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Exponential versus Factorial Revisited

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Abstract

We study the relationship between iterated factorials, the iterated Euler gamma function, and iterated exponentiation. We demonstrate that both the iterated factorial and the iterated gamma function can be represented as a power tower, where all the terms are identical except for the top term. Additionally, we show that the top terms of this representation form a converging sequence. Finally, we prove that this representation can be reversed.

1 Introduction

Consider a sequence of factorials (entry A000142 in the On-Line Encyclopedia of Integer Sequences (OEIS) [7])

$$1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, \ldots$$

and a sequence of iterated exponentiation (tetration) of 2 (entry A014221):

$$1, 2, 4, 16, 65536, 2^{65536}. \tag{1}$$

The next term in this sequence has 19729 decimal digits. Now consider a sequence of iterated factorials:

$$3, 6, 720, 720!, \ldots$$

with the number 720! having 1747 decimal digits (note that the terms $3, 6, 720$ appear in order in the OEIS in four different entries, but not in our context). The last two sequences grow, unsurprisingly, very quickly and have garnered some interest in the past. In particular, the question of how to compare these numbers has received some interest recently. One of the recent and interesting results is that of Velleman [8], who proved, among other things, that the largest number that can be written with five symbols, using positive integers written in decimal notation, exponentiation, and the factorial function, is $9^9!!!$ (where by $n!!!$ we mean $((n!)!)!$).

Iterated exponentiation, as illustrated by the example (1)), dates back to Euler [2]. More recently, Knoebel [4] and Anderson [1] also investigated iterated exponentiation, including infinite iterated exponentiation. Shapiro and Shapiro [6] considered iterated exponents reduced (mod k). Velleman [8] studied iterated factorials and exponentiation to determine the most efficient way to generate the largest numbers using factorials and exponentiation. He also showed that one power tower can dominate another, and that power towers tend to dominate iterated factorials. Later, inspired by Velleman's article, the present author [3] compared iterated exponentiation in many scenarios, including power towers with similar heights and so-called periodic towers.

One of the results obtained in [3] was a form of normalization of power towers. In short, it allows any tetration (power tower with equal terms) to be represented by a power tower whose terms are all equal, except for the top term. Before we state that result, we first introduce the notation used throughout the article.

For given $k \geq 1$[a] and $n \geq 0$ we define

$$F_n(k) = \begin{cases} k, & \text{if } n = 0, \\ F_{n-1}(k!), & \text{if } n \geq 1. \end{cases}$$

Note that $F_n(k)! = F_n(k!)$, so for instance,

$$F_2(3) = (3!)! = 6! = 720, \qquad F_3(4) = F_2(24) = F_1(24!) \approx 10^{10^{25.1611}}.$$

We also adopt a simplified notation for iterated factorials (not to be confused with other notation that use multiple exclamation marks) in order to avoid cumbersome bracketing:

$$k\underbrace{!\cdots!}_{\times n} = F_n(k),$$

meaning, for example $3!!! = 6!! = 720!$. Additionally, we use the following notation for $a > 1$ and $b > 1$ (assuming the described operation makes sense in real numbers):

$$E_n(a, b) = \begin{cases} b, & \text{if } n = 0; \\ a^{E_{n-1}(a,b)}, & \text{if } n > 0, \end{cases}$$

[a]Note that if $k = 1$ or $k = 2$, then the numbers $F_n(k)$ form a constant sequence.

$$a \uparrow\uparrow n = E_{n-1}(a,a),$$

$$\log_a^{(n)} b = \begin{cases} 1, & \text{if } n = 0; \\ \log_a b, & \text{if } n = 1; \\ \log_a(\log_a^{(n-1)} b), & \text{if } n > 1. \end{cases}$$

Let us go back to stating the result from [3].

