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Counting Subwords in Non-Decreasing Dyck Paths

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Abstract

A Dyck path is *non-decreasing* if the y-coordinates of its valleys, which are local minima along the path, form a non-decreasing sequence. To count the number of subpaths (or subwords) within non-decreasing Dyck paths, we use generating functions and recursive relations. These techniques enable us to count subpaths of various lengths, such as two, three, four, and five, while considering both the length and the number of subwords. Using constructive recursive proofs, we establish a combinatorial interpretation of the counting process, expressing the results as linear combinations of Fibonacci and Lucas numbers. In several cases, we provide new interpretations of sequences from the On-Line Encyclopedia of Integer Sequences (OEIS).

1 Introduction

A *Dyck path* of semilength n is a lattice path in the first quadrant of the xy-plane that begins at the origin $(0,0)$ and ends at $(2n,0)$, consisting of North-East steps $U=(1,1)$ and South-East steps $D=(1,-1)$. In a Dyck path P, a *valley* is a subpath of the form DU, while a *peak* is a subpath of the form UD. Informally, valleys correspond to local minima, and peaks correspond to local maxima. To encode a Dyck path of semilength n, we use a word $w=w_1w_2\cdots w_{2n}$ from the alphabet $\{U,D\}$, referred to as a *Dyck word.* The occurrence of a *subword* v in a Dyck word w means that there exist subwords u_1 and u_2 from $\{U,D\}^*$ such that $w=u_1vu_2$. For instance, the valleys and peaks can be identified as the subwords DU and UD, respectively. In enumerative combinatorics, a classic problem involves counting the occurrences of a given subword within a combinatorial structure. For example, Deutsch [4] studied the distribution of subwords of length 2, namely UU, UD, DU, and DD, in the set of Dyck words. The study of subwords of length 3, specifically UUU, UUD, UDU, UDD, DUU, DUD, DDU, and DDD, has been explored by various researchers, including Deutsch [4], Mansour [13], Merlini et al. [14], Sapounakis et al. [18], and Sun [19].

Barcucci et al. [1] introduced the notion of a *non-decreasing* Dyck path, defining it as a Dyck path P where the y-coordinates of its valleys form a non-decreasing sequence. Other recent results on this topic can be found in [2, 3, 5, 6, 8–12, 15, 16]. This paper focuses on enumerating subwords of various lengths, including 2, 3, 4, as well as certain general cases. We present our results using generating functions and recursive relations, expressing the formulas as linear combinations of Fibonacci and Lucas numbers. We recall that the Fibonacci sequence is defined by the recurrence relation $F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$, with initial values $F_0=0$ and $F_1=1$. Similarly, the Lucas sequence is given by $L_n=L_{n-1}+L_{n-2}$ for $n\geq 2$, with initial values $L_0=2$ and $L_1=1$.

Let $\mathcal{D}_n$ denote the set of all non-decreasing Dyck words of length $2n$, and let $\mathcal{D}=\bigcup_{k\geq 1}\mathcal{D}_k$ represent the union of these sets. Barcucci et al. [1] proved that $|\mathcal{D}_n|=F_{2n-1}$.

When considering a subword $v\in\{U,D\}^*$, we use $|w|_v$ to denote the number of occurrences of the subword v in w, including overlapping instances. To analyze this, we introduce the bivariate generating function $\boldsymbol{F}_v(x,y)$ defined as follows:

$$\boldsymbol{F}_v(x,y)=\sum_{n\geq 1}x^n\sum_{w\in\mathcal{D}_n}y^{|w|_v}.$$

Let $\boldsymbol{d}_v(n)$ denote the total number of occurrences of the subword v in the Dyck words of $\mathcal{D}_n$. The generating function for the sequence $\boldsymbol{d}_v(n)$ can be obtained by differentiating $\boldsymbol{F}_v(x,y)$ with respect to y and then setting $y=1$ we have

$$\boldsymbol{D}_v(x):=\left.\frac{\partial\boldsymbol{F}_v(x,y)}{\partial y}\right|_{y=1}.$$

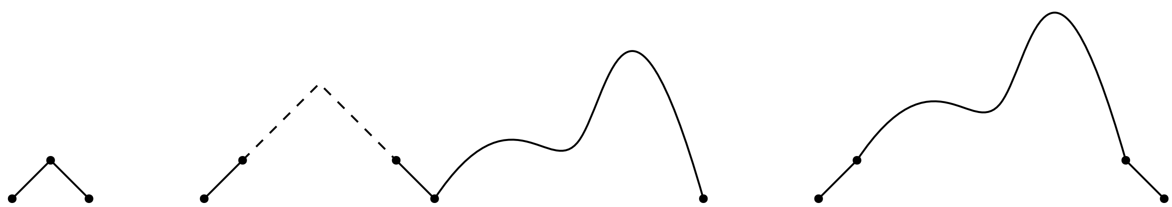
A *pyramid* of semi-length $h\geq 1$ in a Dyck word is a subword of the form U^hD^h. A pyramid is considered *maximal* and is denoted by Δ_h if it cannot be extended to a larger pyramid $U^{h+1}D^{h+1}$. That is, if a symbol appears immediately before or after Δ_h, it must be D or U, respectively. The empty pyramid is represented by Δ_0.

Based on the definition of a non-decreasing Dyck word w, it can be decomposed using the *first return decomposition*. This decomposition yields three cases: $w = UD$, the minimal word, $w = \Delta v$, or $w = UvD$, where $\Delta \neq \Delta_0$ is a pyramid and $v \in \mathcal{D}$. This decomposition is shown in Figure 1. Here, $|w|$ denotes the length of the word w. The first return decomposition leads to the following equation for the generating function $D(x) = \sum_{w \in \mathcal{D}} x^{|w|/2}$:

$$D(x) = x + \frac{x}{1-x} D(x) + xD(x). \tag{1}$$

By solving this equation, we obtain the generating function for non-decreasing Dyck words of length $2n$:

$$D(x) = \frac{x(1-x)}{1-3x+x^2} = \sum_{n=1}^{\infty} F_{2n-1} x^n.$$



(1) (2) (3)

Figure 1: Factorization of non-decreasing Dyck paths.

2 Words of length two

In this section, we focus on counting the number of subwords of length two in $\mathcal{D}_n$. Czabarka et al. [3] obtained the generating function for the subwords UD and DU. Therefore, we present their theorems without providing their proofs. However, we introduce new generating functions and recursive relations specifically for the number of subwords of the form UU and DD.

