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Triangular Numbers With a Single Repeated Digit

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Abstract

The question of which triangular numbers have a decimal representation containing a single repeated digit seemed to be settled since at least the 1970s: Ballew and Weger provided a complete list and a proof that those are the only numbers of that kind. This assertion is referenced by other authors in the field. However, their proof is flawed. We present a new and elementary proof of the statement, which corrects the error.

1 Introduction

The study of triangular numbers—integers of the form $\frac{k(k+1)}{2}$—is a long-standing and well-researched area within the field of number theory. For example, Gauß proved in his *Disquisitiones Arithmeticae* [4] that every positive integer can be expressed as the sum of three triangular numbers.

A question that arises is that of the decimal representation of triangular numbers. In particular, which triangular numbers can be expressed using a single digit? That is to say, are there triangular numbers of the form $d \cdot \frac{10^i-1}{9}$ with some digit $d \in \{1, \ldots, 9\}$ and positive integer i? One instance of this question being posed is Problem 15648 in the *Educational Times* [2], which was proposed by Youngman and subsequently answered by Escott in the same journal. However, Escott only demonstrated the non-existence of triangular numbers with repeated digits exceeding 666, comprising a maximum of 30 digits, do not exist.

Subsequently, Ballew and Weger [1] presented an erroneous proof for the following theorem.

Theorem 1. *The only triangular numbers whose decimal representations consist of a single repeated digit are* $1, 3, 6, 55, 66,$ *and* 666.

In a recent contribution, Kafle, Luca, and Togbé [5] generalized the problem to include repeating blocks with two digits: the only triangular numbers whose decimal representation can be written using only one block of two digits are $10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 5050$, and 5151. However, for the original problem, they just cite the work of Ballew and Weger.

Recently, on the internet platform MathOverflow [6], a discussion regarding this topic was held. In this discussion, the error in Ballew and Weger's proof of Theorem 1 was identified, and a proof using the methods of elliptic curves was presented. In this paper, we provide an elementary proof of Theorem 1 using only Pell equations.

1.1 Outline

In accordance with the methodology proposed by Escott, Ballew, and Weger, we initially reformulate the problem. Subsequently, we provide a concise account of the erroneous assertion present within the proof of Ballew and Weger. In Section 3, a proof is provided for each remaining digit, demonstrating that there are no further triangular numbers consisting of a single repeated digit.

2 Reduction and reformulation of the problem

For every positive integer k let T_k denote the k'th triangular number. In accordance with Escott's approach and the valid part of Ballew's and Weger's paper, we arrive at the following conclusion: if the decimal expression of T_k consists solely of the single repeated digit d, we have

$$T_k = \frac{k(k+1)}{2} = d \cdot \frac{10^i - 1}{9}, i \geq 1, d \in \{1, \ldots, 9\}$$

$$\iff$$

$$(2k+1)^2 = 4k^2 + 4k + 1 = 8d \cdot \frac{10^i - 1}{9} + 1.$$

A solution (k, d, i) in positive integers for the original problem exists, if and only if the quantity $D := 1 + 8d \cdot \frac{10^i-1}{9}$ is a perfect square.

From the previous line, we see that D satisfies the congruence $D \equiv 1+8d \pmod{10}$. Thus, immediately, we can rule out $d \in \{2, 4, 7, 9\}$, because in those cases D would be congruent 7 or 3 modulo 10, which are not quadratic residues. Therefore, the value of d must be $1, 3, 5, 6$, or 8. Now we look at these cases one by one:

$d = 1$: If $i = 1$, it follows that $D = 8 \cdot \frac{10^i-1}{9} + 1 = 9 = 3^2$, which yields $T_1 = 1$, and for $i > 1$, we have $D = 88 \cdots 89$.

$d = 3$: If $i = 1$, then $D = 24 \cdot \frac{10^i-1}{9} + 1 = 25 = 5^2$, which yields $T_2 = 3$. Otherwise, we have $D = 26 \cdots 65$.

$d = 5$: If $i = 1$, we have $D = 40 \cdot \frac{10^i-1}{9} + 1 = 41$. For $i = 2$, we get $D = 441 = 21^2$, which yields $T_{10} = 55$, and for $i > 2$, it follows that $D = 44 \cdots 41$.

$d = 6$: If $i = 1$, 2, or 3, then $D = 48 \cdot \frac{10^i-1}{9} + 1 = 49 = 7^2$, $D = 529 = 23^2$, or $D = 5329 = 73^2$, which yields $T_3 = 6, T_{11} = 66$, and $T_{36} = 666$. If $i > 3$, we have $D = 53 \cdots 329$.

