(i,\pi(i))$ is a breakpoint, see Lemma 10 in the next section. A "transposition" operation, as defined by Bafna and Pevzner [2] replaces the permutation word $\pi_1\cdots\pi_n$ with the word $\pi_1\cdots\pi_{i-1}\pi_j\pi_{j+1}\cdots\pi_{k-1}\pi_i\pi_{i+1}\cdots\pi_{j-1}\pi_k\cdots\pi_n$ for some $i<j<k$. After rewriting the word $\pi_1\cdots\pi_n$ as the cyclic permutation $\pi=(0,\pi_1,\ldots,\pi_n)$, the same rearrangement of

indices corresponds to sending π into $(\pi_i, \pi_k)(\pi_i, \pi_j)\pi = (\pi_i, \pi_j, \pi_k)\pi$, that is, multiplying π with two transpositions. In our language, Bafna and Pevzner [2] study the question, how many such operations are needed to transform π into $\sigma = (1, 2 \ldots, n)$. There seems no point in "breaking" up π between $\pi(i)$ and $\pi(i+1)$ if $\pi(i+1) = \sigma(\pi(i))$ holds. For example, when $n = 4$ and $\pi = (0, 3, 1, 2, 4)$, the numbers 1 and 2 are already cyclically consecutive, it makes sense to rearrange the entries using only operations that keep 1 and 2 consecutive. In the corresponding unicellular hypermonopole we have $\alpha = \sigma\pi^{-1} = (0)(1, 4, 3)(2)$, i.e., 2 is a bud. Note that 0 is also a bud, because the largest numbered point is already at the end of our array. Removing the buds 0 and 2, and renumbering the points consecutively corresponds to replacing σ with $\sigma' = (0, 1, 2)$ and π with $\pi' = (1, 0, 2)$. We arrived at a reduced unicellular hypermonopole, this always happens after removing all buds. To summarize, if we accept as a rule that we only use the interval "transposition" operations of Bafna and Pevzner [2] to break the words at their breakpoints then (after removing the buds in the associated unicellular hypermonopoles) the process is essentially running on a cyclic permutation encoding the unique face of a reduced unicellular hypermonopole.

Remark 9. The careful reader may notice that the smallest numbered point 0 is not listed at the beginning of the cyclic permutation π' above. Furthermore, Bafna and Pevzner only defined transpositions of intervals in permutations written as words, not in cyclic permutations. That said, allowing their operations to be performed on cyclically consecutive blocks of a cyclic permutation does not lead to the introduction of new operations because of the following observation. Consider an array

$$\underline{u} = (a_1, \ldots, a_i, b_1, \ldots, b_j, c_1, \ldots, c_k, d_1, \ldots, d_\ell, a'_1, \ldots, a'_m)$$

where all coordinates are pairwise different numbers. On the one hand, we may think of the above array as a code of a cyclic permutation, which can be decomposed into four intervals of cyclically consecutive entries: $(a'_1, \ldots, a'_m, a_1, \ldots, a_i)$, $(b_1, \ldots, b_j)$, $(c_1, \ldots, c_k)$ and $(d_1, \ldots, d_\ell)$. A move not explicitly considered in [2] is to exchange the "cyclic" interval $(a'_1, \ldots, a'_m, a_1, \ldots, a_i)$ with the adjacent interval $(d_1, \ldots, d_\ell)$. This yields the cyclic permutation encoded by the array

$$\underline{v} = (a_1, \ldots, a_i, d_1, \ldots, d_\ell, b_1, \ldots, b_j, c_1, \ldots, c_k, a'_1, \ldots, a'_m).$$

On the other hand, we can also read $\underline{u}$ as an array representing a permutation as a word that has the following four intervals: $(a_1, \ldots, a_i)$, $(b_1, \ldots, b_j, c_1, \ldots, c_k)$, $(d_1, \ldots, d_\ell)$, and $(a'_1, \ldots, a'_m)$. The array $\underline{v}$ can also be obtained by exchanging $(b_1, \ldots, b_j, c_1, \ldots, c_k)$ and $(d_1, \ldots, d_\ell)$: this is a transposition of intervals allowed by the definition in [2]. The case when we exchange the cyclic interval $(a'_1, \ldots, a'_m, a_1, \ldots, a_i)$ with $(b_1, \ldots, b_j)$ can be dealt with in a completely similar fashion.