Theorem 1. *The number of subwords of the form UD in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{UD}(x,y) = \frac{(1-x)xy}{1-2x+x^2-xy} \quad \textit{and} \quad \boldsymbol{D}_{UD}(x) = \frac{x(1-x)^3}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 1$, we have

$$\boldsymbol{d}_{UD}(n) = \frac{1}{10}((5n-7)F_{2n} - (n-5)L_{2n}).$$

Proof. To derive an expression for $\boldsymbol{F}_{UD}(x,y)$, we refine Equation (1) by introducing a variable y to track the number of occurrences of the subword UD in the word. By applying the same decomposition, we get

$$\boldsymbol{F}_{UD}(x,y) = xy + \frac{xy}{1-x}\boldsymbol{F}_{UD}(x,y) + x\boldsymbol{F}_{UD}(x,y).$$

Solving this functional equation yields the expressions for $\boldsymbol{F}_{UD}(x,y)$. Since $\boldsymbol{d}_{UD}(n)$ corresponds to the number of peaks in $\mathcal{D}_n$, the recursive relation follows from [3, Theorem 2]. The combinatorial formula of $\boldsymbol{d}_{UD}(n)$ follows from some Fibonacci identities. □

Theorem 2. *The number of subwords of the form DU in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{DU}(x,y) = \frac{(1-x)x}{1-2x+x^2-xy} \quad \textit{and} \quad \boldsymbol{D}_{DU}(x) = \frac{(1-x)x^2}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 1$, we have

$$\boldsymbol{d}_{DU}(n) = \frac{1}{10}((5n-2)F_{2n} - nL_{2n}).$$

Since the $\boldsymbol{d}_{DU}(n)$ corresponds to the number of valleys in $\mathcal{D}_n$, the proof of the previous theorem follows from [3, Theorems 14].

Theorem 3. *Let $w \in \{UU, DD\}$. The number of subwords of the form w in $\mathcal{D}_n$ is given by*

$$\boldsymbol{F}_{w}(x,y) = \frac{x(1-xy)}{1-x-2xy+x^2y^2} \quad \textit{and} \quad \boldsymbol{D}_{w}(x) = \frac{x^2\left(1-x+x^2\right)}{\left(1-3x+x^2\right)^2}.$$

Moreover, for $n \geq 1$, we have

$$\boldsymbol{d}_{w}(n) = \frac{1}{10}((7-10n)F_{2n} + (6n-5)L_{2n}). \tag{2}$$

Proof. We prove the case for subwords of the form UU; the case DD is similar and is therefore omitted. From the first return decomposition (as shown in Figure 1), any nonempty non-decreasing Dyck path w can be uniquely decomposed as either UD, $\Delta_1 v$, $\Delta_h v$, or UvD, where $v \in \mathcal{D}$ and $h \geq 2$. Utilizing the symbolic method, see for example Flajolet and Sedgewick [7], we get the functional equation:

$$\boldsymbol{F}_{UU}(x,y) = \underbrace{x}_{(1)} + \underbrace{\left(x + \frac{x^2y}{1-xy}\right)\boldsymbol{F}_{UU}(x,y)}_{(2)} + \underbrace{xy\boldsymbol{F}_{UU}(x,y)}_{(3)}.$$

Here, the labels (1), (2), and (3) in the functional equation correspond to the labels in Figure 1. By solving this functional equation, we obtain the desired generating function for $\boldsymbol{F}_{UU}(x,y)$.

Next, we provide a combinatorial proof of (2). From the previous argument, we can express $\boldsymbol{d}_{UU}(n)$ (for $n > 0$) in terms of Fibonacci and Lucas numbers. Specifically, considering the first return decomposition in Figure 1, any path $w \in \mathcal{D}_n$ can be written as $\Delta_k z$ or UvD, where $v \in \mathcal{D}_{n-1}$, $z \in \mathcal{D}_{n-k}$, and $1 \leq k < n$.

Firstly, we note that the number of subwords of the form UU in Δ_k is $k-1$. This implies that the number of subwords of the form UU in a path of the form $\Delta_k z$ is $(k-1)F_{2(n-k)-1} + \boldsymbol{d}_{UU}(n-k)$, for a fixed $1 \leq k < n$, (since $|\mathcal{D}_{n-k}| = F_{2(n-k)-1}$). Secondly, we observe that the number of subwords UU in the decomposition UvD is equal to the number of subwords UU in v plus one. Thus, for $k = 1, \dots, n-1$, we obtain the recurrence relation:

$$
\begin{aligned}
\boldsymbol{d}_{UU}(n) &= \boldsymbol{d}_{UU}(n-1) + \sum_{k=2}^{n-1}\big((k-1)F_{2(n-k)-1} + \boldsymbol{d}_{UU}(n-k)\big) + \boldsymbol{d}_{UU}(n-1) + F_{2(n-1)-1} \\
&= 2\boldsymbol{d}_{UU}(n-1) + \sum_{k=1}^{n-1}(n-k-1)F_{2k-1} + \sum_{k=1}^{n-1}\boldsymbol{d}_{UU}(k) + F_{2n-3} \\
&= 2\boldsymbol{d}_{UU}(n-1) + F_{2n-3} + \sum_{k=1}^{n-1}\boldsymbol{d}_{UU}(k) + F_{2n-3} \\
&= 2\boldsymbol{d}_{UU}(n-1) + 2F_{2n-3} + \sum_{k=1}^{n-1}\boldsymbol{d}_{UU}(k).
\end{aligned}
$$

Simplifying the equation, we get

$$
\boldsymbol{d}_{UU}(n+1) - \boldsymbol{d}_{UU}(n) = 2\boldsymbol{d}_{UU}(n) - \boldsymbol{d}_{UU}(n-1) + 2(F_{2n-1} - F_{2n-3}).
$$

Therefore, $\boldsymbol{d}_{UU}(n) = 3\boldsymbol{d}_{UU}(n-1) - \boldsymbol{d}_{UU}(n-2) + 2F_{2n-4}$, where $\boldsymbol{d}_{UU}(1) = 0$ and $\boldsymbol{d}_{UU}(2) = 1$.

It is possible to verify that $((7-10n)F_{2n} + (6n-5)L_{2n})/10$ satisfies the previous recurrence relation. □

The sequences $\boldsymbol{d}_{UD}(n)$, $\boldsymbol{d}_{DU}(n)$, and $\boldsymbol{d}_{UU}(n) = \boldsymbol{d}_{DD}(n)$, provide a new combinatorial interpretation for A038731, A001870, A054444, respectively.

3 String of length three

In this section, we focus on counting the number of subwords of length three in $\mathcal{D}_n$. Since there are eight distinct subword types to analyze, we present eight theorems, each addressing a specific type. The proofs for these theorems follow a similar approach based on the first return decomposition illustrated in Figure 1. To ensure a comprehensive understanding, we provide a detailed proof for one representative theorem and outline the main ideas for the remaining proofs.

The proofs of many theorems in this section can naturally be extended to more general cases. For example, the results for U^3, D^3, UD^2, and U^2D extend to subwords of the form

U^k, D^k, UD^k, and U^kD, respectively. However, due to the scope of this paper, we do not develop such generalizations here; interested readers may verify them independently.