$d = 8$: If $i = 1$ or 2, then $D = 64 \cdot \frac{10^i-1}{9} + 1 = 65$ or $D = 705$. Otherwise, $D = 71 \cdots 105$.

Since $\cdots 05$ and $\cdots 65$ cannot be the last two digits of a square number, we can exclude $d = 8$ and $d = 3$, except for T_2.

2.1 The false statement

Ballew and Weger incorrectly asserted that there is no integer z whose square ends in $\cdots 88889$. This assertion is, however, erroneous. For example, $8072917^2 = \cdots 88888889$. In fact, since $z^2 \equiv \cdots 88889 \equiv 1 + 8 \cdot \frac{10^{k+1}-1}{10-1} \equiv 1 - \frac{1}{9} \pmod{10^k}$ is equivalent to $9z^2 - 1 \equiv (3z-1)(3z+1) \equiv 0 \pmod{10^k}$, there are solutions for every k; for example, $z = \cdots 3333$. Consequently, there are square numbers whose last digits are $\cdots 8889$ with an arbitrary number of 8's.

3 A proof using Pell's equations

This section introduces a new line of reasoning, which rules out the remaining cases, namely those with $d \in \{1, 5, 6\}$.

3.1 The case $d = 1$

In this case, we have to find all positive integers p, i with

$$p^2 = 88 \cdots 89 = 8 \cdot \frac{10^i - 1}{9} + 1 \iff (3p)^2 - (8 \cdot 10^i) = 1.$$

Assuming there is a solution for $i \geq 2$ (the case $i = 1$ leads to T_1). Then we can make a case distinction by using the parity of i.

Case A: i is even, i.e., $i = 2r$, $r \in \mathbb{N}$. Subsequently, we have

$$(3p)^2 - 2 \cdot (2 \cdot 10^r)^2 = 1,$$

and we can seek solutions to the Pell equation

$$x^2 - 2y^2 = 1 \tag{1}$$

with the requirements $x = 3p$, $y = 2 \cdot 10^r$, and $r \geq 1$. Thus, y must satisfy $y \equiv 0 \pmod 5$ and $y \not\equiv 0 \pmod 7$.

However, the solutions to (1) are given by $(x_0, y_0) = (3, 2)$, $x_{n+1} = 3x_n + 4y_n$, and $y_{n+1} = 2x_n + 3y_n$. A calculation modulo 5 and modulo 7 as in Table 1 yields $y \equiv 0 \pmod 5 \iff y \equiv 0 \pmod 7$; hence, $y \neq 2 \cdot 10^r$.

n	x_n mod 5	y_n mod 5	x_n mod 7	y_n mod 7
0	3	2	3	2
1	2	2	3	5
2	4	0	1	0
3	2	3	3	2
4	3	3	3	5
5	1	0	1	0
6	3	2	3	2
7		…		

Table 1: Solutions of equation (1) for $d = 1$, Case A, mod 5 and mod 7.

Case B: i is odd, i.e., $i = 2r + 1$, $r \in \mathbb{N}$. Then we have $(3p)^2 - 20 \cdot (2 \cdot 10^r)^2 = 1$, and we can seek solutions to

$$x^2 - 20y^2 = 1 \tag{2}$$

with $x = 3p$, $y = 2 \cdot 10^r$, and $r \geq 1$. Hence, we have $y \equiv 0 \pmod 5$ and $y \not\equiv 0 \pmod{11}$.

But the solutions to (2) are $(x_0, y_0) = (9, 2)$, $x_{n+1} = 9x_n + 40y_n$, and $y_{n+1} = 2x_n + 9y_n$. A calculation modulo 5 and 11 similar to the one in Case A yields $y \equiv 0 \pmod 5 \iff y \equiv 0 \pmod{11}$, so $y \neq 2 \cdot 10^r$ with $r \geq 1$.

3.2 The case $d = 5$

Here we have to find all solutions (p, i) in positive integers with $i > 2$ for the diophantine equation

$$p^2 = 44\cdots41 = 40 \cdot \frac{10^i - 1}{9} + 1.$$

Assume there is a solution (p, i).

Case A: i is even, i.e., $i = 2r$, $r \in \mathbb{N}$, $r \geq 2$. Then we have $(3p)^2 - 10 \cdot (2 \cdot 10^r)^2 = -31$, and we can seek solutions to the equation

$$x^2 - 10y^2 = -31 \tag{3}$$

with $x = 3p$, $y = 2 \cdot 10^r$, $r \geq 2$. Hence, y has to satisfy the congruences $y \equiv 0 \pmod 8$ and $y \not\equiv 0 \pmod 7$.