The other model is the one introduced by Alexeev and Zograf [1]. Consider a $2n$ sided polygon whose boundary consists of n black sides followed by n gray sides (represented by dashed lines). The black sides are oriented in the counterclockwise direction and the gray

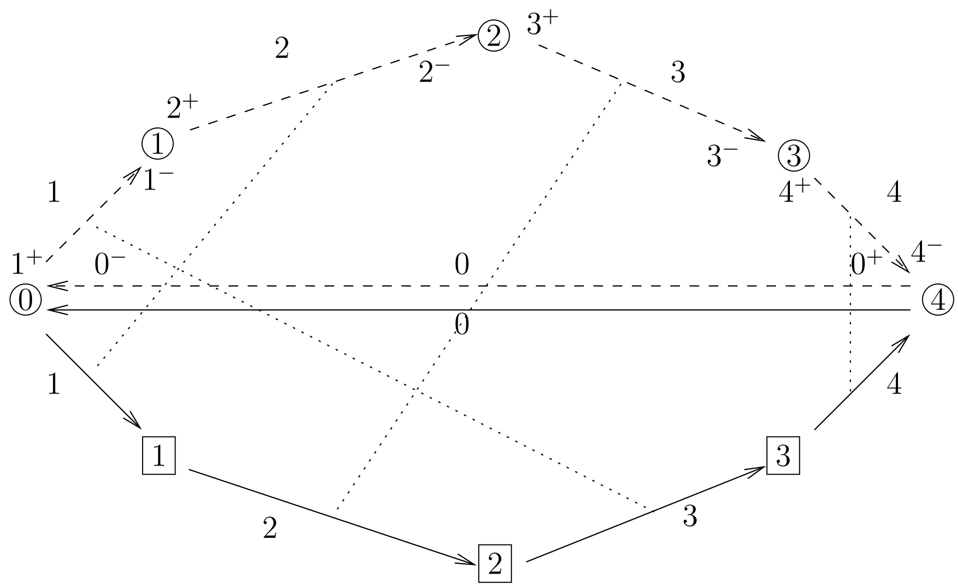

Figure 3: A polygon gluing diagram for $\pi = (0, 2, 3, 1, 4)$.

sides are oriented in the clockwise direction, as shown in Fig. 3. Pairwise gluing of black sides with gray sides (respecting orientation) gives an orientable topological surface without boundary of topological genus $g \geq 0$. The genus g depends on the gluing. We cut the polygon along the diagonal connecting the vertex n and the vertex 0, and we add a directed edge of each color from n to 0. By assuming that these added edges will be glued together we don't change the genus. We number all edges by their tail end, and we use the gluing pattern to define the cyclic permutation $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$: we define π_i as the gray edge that is glued with the black edge i. As in the previous model, we define σ as the cyclic permutation $(0, 1, \ldots, n)$. For the gluing pattern shown in Fig. 3 we obtain $\pi = (0, 2, 3, 1, 4)$. As before, let us define $\alpha = \sigma\pi^{-1}$. In our example we obtain $\alpha = (0)(1, 4, 2)(3)$. According to Alexeev and Zograf [1] it is easy to see that the alternating cycles of $G(\pi)$ are in bijection with the vertices of the glued polygon. We can make it easier to see this by adding the signed labels i^- and i^+ along each gray edge labeled i as shown in Fig. 3. For example, the alternating cycle

$$0^- \xrightarrow{\sigma} 1^+ \xrightarrow{\pi^{-1}} 3^- \xrightarrow{\sigma} 4^+ \xrightarrow{\pi^{-1}} 1^- \xrightarrow{\sigma} 2^+ \xrightarrow{\pi^{-1}} 0^-$$

corresponds to the identification ⓪ = $\boxed{2}$ = ③ = $\boxed{3}$ = ① = ⓪. The verification of the details is left to the reader.