To provide an overview of the results in this section, Table 1 summarizes the first few values of the sequence $\boldsymbol{d}_w(n)$, along with the corresponding generating functions and references to related sequences in the OEIS (On-Line Encyclopedia of Integer Sequences [17]). This table serves as a convenient reference for the results presented throughout the section.

Subword w	$\boldsymbol{d}_w(n)$ for $2 \leq n \leq 10$	$\boldsymbol{D}_w(x)$	OEIS
$UUU \sim DUU$	0, 1, 6, 25, 90, 300, 954, 2939, 8850	$\frac{x^3}{(1-3x+x^2)^2}$	A001871
$UDU \sim UUD$	1, 4, 14, 46, 145, 444, 1331, 3926, 11434	$\frac{x^2(1-x)^2}{(1-3x+x^2)^2}$	A030267
UDD	1, 4, 14, 45, 138, 411, 1200, 3454, 9836	$\frac{(1-x)(1-3x+2x^2-x^3)x^2}{(1-2x)(1-3x+x^2)^2}$	A377670
DUD	1, 4, 13, 40, 120, 354, 1031, 2972, 8495	$\frac{x^2(1-2x)}{(1-3x+x^2)^2}$	A238846
DDU	0, 1, 5, 19, 65, 210, 654, 1985, 5911	$\frac{(1-x)x^3}{(1-3x+x^2)^2}$	A001870
DDD	$0, 1, 6, 26, 97, 333, 1085, 3411, 10448$	$\frac{x^3(1-2x+x^2-x^3)}{(1-2x)(1-3x+x^2)^2}$	A377679

Table 1: Non-decreasing words and subwords of length three.

Theorem 4. *The number of subwords of the form UUU in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{UUU}(x,y) = \frac{(1-xy)x}{1-x(2+y)+x^2y} \quad \text{and} \quad \boldsymbol{D}_{UUU}(x) = \frac{x^3}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 1$, we have

$$\boldsymbol{d}_{UUU}(n) = \frac{1}{10}(-(5n+4)F_{2n} + 3nL_{2n}).$$

Proof. From the first return decomposition, see Figure 1, any non-empty non-decreasing Dyck word w can be uniquely decomposed into one of the following three cases:

Case 1. UD. This case corresponds to the generating function x.

Case 2. $\Delta_k v$, where $k \geq 1$ and $v \in \mathcal{D}$. Notice that for $k \geq 3$ the pyramid Δ_k contains $k-2$ subpaths of the form UUU. Therefore, in this case the corresponding generating function is

$$\left(x + x^2 + x^3y + x^4y^2 + \cdots\right) \boldsymbol{F}_{UUU}(x,y) = \left(x + x^2 + \frac{x^3y}{1-xy}\right) \boldsymbol{F}_{UUU}(x,y).$$

Case 3. UvD, with $v \in \mathcal{D}$. Any path in this case can be written as $U^iD^jwD^{i-j}$, with $w \in \mathcal{D} \cup \{\epsilon\}$ and $i \geq 2$, $j = 1, 2, \ldots, i$. See Figure 2 for a geometric representation of the case $i = 4$.

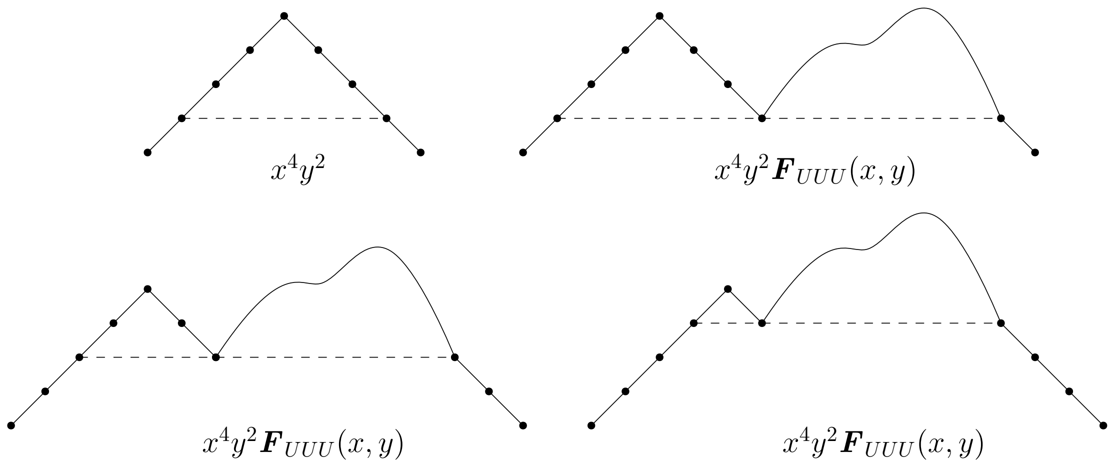


Figure 2: Factorization of non-decreasing Dyck paths in Case 3.

Therefore, the corresponding generating function for this case is

$$x^2(1+\boldsymbol{F}_{UUU}(x,y)) + x^3y(1+2\boldsymbol{F}_{UUU}(x,y)) + x^4y^2(1+3\boldsymbol{F}_{UUU}(x,y)) + \cdots .$$

Simplifying we obtain

$$\frac{x^2}{1-xy} + \frac{x^2}{(1-xy)^2}\boldsymbol{F}_{UUU}(x,y).$$

Summing the expression in cases 1, 2, and 3, we obtain the following functional equation:

$$\boldsymbol{F}_{UUU}(x,y) = x + \left(x + x^2 + \frac{x^3y}{1-xy}\right)\boldsymbol{F}_{UUU}(x,y) + \frac{x^2}{1-xy} + \frac{x^2}{(1-xy)^2}\boldsymbol{F}_{UUU}(x,y).$$

By solving this functional equation, we obtain the desired result.

Next, we provide a combinatorial proof for the sequence $\boldsymbol{d}_{U^3}(n)$. We observe that the number of subwords of the form U^3 in the pyramid Δ_k is $k-2$, as explained in the decomposition described in cases 1, 2, and 3.

For a path of the form $\Delta_k v$, the total number of subwords of the form U^3 is $\boldsymbol{d}_{U^3}(n-k)$, for $k=1,2$. For a fixed $k \geq 3$, the total number of subwords U^3 in a path of the form $\Delta_k v$ is $(k-2)F_{2(n-k)-1} + \boldsymbol{d}_{U^3}(n-k)$, since $|\mathcal{D}_{n-k}| = F_{2(n-k)-1}$.