Theorem 1. *Suppose $a, b > e^{1/e}$ and $a \neq b$. There exists $N \geq 0$, $k \in \mathbb{Z}$ and a converging sequence $(c_n)_{n \geq N}$ such that $1 < c_n < b$ and*

$$a \uparrow\uparrow n = E_{n+k}(b, c_n).$$

In this article, we explore a new scenario that has not been considered before: the comparison of iterated factorials. We also consider the continuous version of factorials, namely, the Euler gamma function. This leads to several interesting results, including the representation of the iterated gamma function of fixed number as a power tower with regular behaviour. Surprisingly, the opposite is also true: iterated exponentiation can be represented as iterations of a converging sequence under the gamma function. This in turn seems both interesting and unexpected, as according to Velleman's results [8] if the power tower just slightly surpasses the iterated factorial, the ratio between further iterations of the respective numbers tends to infinity.

Note that the power tower representation of the iterated factorial or the gamma function (or any other number) is useful because knowing the height of the tower and the top term of the tower is sufficient to compare the two numbers. This, combined with the fact that the mutual comparison between the iterated factorial and the iterated exponentiation seems limited to just [8] motivates us to investigate the topic further.

Recall the following lemmas from [3], which will be useful in the next sections.

Lemma 2. *Suppose $b > a > e^{1/e}$ and define a sequence*

$$a_n = \log_a^{(n)}(b \uparrow\uparrow n).$$

Then $(a_n)_{n \in \mathbb{N}}$ is an increasing sequence and $\lim_{n \to +\infty} a_n < +\infty$.

Lemma 3. *Suppose $a, b > e^{1/e}$ and $n > 0$. Then there is $p \geq 0$ and $c \in (1, b]$ such that*

$$a \uparrow\uparrow n = E_p(b, c).$$

2 Comparing factorials

In this section, we focus on comparing iterated factorials in two ways. First, we directly compare one iterated factorial with another. Then we show that $F_n(k)$ can be replaced by an appropriate power tower.

Theorem 4. *Suppose $n, n' \geq 3$ and $k > k'$ are given. If $F_{k-k'}(n) > n'$, then $F_k(n) > F_{k'}(n')$.*

Proof. We have $F_k(n) = F_{k'}(F_{k-k'}(n)) > F_{k'}(n')$. □

The above result is not surprising, as we are merely comparing a few initial iterations of the factorial.

The following theorem shows that for each iterated factorial, one can find a "representation" as a power tower. This mimics the result obtained in Theorem 1. In the article, when the base of the logarithm is not given, we mean base 10 logarithms.

Theorem 5. *For any $k \geq 3$ there is $\ell \in \mathbb{Z}$, $N > 0$ and the sequence $(c_n)_{n\in\mathbb{N}}$ such that for each $n \geq N$ we have $10 \geq c_n > 1$ and*

$$F_n(k) = E_{n+\ell}(10, c_n).$$

The sequence $(c_n)_{n\geq N}$ is increasing and convergent. Moreover, we have

$$\lim_{n\to+\infty} \frac{E_{n+\ell}(10, c)}{F_n(k)\ln F_n(k)} = \frac{1}{\ln 10}. \tag{2}$$

Proof. Note that if $n \geq 2$, then $F_n(k) > 10$. Set $N_1 = 2$ and for $n \geq N_1$ define

$$b_n := \log^{(n-2)} F_n(k) = \log^{(n-2)} F_{n-2}(F_2(k)). \tag{3}$$

Then by the result of Velleman [8] there is $N_2 > 0$ such that for each $n \geq N_2$ we have

$$F_{n-2}(F_2(k)) < E_{n-2}(F_2(k), F_2(k)) = F_2(k) \uparrow\uparrow (n-1). \tag{4}$$

Then for $n \geq 3$,

$$\begin{aligned}
\log b_n &= \log\left(\log^{(n-2)} F_{n-2}(F_2(k))\right) \\
&< \log\left(\log^{(n-2)} F_2(k) \uparrow\uparrow (n-1)\right) \\
&= \log^{(n-1)} F_2(k) \uparrow\uparrow (n-1).
\end{aligned}$$

and by Lemma 2 the sequence of numbers $a_n = \log^{(n-1)} F_2(k) \uparrow\uparrow (n-1)$ is convergent. To prove that $(b_n)_{n\geq 0}$ is convergent, it is therefore enough to show that (b_n) is increasing.