4 Counting reduced unicellular hypermonopoles

In this section we express the number of reduced unicellular hypermonopoles in terms of the number $H(n, k)$. In doing so, the following lemma will be useful.

Lemma 10. *A unicellular hypermonopole (σ, α) satisfying $\sigma = (1, \ldots, n)$ is reduced if and only if the cyclic permutation $\pi = \alpha^{-1}\sigma$ satisfies $\pi(i) \neq i + 1$ for all $i \in \{1, \ldots, n\}$. Here addition is performed mod n.*

Indeed, $\alpha^{-1}\sigma(i) = i + 1$ is equivalent to $\alpha(i + 1) = i + 1$. Let us also note the following consequence of the proof of Lemma 2.

Corollary 11. *If (σ, α) is a reduced unicellular hypermonopole on n points then*

$$1 \leq z(\alpha) \leq n/2 \quad \textit{holds.}$$

As a consequence, the number $r(n, k)$ of reduced unicellular hypermonopoles (σ, α) satisfying $\sigma = (1, \ldots, n)$ and $z(\alpha) = k$ can be only nonzero for $1 \leq k \leq n/2$.

Proposition 12. *Given $n \geq 2$ and $1 \leq k \leq n/2$, the number of reduced unicellular hypermonopoles (σ, α) satisfying $\sigma = (1, \ldots, n)$ and $z(\alpha) = k$ is given by*

$$r(n, k) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} H(n - i, k - i). \tag{5}$$

Proof. We compute $r(n, k)$ using inclusion-exclusion. Let $\mathcal{H}_{n,k}$ be the set of all unicellular hypermonopoles (σ, α) satisfying $\sigma = (1, \ldots, n)$ and $z(\alpha) = k$. For each $j \in \{1, \ldots, n\}$, let $\mathcal{H}_{n,k,j}$ be the subset of $\mathcal{H}_{n,k}$ also satisfying $\alpha(j) = j$. Clearly we have

$$r(n, k) = \left| \mathcal{H}_{n,k} - \bigcup_{j=1}^{n} \mathcal{H}_{n,k,j} \right|.$$

Using the inclusion-exclusion formula we obtain

$$r(n, k) = \sum_{i=0}^{n} (-1)^i \sum_{\{j_1, \ldots, j_i\} \subseteq \{1, \ldots, n\}} \left| \mathcal{H}_{n,k,j_1} \cap \ldots \cap \mathcal{H}_{n,k,j_i} \right|. \tag{6}$$

First we show that it suffices to perform the summation on the right hand side only up to $i = k - 1$. All hypermaps $(\sigma, \alpha) \in \mathcal{H}_{n,k,j_1} \cap \ldots \cap \mathcal{H}_{n,k,j_i}$ have the property that $j_1, \ldots, j_i$ are fixed points of α. Since $z(\alpha) = k$, we may restrict the summation in (6) to $i \leq k$. Furthermore the case $i = k$ is possible only if the cycles $(j_1), \ldots, (j_k)$ are all the cycles of α in which case $k = n$ in contradiction with $k \leq n/2$.

From now on let us fix a subset $\{j_1, \ldots, j_i\}$ of $\{1, \ldots, n\}$ for some $i \leq k - 1$. Let σ' be the cyclic permutation of $\{1, \ldots, n\} - \{j_1, \ldots, j_i\}$ obtained from $(1, \ldots, n)$ by removing the elements $j_1, \ldots, j_i$. Given any unicellular hypermonopole $(\sigma, \alpha) \in \mathcal{H}_{n,k,j_1} \cap \ldots \cap \mathcal{H}_{n,k,j_i}$, let us define the unicellular hypermonopole (σ', α') on the set of points $\{1, \ldots, n\} - \{j_1, \ldots, j_i\}$ by the following procedure:

1. We define $\pi = \alpha^{-1}\sigma$ as the unique face of (σ, α).

2. We define the unique face π' of (σ', α') as the cyclic permutation obtained by removing the elements $j_1, \ldots, j_i$ from π .