Furthermore, the number of subwords U^3 in a path of the form UvD is equal to the number of subwords U^3 in v. Additionally, we add 1 if the path v is not of the form $\Delta_1 Q$ with $Q \in \mathcal{D}_{n-2}$. Therefore, for $k = 1, \dots, n-1$, we obtain the recurrence relation:

$$\begin{aligned}\boldsymbol{d}_{U^3}(n) &= \left(\sum_{k=3}^{n-1}(k-2)F_{2k-1} + \sum_{i=1}^{n-1}\boldsymbol{d}_{U^3}(i)\right) + (\boldsymbol{d}_{U^3}(n-1) + F_{2n-3}) - F_{2(n-1)-3} \\ &= \sum_{i=1}^{n-2}(n-2-i)F_{2i-1} + \sum_{i=1}^{n-1}\boldsymbol{d}_{U^3}(i) + \boldsymbol{d}_{U^3}(n-1) + F_{2n-3} - F_{2n-5} \\ &= F_{2(n-3)} - 1 + F_{2n-3} - F_{2(n-1)-3} + \sum_{i=1}^{n-1}\boldsymbol{d}_{U^3}(i) + \boldsymbol{d}_{U^3}(n-1).\end{aligned}$$

Subtracting $\boldsymbol{d}_{U^3}(n+1)$ from $\boldsymbol{d}_{U^3}(n)$ we obtain

$$\boldsymbol{d}_{U^3}(n) = 3\boldsymbol{d}_{U^3}(n-1) - \boldsymbol{d}_{U^3}(n-2) + F_{2(n-2)},$$

where $\boldsymbol{d}_{U^3}(1) = \boldsymbol{d}_{U^3}(2) = 0$ and $\boldsymbol{d}_{U^3}(3) = 1$. It is possible to verify that

$$\frac{3nL_{2n} - (5n+4)F_{2n}}{10}$$

satisfies the previous recurrence relation. □

The coefficient array $[x^n y^m]\boldsymbol{F}_{UUU}(x, y)$ corresponds to the matrix sequence A153342. Our combinatorial interpretation is new.

From the proof of Theorem 4, we observe that once the functional equation is established, completing the remaining details becomes straightforward. Therefore, for the other theorems in this section, we will present only the functional equations and leave the detailed proofs to the reader.

As mentioned earlier, all functional equations in this section are derived based on the first return decomposition illustrated in Figure 1. The labels within each functional equation correspond directly to the labels provided in Figure 1.

Theorem 5. *The number of subwords of the form UDU in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{UDU}(x, y) = \frac{(1-x)x}{1 - x(2+y) + x^2 y} \quad \text{and} \quad \boldsymbol{D}_{UDU}(x) = \frac{x^2(1-x)^2}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 1$, we have

$$\boldsymbol{d}_{UDU}(n) = \frac{1}{10}((-5n+11)F_{2n} + (3n-5)L_{2n}).$$

Proof. From decomposition given in Figure 1 we have

$$\boldsymbol{F}_{UDU}(x, y) = \underbrace{x}_{(1)} + \underbrace{xy\boldsymbol{F}_{UDU}(x, y) + \frac{x^2}{1-x}\boldsymbol{F}_{UDU}(x, y)}_{(2)} + \underbrace{x\boldsymbol{F}_{UDU}(x, y)}_{(3)}.$$

Solving this functional equation we obtain the desired generating function. □

The coefficient array $[x^n y^m]\boldsymbol{F}_{UDU}(x,y)$ coincides with the matrix A105306.

Theorem 6. *The number of subwords of the form UDD in $\mathcal{D}_n$ is counted by*

$$\begin{aligned}\boldsymbol{F}_{UDD}(x,y) &= \frac{(1-x)(1-2x)(1-x(1-y))x}{(1-3x+x^2(2-y))(1-2x+x^2(1-y))} \quad \textit{and}\\ \boldsymbol{D}_{UDD}(x) &= \frac{(1-x)(1-3x+2x^2-x^3)x^2}{(1-2x)(1-3x+x^2)^2}.\end{aligned}$$

Furthermore, for $n \geq 2$, we have

$$\boldsymbol{d}_{UDD}(n) = \frac{1}{5}((5n-9)F_{2n} + (-2n+5)L_{2n} - 5\cdot 2^{n-2}).$$

Proof. For $i_j \geq 0$ and $n \geq 0$, the paths of the form $U(\Delta_{i_1}\Delta_{i_2}\cdots\Delta_{i_n}UD)D$, can be obtained from case (3) of the decomposition shown in Figure 1. Recall that when $i_j = 0$, Δ_{i_j} corresponds to an empty pyramid. In these paths, the subword UDD appears at the end of the path (highlighted in red). The generating function for this type of paths is given by

$$x(xy + xy\Delta(x,y) + xy\Delta(x,y)^2 + \cdots) = x\left(\frac{xy}{1-\Delta(x,y)}\right),$$

where $\Delta(x,y)$ is the generating function of the non-empty pyramids according to the number of UDD's. So,

$$\Delta(x,y) = x + x^2y + x^3y + \cdots = \frac{x - x^2 + x^2y}{1-x}.$$

Therefore, we have that $\boldsymbol{F}_{UDD}(x,y)$ is equal to

$$\underbrace{x}_{(1)} + \underbrace{\left(x + \frac{x^2y}{1-x}\right)\boldsymbol{F}_{UDD}(x,y)}_{(2)} + \underbrace{x\left(\boldsymbol{F}_{UDD}(x,y) - \frac{x}{1-\Delta(x,y)} + \frac{xy}{1-\Delta(x,y)}\right)}_{(3)}.$$

Solving this functional equation we obtain the desired result. ◻

The coefficient array $[x^n y^m]\boldsymbol{F}_{UDD}(x,y)$ coincides with the matrix A105306.

Theorem 7. *The number of subwords of the form DUU in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{DUU}(x,y) = \frac{x - x^2 + x^3(1-y)}{1-3x+x^2(3-2y)-x^3(1-y)} \quad \textit{and} \quad \boldsymbol{D}_{DUU}(x) = \frac{x^3}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 3$, we have

$$\boldsymbol{d}_{DUU}(n) = \frac{1}{10}(-(5n+4)F_{2n} + 3nL_{2n}).$$

Proof. From case (2) in the decomposition given in Figure 1, we deduce that the paths of the form $U\Delta Dv$, where Δ may be empty, determine the subpath DUU if v starts with UU. This gives rise to the following functional equation:

$$\boldsymbol{F}_{DUU}(x,y) = \underbrace{x}_{(1)} + \underbrace{\frac{x}{1-x}\left(\boldsymbol{F}_{DUU}(x,y) - \boldsymbol{G}_{DUU}(x,y) + y\boldsymbol{G}_{DUU}(x,y)\right)}_{(2)} + \underbrace{x\boldsymbol{F}_{DUU}(x,y)}_{(3)},$$

where $\boldsymbol{G}_{DUU}(x,y)$ is the generating function for the non-decreasing Dyck path starting with UU according to the number of DUU's. This kind of Dyck paths admit the decomposition given in Figure 3. Therefore, we have the functional equation:

$$\begin{aligned}\boldsymbol{G}_{DUU}(x,y) = \underbrace{x^2}_{(1)} &+ \underbrace{\frac{x^2}{1-x}\left(\boldsymbol{F}_{DUU}(x,y) - \boldsymbol{G}_{DUU}(x,y) + y\boldsymbol{G}_{DUU}(x,y)\right)}_{(2)} \\ &+ \underbrace{\frac{x^2}{1-x}\left(\boldsymbol{F}_{DUU}(x,y) - \boldsymbol{G}_{DUU}(x,y) + y\boldsymbol{G}_{DUU}(x,y)\right)}_{(3\text{-a})} + \underbrace{x^2\boldsymbol{F}_{DUU}(x,y)}_{(3\text{-b})}.\end{aligned}$$