Notice that for $m \geq 28$ we have $\log m! > m$. Indeed, since the latter is equivalent to $m! > 10^m$, using the the well-known bound $m! > (m/e)^m$ we have for $m \geq 28$:

$$m! > \left(\frac{m}{e}\right)^m > 10^m. \tag{5}$$

Furthermore, for $m \geq 3$ we have $m!! \geq 3!! = 720$, which by (5) implies that

$$\log\log\log m!!! > \log\log m!!$$

for any $m \geq 3$. In particular, for $n \geq 5$, we have

$$\begin{aligned} b_n &= \log^{(n-5)}(\log\log\log F_{n-5}(F_2(k))!!!) \\ &> \log^{(n-5)}(\log\log F_{n-5}(F_2(k))!!) \\ &= \log^{(n-3)} F_{n-3}(F_2(k)) = b_{n-1}, \end{aligned}$$

so $(b_n)_{n\geq 5}$ is an increasing sequence. Furthermore,

$$b_5 = \log^{(3)} F_3(F_2(k)) \geq \log^{(3)} F_5(3) > 1749,$$

and thus there is $b > 1749$ such that $b_n \to b$.

By Lemma 3 there is $p \geq 0$ and $c \in (1, 10]$ such that $b = E_p(10, c)$. Set $b_n = E_p(10, c_n)$ for suitable choice of c_n. Since $c_n \to c \in (1, 10]$ and c_n is increasing, there is $N \geq \max\{5, N_2\}$ such that $c_N > 1$. It remains to notice that for $n \geq N$, using the definition (3), we have

$$F_n(k) = E_{n-2}(10, E_p(10, c_n)) = E_{n+p-2}(10, c_n).$$

We now show that (2) holds. The argument used in this proof is similar to the one used by Velleman [8]. First, we show that for $n \geq N$ we have

$$\frac{F_n(k)(\ln F_n(k) - 1)}{\ln 10} \leq E_{n+\ell}(10, c) \leq \frac{F_n(k)\ln F_n(k)}{\ln 10}. \tag{6}$$

Suppose that for some $n \geq N$ we have

$$E_{n+\ell}(10, c) < \frac{F_n(k)(\ln F_n(k) - 1)}{\ln 10}.$$

Then

$$E_{n+\ell+1}(10, c) = e^{E_{n+\ell}(10,c)\ln 10} < e^{F_n(k)(\ln F_n(k)-1)} = \left(\frac{F_n(k)}{e}\right)^{F_n(k)} < F_n(k)! = F_{n+1}(k).$$

As $c_{n+1} \leq c$, we now have

$$F_{n+1}(k) = E_{n+\ell+1}(10, c_{n+1}) \leq E_{n+\ell+1}(10, c) < F_{n+1}(k),$$

which is a contradiction. Next, suppose that for some $n \geq N$ we have

$$E_{n+\ell}(10, c) > \frac{F_n(k)\ln F_n(k)}{\ln 10}.$$

Since $(c_m)_{n\geq N}$ is increasing and the function $x \mapsto E_{n+\ell}(10, x)$ is continuous, there is $M \geq n$ such that for each $m \geq M$ we have

$$E_{n+\ell}(10, c_m) > \frac{F_n(k)\ln F_n(k)}{\ln 10} \qquad \text{and} \qquad F_m(k) \geq 10.$$

Then, using the fact that $r^r > r! \ln r!$ ($r \geq 3$ is a positive integer; see [8] for the proof of that inequality), we get

$$\begin{aligned} E_{n+\ell+1}(10, c_m) = e^{E_{n+\ell}(10,c_m)\ln 10} &> e^{F_n(k)\ln F_n(k)} = (F_n(k))^{F_n(k)} \\ &> F_n(k)! \ln F_n(k)! = F_{n+1}(k) \ln F_{n+1}(k) \\ &> \frac{F_{n+1}(k) \ln F_{n+1}(k)}{\ln 10}. \end{aligned}$$