3. The permutation α' is given by $\alpha' = \sigma'\pi'^{-1}$.

The operation $(\sigma, \alpha) \mapsto (\sigma', \alpha')$ associates to each element of $\mathcal{H}_{n,k,j_1} \cap \ldots \cap \mathcal{H}_{n,k,j_i}$ a unicellular hypermonopole (σ', α'). The operation is invertible: to obtain π from π' we must insert each $j \in \{j_1, \ldots, j_i\}$ right before $j+1$. Hence we obtain a bijection between the hypermaps in $\mathcal{H}_{n,k,j_1} \cap \ldots \cap \mathcal{H}_{n,k,j_i}$ and the set of all unicellular hypermonopoles (σ', α') on the set of points $\{1, \ldots, n\} - \{j_1, \ldots, j_i\}$ satisfying $z(\alpha) = k - i$. Therefore we have

$$|\mathcal{H}_{n,k,j_1} \cap \ldots \cap \mathcal{H}_{n,k,j_i}| = H(n-i, k-i),$$

and the statement is a direct consequence of (6). □

Corollary 13. *If $n - k$ is odd then $r(n,k) = 0$. As a consequence the least value of n for which $r(n,k) > 0$ holds for some $k \leq \frac{n}{2}$ is $n = 3$.*

Indeed, if $n - k$ is odd then all terms $H(n-i, k-i) = 0$ appearing on the right hand side of (5) are zero.

The values of $r(n,k)$ for $3 \leq n \leq 12$ are shown in Table 2. They also listed them as A371665 in the On-Line Encyclopedia of Integer Sequences [15].

n \ k	1	2	3	4	5	6
3	1					
4	0	1				
5	8	0				
6	0	36	0			
7	180	0	49			
8	0	1 604	0	21		
9	8 064	0	5 144	0		
10	0	112 608	0	7 680	0	
11	604 800	0	604 428	0	5 445	
12	0	11 799 360	0	1 669 052	0	1 485

Table 2: The values of $r(n,k)$ for $3 \leq n \leq 12$ and $1 \leq k \leq \lfloor n/2 \rfloor$.

Combining Lemma 6 and Equation (5) we obtain the following formula.

Theorem 14. *The numbers $r(n,k)$ satisfy*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} r(n,k) \cdot x^k = \sum_{i=0}^{n-1} \binom{n}{i} (-x)^i \cdot \frac{(x)^{n-i+1} - (x)_{n-i+1}}{(n-i)(n-i+1)}.$$

Proof. By Lemma 6 the number $H(n-i,k-i)$ is given by

$$H(n-i,k-i) = [x^{k-i}]\frac{(x)^{n-i+1}-(x)_{n-i+1}}{(n-i)(n-i+1)} = [x^k]x^i\frac{(x)^{n-i+1}-(x)_{n-i+1}}{(n-i)(n-i+1)}.$$

The statement now follows from Equation (5) after noticing that we may extend the upper limit of the summation to n. The equation

$$r(n,k) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}H(n-i,k-i)$$

also holds if we set $H(n,k)=0$ for $k\le 0$. Lemma 6 is still applicable: the expression $((x)^{(n+1)}-(x)_{n+1})/((n+1)n)$ is a polynomial of x with zero constant term, containing no negative powers of x. □

Lemma 2 and Proposition 12 allow us to compute the number of all reduced unicellular hypermonopoles of a fixed genus, using the following result.

Proposition 15. *The number $u(g)$ of all reduced unicellular hypermonopoles of genus g is given by*

$$u(g) = \sum_{n=2g+1}^{4g} r(n,n-2g).$$

Proof. By (2) a unicellular hypermonopole with k cycles has genus g if and only if $k=n-2g$ holds. As seen in Lemma 2, n must be at least $2g+1$ and at most $4g$. □

The first 10 entries of the sequence $\{u(g)\}_{g=1}^{\infty}$ are the following:

$$2, 114, 21538, 8698450, 6113735682, 6641411533106,$$
$$10323616703610338, 21755183272319116818,$$
$$59718914489141881419202, 207083242485963591169089778.$$

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