Solving the system of functional equations we obtain the desired result. □

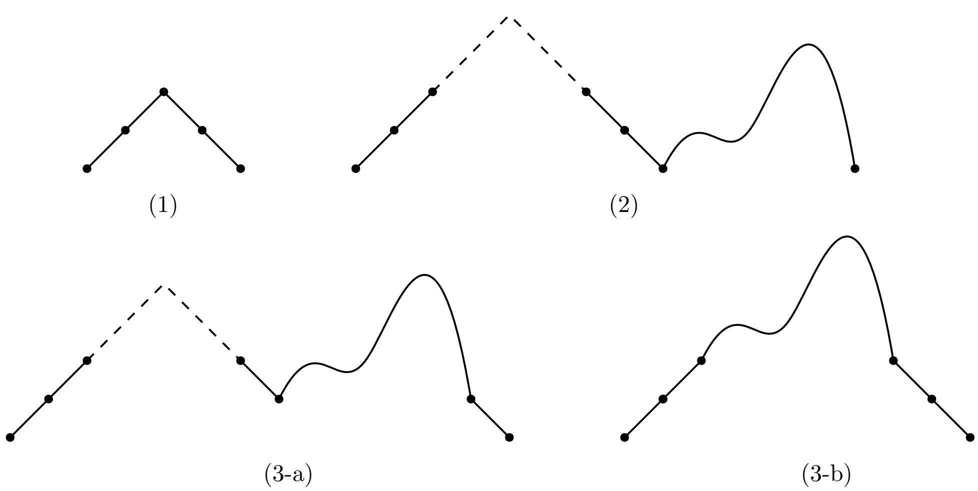


Figure 3: Factorization of non-decreasing Dyck paths starting with UU.

The coefficient array $[x^n y^m]\boldsymbol{F}_{DUU}(x,y)$ coincides with the matrix A105306.

Theorem 8. *The number of subwords of the form DUD in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{DUD}(x,y) = \frac{x(1-x-x^2(1-y))}{1-x^2(1-2y)+x^3(1-y)-x(2+y)} \quad \textit{and} \quad \boldsymbol{D}_{DUD}(x) = \frac{x^2-2x^3}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 3$, we have

$$\boldsymbol{d}_{DUD}(n) = \frac{1}{5}((5n+1)F_{2n} - 2nL_{2n}).$$

Proof. In case (2) of the decomposition given in Figure 1, paths of the form $U\Delta DUDv$, where Δ may be empty, determine the subpath DUD. Therefore, we have the functional equation:

$$\boldsymbol{F}_{DUD}(x,y) = \underbrace{x}_{(1)} + \underbrace{\frac{x}{1-x}\left(\boldsymbol{F}_{DUD}(x,y) - \boldsymbol{G}_{DUD}(x,y) + y\boldsymbol{G}_{DUD}(x,y)\right)}_{(2)} + \underbrace{x\boldsymbol{F}_{DUD}(x,y)}_{(3)},$$

where $\boldsymbol{G}_{DUD}(x,y) = x + x(\boldsymbol{F}_{DUD}(x,y) - \boldsymbol{G}_{DUD}(x,y) + y\boldsymbol{G}_{DUD}(x,y))$. Solving this system of equations yields the desired result. □

The coefficient array $[x^n y^m]\boldsymbol{F}_{DUD}(x,y)$ does not appear in the OEIS.

Theorem 9. *The number of subwords of the form DDU in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{DDU}(x,y) = \frac{(1-x)x}{1-3x+x^2(2-y))} \quad \text{and} \quad \boldsymbol{D}_{DDU}(x) = \frac{(1-x)x^3}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 3$, we have

$$\boldsymbol{d}_{DDU}(n) = \frac{1}{10}((10n-13)F_{2n} + (-4n+5)L_{2n}).$$

Proof. The decomposition presented in Figure 1 yields the following functional equation:

$$\boldsymbol{F}_{DDU}(x,y) = \underbrace{x}_{(1)} + \underbrace{\left(x + \frac{x^2y}{1-x}\right)\boldsymbol{F}_{DDU}(x,y)}_{(2)} + \underbrace{x\boldsymbol{F}_{DDU}(x,y)}_{(3)}.$$

Solving this equation we obtain the desired result. □

The coefficient array $[x^n y^m]\boldsymbol{F}_{DDU}(x,y)$ coincides with the matrix A121466.

Theorem 10. *The number of subwords of the form DDD in $\mathcal{D}_n$ is counted by*

$$\begin{aligned}
\boldsymbol{F}_{DDD}(x,y) &= \frac{x(1-xy)(1-2xy-x^2(1-y^2))}{(1-x(1+y)-x^2(1-y))(1-x(1+2y)-x^2(1-y-y^2))} \quad \text{and} \\
\boldsymbol{D}_{DDD}(x) &= \frac{x^3(1-2x+x^2-x^3)}{(1-2x)(1-3x+x^2)^2}.
\end{aligned}$$

Furthermore, for $n \geq 3$, we have

$$\boldsymbol{d}_{DDD}(n) = \frac{1}{4}((-8n+10)F_{2n} + (4n-6)L_{2n} + 2^n).$$

Proof. In case (3) of the decomposition presented in Figure 1, we consider the paths of the form UvD, where $v \in \mathcal{D}$. We classify v according to the number of ending $D's$. Note that if the last return of v contains exactly k D's, where $k \geq 2$, these paths generate $k-1$ subpaths of the form DDD.

For example, if the last return of v is represented by DDD, the path UvD can be decomposed as $U(P_1UP_2UP_3UDDD)D$, where each P_i $(i = 1, 2, 3)$ is a non-decreasing Dyck path (possibly empty) with only valleys at the floor level, see Figure 4.