To solve equation (3), we observe the identity $1 = 19^2 - 10\cdot 6^2 = (19+6\sqrt{10})\cdot(19-6\sqrt{10})$. Thus, if (x_n, y_n) is a solution of equation (3), we have

$$\begin{aligned}
-31 &= x_n^2 - 10y_n^2 \\
&= (x_n + y_n\sqrt{10}) \cdot (x_n - y_n\sqrt{10}) \\
&= (x_n + y_n\sqrt{10}) \cdot (19 + 6\sqrt{10}) \cdot (x_n - y_n\sqrt{10}) \cdot (19 - 6\sqrt{10}) \\
&= (19x_n + 60y_n + (6x_n + 19y_n)\sqrt{10}) \cdot (19x_n + 60y_n - (6x_n + 19y_n)\sqrt{10}) \\
&= (19x_n + 60y_n)^2 - 10(6x_n + 19y_n)^2 \\
&= x_{n+1}^2 - 10y_{n+1}^2
\end{aligned}$$

with $x_{n+1} := 19x_n + 60y_n$, and $y_{n+1} := 6x_n + 19y_n$.

By recursion, we can generate entire sequences of solutions starting from an initial solution. But considering this recursion as a linear equation system with x_n and y_n as unknowns, we can go backwards, too: if (x_{n+1}, y_{n+1}) is a solution of equation (3), then (x_n, y_n) with $x_n = 19x_{n+1} - 60y_{n+1}$ and $y_n = 19y_{n+1} - 6x_{n+1}$ is also a solution.

As long as $60y_{n+1} < 19x_{n+1}$, and $6x_{n+1} < 19y_{n+1}$, both x_n and y_n are positive. The second inequality follows for all positive solutions of equation (3) via

$$\begin{aligned}
&x_{n+1}^2 - 10y_{n+1}^2 = -31 < 0 \\
&\Rightarrow 36x_{n+1}^2 < 360y_{n+1}^2 < 361y_{n+1}^2 \\
&\Rightarrow 6x_{n+1} < 19y_{n+1}.
\end{aligned}$$

Provided that $y_{n+1} \geq 34$, the first inequality is also true. If it was not, we would have $361x_{n+1} \leq 3600y_{n+1}$, and thus $x_{n+1}^2 \leq \left(10 - \frac{10}{361}\right) y_{n+1}^2$ and

$$-31 = x_{n+1}^2 - 10y_{n+1}^2 \leq -\frac{10}{361} \cdot y_{n+1}^2 \leq -\frac{10}{361} \cdot 34^2 = -\frac{11560}{361} < -\frac{11191}{361} = -31.$$

Hence, every solution (x_{n+1}, y_{n+1}) of equation (3) in positive integers with $y_{n+1} \geq 34$ leads to another such solution (x_n, y_n). And with $y_{n+1} \geq 34$. we have $x_{n+1} > 3y_{n+1}$, since otherwise we would get the contradiction $-31 = x_{n+1}^2 - 10y_{n+1}^2 \leq -y_{n+1}^2 \leq -34^2 < -31$. Thus, $y_n = 19y_{n+1} - 6x_{n+1} < 19y_{n+1} - 18y_{n+1} = y_{n+1}$.

Thus, provided that (x_{n+1}, y_{n+1}) is a solution of equation (3) in positive integers with $y_{n+1} \geq 34$, we get a new solution (x_n, y_n) of this equation in positive integers with a smaller second component.

This can now be considered in the other direction, again: every solution of equation (3) in positive integers can be obtained via the recursion $x_{n+1} := 19x_n + 60y_n$ and $y_{n+1} := 6x_n + 19y_n$ from an initial solution (x_n, y_n) with $0 < y_n < 34$. Checking all possibilities with $0 < x$ and $0 < y \leq 34$ yields the two solutions $(x_0, y_0) \in \{(3, 2), (63, 20)\}$. From there, we get all other solutions of (3) via recursion.

As in the case $d = 1$, a calculation modulo 8 and 7 now yields $y \equiv 0 \pmod 8 \iff y \equiv 0 \pmod 7$, so $y \neq 2 \cdot 10^r$.