Thus, by induction,

$$E_{n+\ell+s}(10, c_m) > \frac{F_{n+s}(k) \ln F_{n+s}(k)}{\ln 10}$$

for all $s \geq 0$. This implies that, as $m \geq n$, we also have

$$F_m(k) = E_{m+\ell}(10, c_m) > \frac{F_m(k) \ln F_m(k)}{\ln 10} \geq F_m(k),$$

which is another contradiction. Summing up, we have established that (6) hold for all n large enough. But that bounds can be rewritten to

$$\frac{1}{\ln 10} - \frac{1}{\ln 10 \ln F_n(k)} \leq \frac{E_n(10, c)}{F_n(k) \ln F_n(k)} \leq \frac{1}{\ln 10},$$

from which (2) follows. This concludes the proof. □

Remark 6. It is clear that the decimal base in Theorem 5 is not special: the same result can be proved for any base $b > e^{1/e}$ instead of 10. The only significant difference is the estimation for m such that $\log_b m! > m$, or $m/e > b$.

Example 7. Let $k = 9$, then (with a little help from WolframAlpha [9]), we get

$$\begin{aligned} 9! &= 362\,880, \\ 9!! &= 10^{10^{6.269497348270999\ldots}} \approx E_2(10, 6.269497348270999), \\ 9!!! &= 10^{10^{10^{6.269498812196401\ldots}}} \approx E_3(10, 6.269498812196401), \\ 9!!!! &\approx E_4(10, 6.269498812196401). \end{aligned}$$

This suggests that

$$F_n(9) \approx E_n(10, 6.269498812196401)$$

for all $n \geq 3$. This approximation is correct and justified by Theorem 5. Furthermore, a stronger approximation can be provided:

$$\frac{E_n(10, 6.269498812196401)}{F_n(9) \ln F_n(9)} \approx 1.$$

We can easily find out that the following approximation is also correct:

$$336\,500 \uparrow\uparrow n \approx E_n(10, 6.2694) \approx F_n(9).$$

Thus it is interesting to formalize the approximation. We present the result below.

Theorem 8. *For any $k \geq 3$ there is $b_n > e^{1/e}$ such that $F_n(k) = b_n \uparrow\uparrow n$. The sequence $(b_n)_{n\in\mathbb{N}}$ converges to some $b \leq k!$.*

Proof. Write $a_n := F_n(k)$. It follows from a standard intermediate-value argument that one can find b_n such that $b_n \uparrow\uparrow n = F_n(k)$ (for similar reasoning we refer to [3]). We are thus left to show that $(b_n)_{n\in\mathbb{N}}$ is bounded and eventually increasing.

Using an argument similar to (3) and (4) we get for sufficiently large n that

$$F_n(\lfloor b_{n+1} \rfloor) < b_{n+1} \uparrow\uparrow (n+1) = F_n(k!)$$

and thus $b_n \leq k!$ for all n large enough.

We now show that $b_n < b_{n+1}$ for all n large enough. First, we show that for sufficiently large n we have

$$\frac{b_n \uparrow\uparrow n}{e} > b_n. \tag{7}$$

This is equivalent to

$$b_n^{b_n \uparrow\uparrow (n-1) - 1} > e,$$

or to

$$b_n \uparrow\uparrow (n-1) > \log_{b_n} e + 1. \tag{8}$$

Recall that if $s > e^{1/e}$, then $s \uparrow\uparrow n \to +\infty$ and if $e^{-1/e} \leq s \leq e^{1/e}$, then $s \uparrow\uparrow n$ converges (see [1]). As $F_n(k)$ diverges for $k \geq 3$, from that we get $b_n > e^{1/e}$ and $\log_{b_n} e < e$, which implies (8) and thus (7) is established for all sufficiently large n.

Finally, we write

$$b_{n+1} \uparrow\uparrow (n+1) = (b_n \uparrow\uparrow n)! > \left(\frac{b_n \uparrow\uparrow n}{e}\right)^{b_n \uparrow\uparrow n} > b_n^{b_n \uparrow\uparrow n} = b_n \uparrow\uparrow (n+1),$$

conclude the inequality $b_n < b_{n+1}$, and finish the proof. □

3 The case of Euler gamma function

We now move our attention to the continuous version of the factorial. Here, the Euler gamma function is considered on positive real numbers