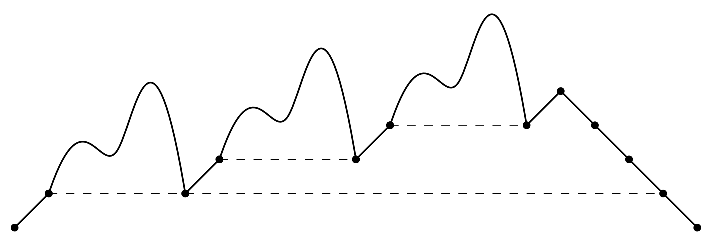


Figure 4: Factorization of non-decreasing Dyck paths of the form $U(P_1UP_2UP_3UDDD)D$.

In general, this kind of paths have this generating function:

$$x(x\boldsymbol{V}_{DDD}(x,y) + x^2y\boldsymbol{V}_{DDD}(x,y)^2 + x^3y^2\boldsymbol{V}_{DDD}(x,y)^3 + \cdots) = x\left(\frac{x\boldsymbol{V}_{DDD}(x,y)}{1-xy\boldsymbol{V}_{DDD}(x,y)}\right),$$

where $\boldsymbol{V}_{DDD}(x,y)$ is the generating function of the non-decreasing Dyck paths (possible empty) with only valleys at floor level, according to the number of DDD's. So,

$$\boldsymbol{V}_{DDD}(x,y) = 1 + \left(x + x^2 + \frac{x^3y}{1-xy}\right)\boldsymbol{V}_{DDD}(x,y).$$

Therefore, we have this functional equation:

$$\boldsymbol{F}_{DDD}(x,y) = \underbrace{x}_{(1)} + \underbrace{\left(x + x^2 + \frac{x^3y}{1-xy}\right)\boldsymbol{F}_{DDD}(x,y)}_{(2)} + \underbrace{x\left(\frac{x\boldsymbol{V}_{DDD}(x,y)}{1-xy\boldsymbol{V}_{DDD}(x,y)}\right)}_{(3)}.$$

Solving this system of equations we obtain the desired result. $\square$

Theorem 11. *The number of subwords of the form UUD in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{UUD}(x,y) = \frac{(1-x)x(1-x(1-y))}{1-3x+x^2(3-2y)-x^3(1-y)} \quad \textit{and} \quad \boldsymbol{D}_{UUD}(x) = \frac{(1-x)^2x^2}{(1-3x+x^2)^2}.$$

Furthermore, for $n \geq 3$, we have

$$\boldsymbol{d}_{UUD}(n) = \frac{1}{10}((-5n+11)F_{2n} + (3n-5)L_{2n}). \tag{3}$$

Proof. The decomposition presented in Figure 1 yields the following functional equation:

$$\boldsymbol{F}_{UUD}(x,y) = \underbrace{x}_{(1)} + \underbrace{\left(x + \frac{x^2y}{1-x}\right)\boldsymbol{F}_{UUD}(x,y)}_{(2)} \\ + \underbrace{x\left(\boldsymbol{F}_{UUD}(x,y) - (x + x\boldsymbol{F}_{UUD}(x,y)) + y(x + x\boldsymbol{F}_{UUD}(x,y))\right)}_{(3)}.$$

Solving this functional equation we obtain the desired result. □

The coefficient array $[x^n y^m]\boldsymbol{F}_{UUD}(x,y)$ coincides with the matrix A304429.

4 String of length four

In this section, we focus on counting the number of subwords of length four in $\mathcal{D}_n$. Given that there are 16 distinct subword types, we have compiled the results in Table 2, rather than presenting each result as a separate theorem. We provide a detailed proof for one representative theorem and omit the remaining proofs, as they follow a similar structure or have been addressed in earlier sections.

Theorem 12. *The number of subwords of the form U^3D in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{F}_{U^3D}(x,y) = \frac{(1-x)\,(x + x^3(y-1))}{1 - 3x + x^2 + (3-3y)x^3 - 2(1-y)x^4} \quad \text{and} \quad \boldsymbol{D}_{U^3D}(x) = \frac{(x-1)^2x^3(x+1)}{(1+x(x-3))^2}.$$

Furthermore, for $n \geq 3$, we have

$$\boldsymbol{d}_{U^3D}(n) = nF_{2n-5} - L_{2n-6}.$$

Proof. For simplicity, and when there is no ambiguity, we use $\boldsymbol{F}$ to denote $\boldsymbol{F}_{U^3D}(x,y)$ throughout this proof. From the first return decomposition any non-empty non-decreasing Dyck word w may be uniquely decomposed in one of these three cases.

Case 1. UD. This case corresponds to the generating function x.

Case 2. $\Delta_k v$, where $v \in \mathcal{D}$ and $k \geq 1$. In this case the corresponding generating function is

$$\left(x + x^2 + \frac{x^3y}{1-x}\right)\boldsymbol{F}.$$

Case 3. UvD, with $v \in \mathcal{D}$. Any path in this case can be written as $U^iD^jwD^{i-j}$, with $w \in \mathcal{D} \cup \{\epsilon\}$, for all $i \geq 2$ and $j = 1, 2, \ldots, i$. This gives rise to the following generating function:

$$x\left(\boldsymbol{F} - (x^2 + x^2\boldsymbol{F} + x^2\boldsymbol{F}) + y(x^2 + x^2\boldsymbol{F} + x^2\boldsymbol{F})\right).$$

Summing and simplifying the generating functions from cases 1, 2 and 3 we obtain the functional equation:

$$\boldsymbol{F} = \underbrace{x}_{(1)} + \underbrace{x\left(x + x^2 + \frac{x^3 y}{1-x}\right)\boldsymbol{F}}_{(2)} + \underbrace{x\left(\boldsymbol{F} - (x^2 + x^2\boldsymbol{F} + x^2\boldsymbol{F}) + y(x^2 + x^2\boldsymbol{F} + x^2\boldsymbol{F})\right)}_{(3)}.$$

Solving this functional equation for $\boldsymbol{F}$ we obtain the desired generating function.

Next, we present a constructive and combinatorial proof of the sequence $\boldsymbol{d}_{U^3D}(n)$. We observe that the number of subwords of the form U^3D in Δ_k is 1 if $k > 2$.

A path $\mathcal{D}_n$ can be uniquely decomposed as $\Delta_k v$ or UzD where $v \in \mathcal{D}_{n-i}$ and $z \in \mathcal{D}_{n-1}$. It is evident that the number of subwords of the form U^3D in a path of the form $\Delta_k v$ is $\boldsymbol{d}_{U^3D}(n-k)$ for $k = 1, 2$. For a fixed $k \geq 3$ the total number of subwords U^3D in a path of the form $\Delta_k v$ is $F_{2(n-k)-1} + \boldsymbol{d}_{U^3D}(n-k)$.

We observe that the number of subwords U^3D in a path in the decomposition UzD is equal to the number of subwords U^3D in v. Additionally, we add 1 in cases where the path z takes one of the forms $\Delta_2 Q \in \mathcal{D}_{n-1}$ or $U\Delta_1 QD \in \mathcal{D}_{n-1}$, where $Q \in \mathcal{D}_{n-3}$.