Case B: i is odd, i.e., the number of occurrences of the digit 4 in the decimal representation of p^2 is odd. In this case, according to a basic rule

$$44 \cdots 41 \equiv 1 - 4 + (4 - 4) + \cdots + (4 - 4) \equiv -3 \equiv 8 \pmod{11},$$

which is not a quadratic residue modulo 11.

(This is Problem 2 from the first round of the 2024 Bundeswettbewerb Mathematik in Germany [3]. When we tried to solve the related open question for Case A, i.e., whether 1 and 441 are the only squares with an even number of occurrences of the digit 4, we came across Ballew's and Weger's flawed proof and ended up with this paper.)

3.3 The case $d = 6$

Assume there is a solution (p, i) with $i \geq 4$. (The cases with $i \leq 3$ yield the solutions T_3, T_{11}, and T_{36}.) Then we have

$$p^2 = 53 \cdots 329 = 48 \cdot \frac{10^i - 1}{9} + 1 \iff 9p^2 - 3 \cdot (4^2 \cdot 10^i) = -39.$$

Case A: i is even, i.e., $i = 2r$ with $r \geq 2$. Thus, $3p^2 - (4 \cdot 10^r)^2 = -13$ and we can seek solutions to the Pell equation

$$x^2 - 3y^2 = 13 \tag{4}$$

with the additional requirement of $x = 4 \cdot 10^r$, $y = 3p$, and $r \geq 2$. Hence, it has to be $x \equiv 0 \pmod{50}$. We can obtain all solutions of (4), starting from $(x_0, y_0) = (4, 1)$ or

$(x_0, y_0) = (5, 2)$, through the recursion $x_{n+1} = 2x_n + 3y_n$, and $y_{n+1} = x_n + 2y_n$ by the same method used in the case $d = 5$. A calculation modulo 50 and 241 as in the case $d = 1$ yields $x_i \equiv 0 \pmod{50} \iff x_i \equiv 94 \pmod{241}$. Thus, in every solution, we have $94 \equiv 4 \cdot 10^r \pmod{241}$ and, therefore, $10^r \equiv 144 \pmod{241}$. However, this is not the case for any positive integer r.

Case B: i is odd, i.e., $i = 2r + 1$ with $r \geq 2$. This leads to $(3p)^2 - 30 \cdot (4 \cdot 10^r)^2 = -39$ and therefore, the Pell equation

$$x^2 - 30y^2 = -39 \tag{5}$$

with $x = 3p$, $y = 4 \cdot 10^r$, and $r \geq 2$. All solutions of (5) can be obtained from the starting solutions $(x_0, y_0) = (9, 2)$ or $(x_0, y_0) = (21, 4)$ through the recursion $x_{n+1} = 11x_n + 60y_n$, $y_{n+1} = 2x_n + 11y_n$ using the same methods as above.

Now assume that there is a solution with $r \geq 4$. It follows that $y = 4 \cdot 10^r \equiv 0 \pmod{64}$. A calculation modulo 64 and 31, as in the case $d = 1$, reveals that $y_i \equiv 0 \pmod{64}$ is equivalent to $y_i \equiv 3 \pmod{31}$, and, therefore, $10^r \equiv 24 \pmod{31}$, which is not the case for any positive integer r.

In the cases with $r \in \{2, 3\}$, it can be verified that in both families of solutions of (5) we have $y_3 > 4000$. Consequently, the only remaining candidates for additional solutions are (x_1, y_1) and (x_2, y_2) in both families. However, a brief calculation reveals that they are not solutions to the original problem.

This demonstrates that, in each case, there are no additional solutions beyond those initially presented in the paper, thereby establishing the proof of Theorem 1.

Remark 2. The method can be applied in the generalized version with repeated blocks of digits as well. In consideration of blocks of length 2, as presented in [5], we have to solve the equation

$$\begin{aligned}
\frac{k(k+1)}{2} &= c \cdot \frac{100^i - 1}{99}, \; c \in \{10, \dots, 99\} \\
\iff \quad (2k+1)^2 &= 8c \cdot \frac{100^i - 1}{99} + 1 \\
\iff (33 \cdot (2k+1))^2 &= 88c \cdot 100^i + 1089 - 88c \\
\iff \quad x^2 - 22c \cdot y^2 &= 1089 - 88c,
\end{aligned} \tag{6}$$

with $x := 33 \cdot (2k + 1)$, $y := 2 \cdot 10^i$ in positive integers. Now we can compute the set of all integer solutions (x, y) to the Pell equation (6) and find moduli that disprove that y can be of the form $y = 2 \cdot 10^i$.

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