$$\Gamma(x) := \int_0^{+\infty} t^{x-1} e^{-x} dt.$$

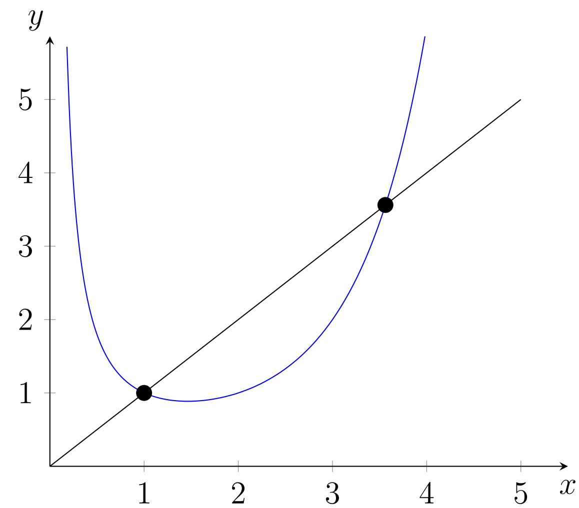

Figure 1: Solutions to $\Gamma(x) = x$.

The equation $\Gamma(x) = x$ has two positive solutions (see Figure 1). There is one trivial solution $\Gamma(1) = 1$, the second one is $z \approx 3.562$. If we let $\Gamma^n(x)$ denote the n-th iteration of x under Γ (i.e., $\Gamma^0(x) = x, \Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$ for $n > 0$), then for each $x > z$ we have $\lim_{n\to+\infty} \Gamma^n(x) = +\infty$. Thus, from now on we consider the case $x > z$.

Our goal is to extend Theorem 5 to the gamma function. Note that since $\Gamma(x) = (x-1)!$ for $x \in \mathbb{N} \setminus \{0\}$, the extension improves the theorem to real values above $x = 3$.

We also highlight to the reader that throughout the rest of the paper, the constant "z" is reserved for the non-trivial solution of $\Gamma(x) = x$.

We now establish a generalization of Theorem 5 to the gamma function. We begin with the following lemma, which provides a weaker (yet more useful for the purpose of this article) version of the inequality obtained in [5], but with a proof that is considerably more elementary.

Lemma 9. *For all $x \geq 1$ we have*

$$\Gamma(x+1) > \left(\frac{x}{e}\right)^x. \tag{9}$$

For all $x > 4$ we have

$$\Gamma(x+1)\ln\Gamma(x+1) \leq \Gamma(x)^{\Gamma(x)}. \tag{10}$$

Proof. First, we show that the inequality (9) holds for $x \in [1, 2]$. For that, let us define the function $f(x) = \ln\left(\frac{x}{e}\right)^x$. Then for $x \in [1, 2]$ we have $f(x) < 0$. On the other hand, $\ln\Gamma(x+1) \geq \ln\Gamma(2) \geq 0$ (see Figure 2). This implies that inequality (9) is established for $x \in [1, 2]$.

Notice that since $(1 - \frac{1}{x})^{x-1} > \frac{1}{e}$, we have

$$x \cdot \frac{(x-1)^{x-1}}{e^{x-1}} > \frac{x^x}{e^x}.$$

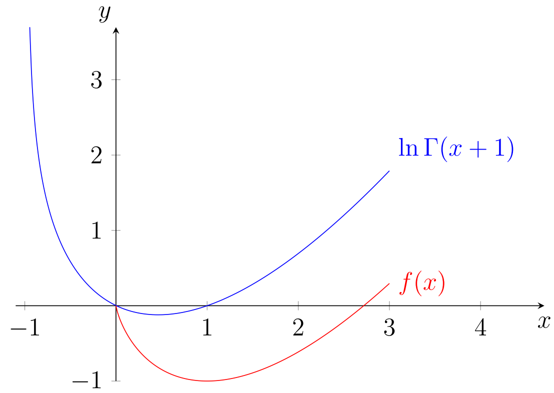

Figure 2: Graph of functions $\ln\Gamma(x+1)$ and $f(x)$ for $x > 0$.

This means that if the inequality is established for $x \in [n-1, n]$ with integer $n \geq 2$, it is also valid in the interval $[n, n+1]$, since

$$\Gamma(x+1) = x\Gamma(x) > x \cdot \frac{(x-1)^{x-1}}{e^{x-1}} > \frac{x^x}{e^x}.$$

This remark and simple induction completes the proof of (9).

