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Arbitrarily Long Sequences of Sierpiński Numbers that are the Sum of a Sierpiński Number and a Mersenne Number

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Abstract

Y.-G. Chen conjectured that for any positive integer r, there are infinitely many sets of r consecutive integers that are all Sierpiński and Riesel, and he demonstrated this result for $r = 5$. In this paper, we prove a variation of Chen's conjecture. In particular, we show that for any positive integer r, there exists an arithmetic progression of positive integers k such that the set $\{k + 2^t - 1 : 0 \leq t \leq r - 1\}$ contains r Sierpiński integers.

1 Introduction

In the search for factors of large Mersenne numbers, Riesel [20] stumbled on an interesting phenomenon: $509203 \cdot 2^n - 1$ is composite for all natural numbers n. A few years later, Sierpiński published a similar result [21]: $15511380746462593381 \cdot 2^n + 1$ is composite for all natural numbers n. Based on these results, we create some definitions. If k is an odd natural number such that $k \cdot 2^n - 1$ is composite for all $n \in \mathbb{N}$, then we say k is a *Riesel number*. If k has the property that $k \cdot 2^n - 1$ is composite for all natural numbers (but not necessarily odd), then we call k a *Riesel-type number*. We define *Sierpiński numbers* and *Sierpiński-type numbers* similarly, instead using that the expression $k \cdot 2^n + 1$ is composite for all $n \in \mathbb{N}$. The smallest known Sierpiński number is 78557, an unpublished result from 1962, and the smallest known Riesel number is 509203, from Riesel's germinal result in 1956.

Many articles have been published about Riesel numbers, Sierpiński numbers, or Riesel-Sierpiński numbers occurring within other interesting sequences of numbers – Fibonacci numbers [17, 19], Lucas numbers [2], Cullen numbers [5], polygonal numbers [1], Carmichael numbers [4], Ruth-Aaron numbers [9], perfect powers [6, 12, 15], and others (e.g., [14]).

Simply putting together the work that led to finding 78557 and Sierpiński's original result, we see that there are infinitely many pairs of consecutive integers k and $k+1$ with the property that both

$$k \cdot 2^n + 1 \quad \text{and} \quad (k+1) \cdot 2^n + 1$$

are composite for all natural numbers n. Note that since one of k and $k+1$ is even, we have found two consecutive integers that are both Sierpiński-type numbers. Chen [7] extended the observation about 78557 and Sierpiński's original result to find infinitely many values of k such that if $\kappa \in \{k, k+1, k+2, k+3, k+4\}$, then both $\kappa - 2^n$ and $\kappa \cdot 2^n + 1$ are composite for all natural numbers n. In addition, Chen [8] showed that there are infinitely many odd positive integers k such that the eight odd integers $k, k+2, k+4, k+6, \ldots, k+14$ are all Riesel numbers.

In this paper, we extend this result in a different direction. In particular, we show that there are arbitrarily long sequences of Sierpiński-type numbers of the form $k + M_t$, where $M_t = 2^t - 1$. That is, M_t denotes the tth Mersenne number. Observe that if k is an odd number, then for $r > 0$, all of the integers of the form $k + M_r$ are even, and the numbers we produce from Theorem 1 are Sierpiński-type numbers. On the other hand, if we construct an even k with the Sierpiński property, then the integers of the form $k + M_r$ with $r > 0$ are all odd, and hence they are truly Sierpiński numbers.

Theorem 1. *Let t be a fixed natural number. Then there are infinitely many integers k (in arithmetic progression) with the property that for each integer κ in the set*

$$\{k + M_0, k + M_1, k + M_2, \ldots, k + M_t\},$$

the integer $\kappa \cdot 2^n + 1$ is composite for all nonnegative integers n.

The remainder of this paper is organized into sections which we now describe. In Section 2, we present the notion of a covering system and the notation conventions we have adopted.

We also present some expository work on creating new coverings from existing coverings (cf. [18]). In Section 3, we use these ideas to demonstrate the proof of Theorem 1.

2 Preliminaries

2.1 Coverings

The standard technique used in this area of mathematics is to build a covering system, and we follow this well-worn path in this article. A *covering system* (or simply *covering*) is a set of congruences with the property that every integer satisfies at least one of the congruences. A very simple example of a covering system is the system of residue classes shown below.

$$\begin{cases} 0 \pmod 2 \\ 1 \pmod 2 \end{cases}$$

This is, of course, a covering system since every integer is either even or odd. There are infinitely many similarly constructed covering systems; let $m \in \mathbb{Z}$ with $m > 1$. Then the residue classes

$$\begin{cases} 0 \pmod m \\ 1 \pmod m \\ \vdots \\ m-1 \pmod m \end{cases} \tag{1}$$

form a covering system.