Therefore, for $k = 1, \dots, n-1$, we obtain the recurrence relation:

$$\boldsymbol{d}_{U^3D}(n) = \sum_{k=1}^{n-1} \boldsymbol{d}_{U^3D}(k) + \boldsymbol{d}_{U^3D}(n-1) + \sum_{k=1}^{n-3} F_{2k-1} + 2F_{2n-7}.$$

By simplifying similarly as in the proof of Theorem 4, we obtain the recurrence relation:

$$\boldsymbol{d}_{U^3D}(n) = 3\boldsymbol{d}_{U^3D}(n-1) - \boldsymbol{d}_{U^3D}(n-2) + L_{2n-7},$$

where $\boldsymbol{d}_{U^3D}(1) = \boldsymbol{d}_{U^3D}(2) = 0$, $\boldsymbol{d}_{U^3D}(3) = 1$, and $\boldsymbol{d}_{U^3D}(4) = 5$. It is possible to verify that $nF_{2n-5} - L_{2n-6}$ satisfies the previous recurrence relation. □

From Table 2, we observe that the subwords $UDUU$, $DUDU$, and $DDDU$ correspond to the same sequence A001870, while the subwords $UUDU$, $UDDU$ correspond to the same sequence A030267.

5 Strings of length five and some generalizations

In this section, we provide combinatorial expressions for counting the number of subwords of length five in $\mathcal{D}_n$ in terms of Fibonacci and Lucas numbers. Table 3 presents the 32 distinct subword types along with their corresponding formulas. The results were obtained using the computer software *Mathematica*®. While we have proven some of these results, not all have been rigorously demonstrated. However, based on the proofs we have established, we strongly believe that the remaining results can be proven using the same techniques applied in previous sections.

For the subword DUD^2U, we have $\boldsymbol{d}_{DUD^2U}(n) = 0$, since it does not appear in non-decreasing Dyck paths. To simplify notation, we omit the argument n in the functions presented in Table 3.

Subword w	$\boldsymbol{D}_w(x)$	Sequence	OEIS
$UUUU$	$\frac{x^4\left(-x^2+x+1\right)}{(x^2-3x+1)^2}$	$\frac{2(n-3)L_{2n-5}-3F_{2n-6}}{5}$	A375995
$UUUD$	$\frac{(x-1)^2x^3(x+1)}{(x^2-3x+1)^2}$	$nF_{2n-5}-L_{2n-6}$	A377857
$UUDU$	$\frac{(x-1)^2x^3}{(x^2-3x+1)^2}$	$\frac{(n-1)L_{2n-4}-F_{2n-7}}{5}$	A030267
$UDUU$	$\frac{(1-x)x^3}{(x^2-3x+1)^2}$	$\frac{(n-1)L_{2n-3}-F_{2n-2}}{5}$	A001870
$DUUU$	$\frac{x^4(x+1)}{(x^2-3x+1)^2}$	$nF_{2n-5}-L_{2n-4}$	A197649
$UUDD$	$\frac{(1-x)^3x^2}{(x^2-3x+1)^2}$	$\frac{nL_{2n-3}+3F_{2n-5}}{5}$	A038731
$UDUD$	$\frac{x^2\left(2x^2-3x+1\right)}{(x^2-3x+1)^2}$	$\frac{nL_{2n-4}+2L_{2n-3}-F_{2n-3}}{5}$	A059502
$UDDU$	$\frac{(x-1)^2x^3}{(x^2-3x+1)^2}$	$\frac{(n-1)L_{2n-4}-F_{2n-7}}{5}$	A030267
$DDUU$	$\frac{x^3}{(x^2-3x+1)^2}$	$\frac{(n-1)L_{2n-4}-2F_{2n-2}}{5}$	A001871
$DUDU$	$\frac{(1-x)x^3}{(x^2-3x+1)^2}$	$\frac{(n-1)L_{2n-3}-F_{2n-2}}{5}$	A001870
$UDDD$	$\frac{\left(-x^5+2x^4-5x^3+8x^2-5x+1\right)x^3}{(2x^3-7x^2+5x-1)^2}$	$\frac{(n-3)L_{2n-5}+L_{2n-3}+F_{2n+2}-5(n+5)2^{n-4}}{5}$	A378383
$DUDD$	$\frac{(x-1)x^3}{2x^3-7x^2+5x-1}$	$F_{2n-2}-2^{n-2}$	A105693
$DDUD$	$\frac{(1-2x)x^2}{(x^2-3x+1)^2}$	$\frac{(n-2)L_{2n-5}+4F_{2n-4}}{5}$	A238846
$DDDU$	$\frac{(1-x)x^4}{(x^2-3x+1)^2}$	$\frac{(n-2)L_{2n-5}-F_{2n-4}}{5}$	A001870
$DUUD$	$\frac{-x^3\left(x^2+x-1\right)}{(x^2-3x+1)^2}$	$\frac{2nL_{2n-5}-6F_{2n-6}-F_{2n-7}}{5}$	A377866
$DDDD$	$\frac{x^4\left(x^4+2x^2-3x+1\right)}{(2x^3-7x^2+5x-1)^2}$	$\frac{3(n-2)L_{2n-4}-3F_{2n+1}+5(n+9)2^{n-4}}{5}$	A377867

Table 2: Subwords of length four in $\mathcal{D}_n$.

5.1 Some general cases

In this section, we analyze generalizations of certain subwords of $\mathcal{D}_n$. Since the proofs follow the same structure as those for shorter lengths, we present only the theorems without detailed proofs.

Theorem 13. *If $n > m$, then the number of subwords of the form U^m in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{d}_{U^m}(n) = 3\boldsymbol{d}_{U^m}(n-1) - \boldsymbol{d}_{U^m}(n-2) + F_{2(n-m)} + F_{2(n-2)} - \sum_{k=1}^{m-2} kF_{2(n-k-2)}.$$

If $n \leq m$, then $\boldsymbol{d}_{U^m}(n) = 0$ and $\boldsymbol{d}_{U^n}(n) = 1$.