The inequality (10) is equivalent to

$$\ln(\Gamma(x+1))^x \leq \Gamma(x)^{\Gamma(x)-1}. \tag{11}$$

We apply a simple bound $\Gamma(y) \leq y^{y-1}$ (valid for $y \geq 1$) to have

$$(\Gamma(x+1))^x \leq ((x+1)^x)^x = e^{x^2\ln(x+1)} \leq e^{x^3}$$

and we apply simple bounds $\Gamma(x) > (3/2)^x$, $\Gamma(x) > x$, both valid for $x > 4$, to obtain

$$(\Gamma(x))^{\Gamma(x)-1} > \left(\left(\frac{3}{2}\right)^x\right)^{x-1} = \left(\frac{3}{2}\right)^{x^2-x}.$$

To show that (11) holds, it is now sufficient to show that for $x > 4$ we have

$$x^3 < \left(\frac{3}{2}\right)^{x^2-x},$$

which is easily verifiable. □

Theorem 10. *For any $b > e^{1/e}$ and any $x \geq z$ there is $\ell \in \mathbb{Z}$, $N > 0$ and a sequence $(c_n)_{n\in\mathbb{N}}$ such that for $n \geq N$ we have $10 \geq c_n > 1$ and*

$$\Gamma^n(x) = E_{n+\ell}(b, c_n).$$

The sequence $(c_n)_{n\geq N}$ is increasing and convergent. Finally, we have

$$\lim_{n\to+\infty} \frac{E_{n+\ell}(b,c)}{((\Gamma^n(k)-1)\ln(\Gamma^n(k)-1)} = \frac{1}{\ln b}. \tag{12}$$

Proof. Let $Y > 0$ be such that for all $y \geq Y$ we have $\log_b \Gamma(y) > y$. The number Y exists because we have, using (9), the following chain of inequalities

$$\log_b \Gamma(y) > \log_b \left(\frac{y-1}{e}\right)^{y-1} > y,$$

while the last one is true as it is equivalent to

$$\left(\frac{y-1}{e}\right)^{y-1} > b^y,$$

which clearly holds for all $y \geq Y$ with $Y \geq b$ sufficiently large.

Let $N > 0$ be such that $\Gamma^N(x) > Y$ and set

$$b_n = \log_b^{(n)} \Gamma^{N+n}(x). \tag{13}$$

This sequence is well-defined and increasing. Indeed, notice that $b_0 = \Gamma^N(x)$ and if b_n is well-defined for some $n \geq 0$, then since we can apply $\log_b(\,\cdot\,)$ n times to $\Gamma^{N+n}(x)$ and

$$\log_b \Gamma(\Gamma^{N+n}(x)) > \Gamma^{N+n}(x), \tag{14}$$

we can apply $\log_b(\,\cdot\,)$ n times to $\log_b \Gamma(\Gamma^{N+n}(x))$ so that b_{n+1} is well-defined. We also notice that (14) implies $b_{n+1} > b_n$.

Set $k = \lceil x \rceil$. Then $k \geq 4$ and

$$\Gamma(x) \leq \Gamma(k) = (k-1)!.$$

Furthermore,

$$\Gamma^2(x) < \Gamma^2(k) = \Gamma((k-1)!) = ((k-1)! - 1)! < k!!$$

and by simple induction,

$$\Gamma^n(x) < F_n(k) \tag{15}$$

for all $n \geq 1$ (for $n = 0$ we can have equality).

Consider the sequence $(b_n)_{n\geq 0}$ again and notice that by (15),

$$b_n = \log_b^{(n)} \Gamma^{N+n}(x) < \log_b^{(n)} F_{N+n}(k) = \log_b^{(n)} F_n(F_N(k))$$

for all $n \geq 1$. Finally, we have $F_N(k) > \Gamma^N(x) \geq Y$, thus, by the proof of Theorem 5, the sequence $d_n = \log_b^{(n)} F_n(F_N(k))$ is convergent.