Erdős [10] was the first to use coverings in 1950. Coverings are interesting in their own right, and questions about coverings [11, 13, 16] can be related to questions is other areas of mathematics. More interesting covering systems use several different moduli, and we now walk through building such a covering. Suppose we start with a covering of the form shown in (1) with $m = 12$. All even integers satisfy one of the congruences of the form $2k \pmod{12}$, so we replace these six congruences with the congruence $0 \pmod 2$. Similarly, all multiples of 3 satisfy one of the congruences of the form $3k \pmod{12}$, so we replace these congruences with $0 \pmod 3$.

We represent the progress we have made using the diagram below. In the top row, we have simplified $k \pmod{12}$ to simply the residue k. We show in the first column the simplified congruence and place a $\bullet$ in the column for each congruence that is subsumed. For example, in the first row, we write the congruence $0 \pmod 2$, and place a $\bullet$ in the columns for the congruences of the form $2k \pmod{12}$.

	0	1	2	3	4	5	6	7	8	9	10	11
0 (mod 2)	•		•		•		•		•		•	
0 (mod 3)	•			•			•			•		

At this point, the covering has the congruences shown below.

$$\begin{cases} 0 \pmod{2} \\ 0 \pmod{3} \\ 1 \pmod{12} \\ 5 \pmod{12} \\ 7 \pmod{12} \\ 11 \pmod{12} \end{cases}$$

We see that we can replace 1 (mod 12) and 5 (mod 12) with 1 (mod 4). We can also replace 7 (mod 12) with 1 (mod 6), and now the diagram becomes what we see below.

	0	1	2	3	4	5	6	7	8	9	10	11
0 (mod 2)	•		•		•		•		•		•	
0 (mod 3)	•			•			•			•		
1 (mod 4)		•				•				•		
1 (mod 6)		•						•				

We have thus constructed the following covering.

$$\mathcal{C} = \begin{cases} 0 \pmod{2} \\ 0 \pmod{3} \\ 1 \pmod{4} \\ 1 \pmod{6} \\ 11 \pmod{12} \end{cases} \tag{2}$$

Moreover, we can represent the diagram in the simplified format shown below.

0	1	2	3	4	5	6	7	8	9	10	11
•		•		•		•		•		•	
•			•			•			•		
	•				•				•		
	•						•				
											•

Since both of these representations use an ample amount of space on the page, we often condense the notation and simply write

$$\mathcal{C} = \{(0,2),(0,3),(1,4),(1,6),(11,12)\}$$

to present this covering.

2.2 Using coverings

The covering system that establishes Riesel's 1956 result is

$$\mathcal{R} = \{(0,2), (1,4), (2,3), (7,12), (7,8), (3,24)\}.$$

The details are shown in the the implications below.

$$\begin{array}{lcl}
n \equiv 0 \pmod 2 & \Longrightarrow & 509203 \cdot 2^n - 1 \equiv 0 \pmod 3. \\
n \equiv 1 \pmod 4 & \Longrightarrow & 509203 \cdot 2^n - 1 \equiv 0 \pmod 5. \\
n \equiv 2 \pmod 3 & \Longrightarrow & 509203 \cdot 2^n - 1 \equiv 0 \pmod 7. \\
n \equiv 7 \pmod{12} & \Longrightarrow & 509203 \cdot 2^n - 1 \equiv 0 \pmod{13}. \\
n \equiv 7 \pmod 8 & \Longrightarrow & 509203 \cdot 2^n - 1 \equiv 0 \pmod{17}. \\
n \equiv 3 \pmod{24} & \Longrightarrow & 509203 \cdot 2^n - 1 \equiv 0 \pmod{241}.
\end{array}$$

Since the congruences for n in the left column form a covering system, we see that for all natural numbers n, the integer $509203 \cdot 2^n - 1$ is divisible by one of the primes in the set $\{3, 5, 7, 13, 17, 241\}$. This set of primes is called a *covering set*. There is one detail that is suppressed in the implications shown above above, and this detail is important for concluding that there are infinitely many Riesel numbers.

$$\begin{array}{lclcl}
n \equiv 0 \pmod 2 & \& & k \equiv 1 \pmod 3 & \Longrightarrow & k \cdot 2^n - 1 \equiv 0 \pmod 3 \\
n \equiv 1 \pmod 4 & \& & k \equiv 3 \pmod 5 & \Longrightarrow & k \cdot 2^n - 1 \equiv 0 \pmod 5 \\
n \equiv 2 \pmod 3 & \& & k \equiv 2 \pmod 7 & \Longrightarrow & k \cdot 2^n - 1 \equiv 0 \pmod 7 \\
n \equiv 7 \pmod{12} & \& & k \equiv 6 \pmod{13} & \Longrightarrow & k \cdot 2^n - 1 \equiv 0 \pmod{13} \\
n \equiv 7 \pmod 8 & \& & k \equiv 2 \pmod{17} & \Longrightarrow & k \cdot 2^n - 1 \equiv 0 \pmod{17} \\
n \equiv 3 \pmod{24} & \& & k \equiv 211 \pmod{241} & \Longrightarrow & k \cdot 2^n - 1 \equiv 0 \pmod{241}
\end{array}$$