$\boldsymbol{d}_{U^5} = \frac{(5n-24)F_{2n-8}+(n-4)L_{2n-7}}{5}$	$\boldsymbol{d}_{U^4D} = \frac{(n-2)(L_{2n-6}+2L_{2n-8})-9F_{2n-9}}{5}$
$\boldsymbol{d}_{U^3DU} = \frac{(n-2)L_{2n-6}-F_{2n-9}}{5}$	$\boldsymbol{d}_{U^3D^2} = \frac{(n-1)L_{2n-4}-F_{2n-7}}{5}$
$\boldsymbol{d}_{U^2DU^2} = \frac{(n-2)L_{2n-5}-F_{2n-4}}{5}$	$\boldsymbol{d}_{U^2DUD} = \frac{(n-1)L_{2n-6}+2L_{2n-5}-F_{2n-5}}{5}$
$\boldsymbol{d}_{U^2D^2U} = \frac{(n-1)L_{2n-4}-F_{2n-7}}{5}$	$\boldsymbol{d}_{U^2D^3} = \frac{2F_{2n-5}+2L_{2n-4}+(n-1)L_{2n-5}}{5} - 2^{n-3}$
$\boldsymbol{d}_{UDU^3} = (n-3)F_{2n-6}$	$\boldsymbol{d}_{UDU^2D} = \frac{(2n-3)L_{2n-6}+6F_{2n-6}-F_{2n-7}}{5}$
$\boldsymbol{d}_{UDUDU} = \frac{(n-1)L_{2n-4}-F_{2n-7}}{5}$	$\boldsymbol{d}_{UDUD^2} = F_{2n-3} - 2^{n-3}$
$\boldsymbol{d}_{UD^2U^2} = \frac{(n-2)L_{2n-5}-F_{2n-4}}{5}$	$\boldsymbol{d}_{UD^2UD} = \frac{(n-1)L_{2n-6}+2L_{2n-5}-F_{2n-5}}{5}$
$\boldsymbol{d}_{UD^3U} = \frac{(n-2)L_{2n-6}-F_{2n-9}}{5}$	$\boldsymbol{d}_{DU^2DU} = \frac{(n-2)L_{2n-5}-F_{2n-4}}{5}$
$\boldsymbol{d}_{DU^2D^2} = \frac{(n-2)L_{2n-5}+4F_{2n-4}}{5}$	$\boldsymbol{d}_{DUDU^2} = \frac{(n-1)L_{2n-4}-2F_{2n-2}}{5}$
$\boldsymbol{d}_{DUDUD} = \frac{(n-2)L_{2n-5}+4F_{2n-4}}{5}$	$\boldsymbol{d}_{DUD^3} = F_{2n-2} - (n+3)2^{n-4}$
$\boldsymbol{d}_{D^2U^3} = (n-1)F_{2n-7} - L_{2n-6}$	$\boldsymbol{d}_{D^2U^2D} = \frac{2(n-1)L_{2n-7}-6F_{2n-8}-F_{2n-9}}{5}$
$\boldsymbol{d}_{D^2UDU} = \frac{(n-2)L_{2n-5}-F_{2n-4}}{5}$	$\boldsymbol{d}_{D^2UD^2} = F_{2n-4} - 2^{n-3}$
$\boldsymbol{d}_{D^3U^2} = \frac{(n-2)L_{2n-6}-2F_{2n-4}}{5}$	$\boldsymbol{d}_{D^3UD} = \frac{(n-3)L_{2n-7}+4F_{2n-6}}{5}$
$\boldsymbol{d}_{D^4U} = \frac{(n-3)L_{2n-7}-F_{2n-6}}{5}$	$\boldsymbol{d}_{DU^4} = \frac{(-35n+84)F_{2n-4}+(17n-44)L_{2n-4}}{10}$
$\boldsymbol{d}_{DU^3D} = \frac{(15n-42)F_{2n-4}-(6n-17)L_{2n-4}}{5}$	
$\boldsymbol{d}_{UD^4} = \frac{64(10n-13)F_{2n-4}-64(4n-23)L_{2n-4}-(5n^2+55n+180)2^n}{640}$	
$\boldsymbol{d}_{D^5} = (n-5)L_{2n-4} - (n+2)F_{2n-4} - 64(4n-23)L_{2n-4} + (n^2+19n+108)2^{n-7}$	

Table 3: Number of subwords of length five in $\mathcal{D}_n$.

Theorem 14. *If $n > m$, then the number of subwords of the form D^m in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{d}_{D^m}(n) = 3\boldsymbol{d}_{D^m}(n-1) - \boldsymbol{d}_{D^m}(n-2) + F_{2(n-m)} + F_{2(n-2)} - \sum_{k=1}^{m-2}(\boldsymbol{d}_{D^k}(n-1) - \boldsymbol{d}_{D^k}(n-2)).$$

If $n \leq m$, then $\boldsymbol{d}_{D^m}(n) = 0$ and $\boldsymbol{d}_{D^n}(n) = 1$.

Theorem 15. *If $n \geq m$, then the number of subwords of the form U^mD in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{d}_{U^mD}(n) = 3\boldsymbol{d}_{U^mD}(n-1) - \boldsymbol{d}_{U^mD}(n-2) + F_{2(n-m)-3} + mF_{2(n-m-1)}.$$

If $n < m$, then $\boldsymbol{d}_{U^mD}(n) = 0$, $\boldsymbol{d}_{U^nD}(n) = 1$, and $\boldsymbol{d}_{U^mD}(m+1) = m+2$.

Theorem 16. *If $n \geq m$, then the number of subwords of the form UD^m in $\mathcal{D}_n$ is counted by*

$$\boldsymbol{d}_{UD^m}(n) = 3\boldsymbol{d}_{UD^m}(n-1) - \boldsymbol{d}_{UD^m}(n-2) + F_{2(n-m)-1} + l_{n-1,m-1} - l_{n-2,m-1},$$

where $l_{n,k}$ is the total number of paths where the last peak is of height k. If $n < m$, then $\boldsymbol{d}_{UD^m}(n) = 0$ and $\boldsymbol{d}_{UD^n}(n) = 1$.

5.2 Subword that share the same sequence

In this section, we present Table 4, which summarizes the subwords that share the same counting sequence. However, depending on the subword, the sequence may be shifted. Of particular interest is the potential to find bijections between these subwords.

Subwords	The OEIS
UU; DD	A054444
UUU; DUU; $DDUU$; $DUDUU$; $DDDUU$	A001871
UUD; UDU; $UUDU$; $UDDU$; $UUUDU$; $UUUDD$; $UUDDU$; $UDUDU$; $UUUDU$; $UUUDD$; $UUDDU$; $UDUDU$; $UDDDU$	A030267
UDD; $UUDDD$	A377670
DUD; $DDUD$; $DUUDD$; $DUDUD$; $DDDUD$.	A238846
DU; DDU; $UDUU$; $DUDU$; $DDDU$; $UUDUU$; $UDDUU$ $DUUDU$; $DDUDU$; $DDDDU$	A001870
UD; $UUDD$	A038731
$DUUU$; $DDUUU$	A197649
$UDUD$; $UDDUD$; $UUDUD$	A059502
$DUUD$; $DDUUD$	A377866
$DUDD$; $DDUDD$	A105693

Table 4: Subwords that share the same sequence.

These subwords have their unique sequence: DDD, $UUUU$, $UDDD$, $DDDD$, $UUUUU$, $UDUUU$, $UDUDD$, $UDDDD$, $DUUUU$, $DUUUD$, $DUDDD$, and $DDDDD$.

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