We have constructed two sequences of numbers $(b_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ such that

- $(b_n)_{n\geq 0}$ is increasing,
- $b_n < d_n$ for $n \geq 1$,

- $(d_n)_{n\geq 0}$ is convergent.

Thus $(b_n)_{n\geq 0}$ is convergent. By (13),

$$\Gamma^{N+n}(x) = E_n(b, b_n).$$

Let $c_n \in (1, b]$ and p_n be the numbers defined via relation (see Lemma 3)

$$b_n = E_{p_n}(b, c_n).$$

Since $(b_n)_{n\geq 0}$ is convergent, so is $(c_n)_{n\geq 0}$ and the sequence $(p_n)_{n\geq 0}$ is eventually constant. More precisely, let $c = \lim_{n\to+\infty} c_n \in (1, b]$ and $p = \lim_{n\to+\infty} p_n$. Then there is $N_1 \geq N$ such that $p_n = p$ and $c_n \in (1, 10]$ for each $n \geq N_1$. Then for such n,

$$\Gamma^{N+n}(x) = E_{n+p}(b, c_n)$$

and finally,

$$\Gamma^n(x) = E_{n+p-N}(b, c_n).$$

Now we show that (12) holds. For that, the argument is similar to the one presented in the proof of Theorem 5. The goal is to show that

$$\frac{(\Gamma^n(k) - 1)(\ln(\Gamma^n(k) - 1) - 1)}{\ln b} \leq E_{n+\ell}(b, c) \leq \frac{(\Gamma^n(k) - 1)\ln(\Gamma^n(k) - 1)}{\ln b}. \tag{16}$$

We apply inequality (9) to show the lower bounds of (16) and we apply inequality (10) for the upper bounds. We omit the details. □

Let us perform several calculation of the following form:

$$10 \approx \Gamma(4.39007), \qquad 10^{10} \approx \Gamma^2(4.64095), \qquad 10^{10^{10}} \approx \Gamma(1160000000) \approx \Gamma^3(4.5982).$$

We observe that the arguments of the iterated gamma function are "not that high", which may suggest that one can replace power towers with the iterated gamma function. This turns out to be true: the process described by Theorem 10 can be reversed. Before addressing this problem, let us prove the following auxiliary lemma.

Lemma 11. *For any $n > 0$ and any $b \geq e$ we have*

$$\Gamma(E_n(b, b)) > E_{n+1}(b, b).$$

Proof. We apply the inequality (9) in the following form

$$\Gamma(y) > \left(\frac{y-1}{e}\right)^{y-1}$$

again with $y = E_n(b,b)$. Then, using $b \geq e$ and the Bernoulli inequality,

$$\begin{aligned}
\Gamma(E_n(b,b)) &> \left(\frac{E_n(b,b)-1}{e}\right)^{E_n(b,b)-1} \\
&\geq \left(\frac{E_n(b,b)-1}{b}\right)^{E_n(b,b)-1} \\
&= \left(\frac{E_n(b,b)}{b}\right)^{E_n(b,b)-1} \left(1 - \frac{1}{E_n(b,b)}\right)^{E_n(b,b)-1} \\
&\geq \left(\frac{E_n(b,b)}{b}\right)^{E_n(b,b)-1} \left(1 - \frac{E_n(b,b)-1}{E_n(b,b)}\right) \\
&= \left(\frac{E_n(b,b)}{b}\right)^{E_n(b,b)-1} \cdot \frac{1}{E_n(b,b)} \\
&= \left(b^{E_{n-1}(b,b)-1}\right)^{E_n(b,b)-1} \cdot b^{-E_{n-1}(b,b)} \\
&= b^{(E_{n-1}(b,b)-1)(E_n(b,b)-1)-E_{n-1}(b,b)} \\
&= b^{E_{n-1}(b,b)E_n(b,b)-E_n(b,b)-2E_{n-1}(b,b)+1}.
\end{aligned}$$

To finish the proof it is enough to show that

$$E_{n-1}(b,b)E_n(b,b) - E_n(b,b) - 2E_{n-1}(b,b) + 1 > E_n(b,b),$$

or, equivalently,

$$E_{n-1}(b,b) - \frac{2E_{n-1}(b,b)-1}{E_n(b,b)} > 2.$$

Notice that $E_{n-1}(b,b) \geq e$ for each $n > 0$, thus it is enough to show that

$$\frac{2E_{n-1}(b,b)-1}{E_n(b,b)} < e - 2. \tag{17}$$

Let $x = E_{n-1}(b,b)$. Then the inequality can be seen as

$$\frac{2x-1}{b^x} < e - 2 \qquad \text{or} \qquad 2x - 1 < (e-2)b^x.$$

Notice again that $b \geq e$. Hence

$$(e-2)b^x \geq (e-2)e^x > \frac{e^x}{2} > 2x - 1,$$

which implies (17) and completes the proof. □

We note that this lemma actually shows the generalization to arbitrary base $b > z$ of the problem investigated by Velemann [8], who solved a simplified version of the above for factorials and the case $b = 10$.

We are now ready to prove our final result.

Theorem 12. *Let $b > z$. There is a converging (and decreasing for sufficiently large n) sequence $(x_n)_{n\geq 0}$ such that $b \uparrow\uparrow n = \Gamma^n(x_n)$.*

Proof. Fix $x > z$. By Theorem 10 we can find ℓ and $N > 0$ such that for $n \geq N$,

$$\Gamma^n(x) = E_{n+\ell}(b, c_n).$$

Here, $b \geq c_n > 1$ and $c_n \to c$. After suitable substitution $x \mapsto \Gamma^\ell(x)$ and correction of N (this substitution preserves the assumption that $x > z$) we can assume that $\ell = 0$, i.e.,

$$\Gamma^n(x) = E_n(b, c_n).$$

In case $\ell < 0$ we are interpreting $\Gamma^{-1}(x)$ to be the unique $x' > z$ such that $\Gamma(x') = x$. Notice that

$$\Gamma^n(\Gamma(x)) = \Gamma^{n+1}(x) = E_{n+1}(b, c_{n+1}) > E_{n+1}(b, 1) = E_n(b, b) \geq E_n(b, c_n) = \Gamma^n(x),$$

hence by the intermediate value theorem for each suitable n there is $x_n \in [x, \Gamma(x))$ such that

$$\Gamma^n(x_n) = E_n(b, b).$$

Let $x'_n = \Gamma^{-1}(x_n)$. Thus

$$\Gamma^{n+1}(x'_n) = E_n(b, b).$$

Shifting indices (i.e., we rename x'_n to x'_{n+1}) and simplifying notation $x'_n \to x_n$, we have

$$\Gamma^n(x_n) = E_{n-1}(b, b) = b \uparrow\uparrow n.$$

Notice that the numbers x_n form a bounded sequence and thus to finish the proof it is enough to show that this sequence is monotone.

We have by Lemma 11,

$$\Gamma^{n+1}(x_n) = \Gamma(\Gamma^n(x_n)) = \Gamma(E_{n-1}(b, b)) > E_n(b, b) = \Gamma^{n+1}(x_{n+1}),$$

thus $x_n > x_{n+1}$ and hence $(x_n)_{n\geq 0}$ is a decreasing sequence. This completes the proof. □

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References

[1] J. Anderson, Iterated exponentials, *Amer. Math. Monthly* **111** (2004), 668–679.

[2] L. Euler, De formulis exponentialibus replicatis, *Acta Acad. Sci. Petropolitanae* **1** (1778), 38–60.

[3] K. Gryszka, How to compare power towers? *Lith. Math. J.* **62** (2022), 192–206.

[4] R. A. Knoebel, Exponentials reiterated, *Amer. Math. Monthly*, **88** (1981), 235–252 .

[5] X. Li and Chao-Ping Chen, Inequalities for the gamma function, *Journal of Inequalities in Pure and Applied Mathematics* **8** (2007), Article 28.

[6] D. B. Shapiro and S. D. Shapiro, Iterated exponents in number theory, *Integers* **7** (2007), # A23.

[7] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, available at `https://oeis.org`, 2025.

[8] D. J. Velleman, Exponential vs. factorial, *Amer. Math. Monthly* **113** (2006), 689–704.

[9] Wolfram Alpha LLC. 2021. Wolfram|Alpha. `https://www.wolframalpha.com/`.

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