Notice now that $k = 509203$ satisfies all of the congruences for k. On the other hand, applying the Chinese remainder theorem to the set of congruences for k shown above, we find that $k \equiv 509203 \pmod{11184810}$ satisfies all of these congruences. This means that 509203 is simply the smallest example of a Riesel number in an infinite arithmetic progression of Riesel numbers. We present all of the information in the table shown above in the condensed notation.

$$\mathcal{R} = \{(0,2), (1,4), (2,3), (7,12), (7,8), (3,24)\}$$
$$\mathcal{K} = \{(1,3), (3,5), (2,7), (6,13), (2,17), (211,241)\}$$

Alternatively, we often present this covering in the form of ordered triples, (a, m, p), where $a \pmod m$ is a congruence in the covering, and p is the prime from the covering set associated with this congruence. (That is, the multiplicative order of 2 modulo p is m.) This is shown in the set below.

$$\mathcal{C} = \{(0,2,3), (1,4,5), (2,3,7), (7,12,13), (7,8,17), (3,24,241)\}$$

Notice that the information in the set $\mathcal{K}$ can be easily computed from the information given in $\mathcal{C}$, so we have not really lost any information by presenting the covering in this condensed notation.

2.3 New coverings from old coverings

We return now to the covering we built in Section 2.1, $\mathcal{C} = \{(0,2),(0,3),(1,4),(1,6),(11,12)\}$. We note that algebraic operations on the space of coverings are described in general in Theorem 4.3 in [18], but here we discuss operations on coverings in an intuitive way that relies on the geometry of the covering diagram. Observe that we can verify that $\mathcal{C}$ covers all residue classes modulo 12 by noticing that there is at least one $\bullet$ in each column of the diagram. Notice that if each $\bullet$ in the diagram is shifted to the right by a fixed number of units (wrapping around from the column labeled 11 to the column labeled 0, as necessary), then the resulting diagram also represents a covering. Below we see two examples of applying this kind of shift to the diagram above; in the first diagram, we see a shift two units to the right, and in the second diagram below, we see a shift of seven units to the left.

0	1	2	3	4	5	6	7	8	9	10	11
•		•		•		•		•		•	
		•			•			•			•
			•				•				•
			•						•		
	•										

0	1	2	3	4	5	6	7	8	9	10	11
	•		•		•		•		•		•
		•			•			•			•
		•				•				•	
•						•					
				•							

Notice that 12 distinct coverings are created by shifting the dots in the diagram for $\mathcal{C}$; each of these coverings has a unique congruence of the form $a \pmod{12}$. If we shift the covering $\mathcal{C}$ one unit to the right, we obtain a new covering, shown below, which we denote $\mathcal{C}_1$:

$$\mathcal{C}_1 = \{(1,2),(1,3),(2,4),(2,6),(0,12)\}.$$

Another way to produce a new covering from an existing covering system is to reflect the diagram for the covering. In the example shown below, we have reflected the diagram for the covering $\mathcal{C}$ over the vertical line that separates the column representing 5 (mod 12) from the column representing 6 (mod 12). The new covering produced is

$$\mathcal{C}' = \{(1,2),(2,3),(2,4),(4,6),(0,12)\}.$$

0	1	2	3	4	5	6	7	8	9	10	11
	•		•		•		•		•		•
		•			•			•			•
		•				•				•	
				•						•	
•											

We could also reflect the diagram over the column for 6 (mod 12). To do this, we add a column to the diagram so that the column for 6 (mod 12) is in the middle.

0	1	2	3	4	5	6	7	8	9	10	11	12
•		•		•		•		•		•		•
•			•			•			•			•
	•				•				•			
	•						•					
											•	

Now we reflect over the middle column to obtain the figure below.

0	1	2	3	4	5	6	7	8	9	10	11	12
•		•		•		•		•		•		•
•			•			•			•			•
			•				•				•	
					•						•	
	•											

To read off the resulting covering, we simply ignore the extra column for the residue class 12 (mod 12). This results in the covering

$$\mathcal{C}' = \{(0,2),(0,3),(3,4),(5,6),(1,12)\}.$$

We observe that this covering can also be constructed from $\mathcal{C}$ by multiplying the residue in each congruence in $\mathcal{C}$ by -1, and then reducing the result according to the modulus of the congruence.

We use these approaches to coverings as tools to find families of Sierpinski numbers in the remainder of this paper.

3 A sequence of Sierpiński numbers of the form $k + M_t$

In this section, we prove that if r is a fixed natural number, then there are infinitely many integers k (in arithmetic progression) with the property that for each integer κ in the set

$$\{k + M_0, k + M_1, k + M_2, \ldots, k + M_r\}, \tag{3}$$

the integer $\kappa \cdot 2^n + 1$ is composite for all nonnegative integers n. To start the process, we first construct k, a Sierpiński number. The table below records the congruences used in the covering and the congruences for $k + M_0 = k$. Notice that this is the covering discovered by Sierpiński [21].

$$\begin{aligned}
n &\equiv 1 \pmod{2} & k &\equiv 1 \pmod{3}\\
n &\equiv 2 \pmod{4} & k &\equiv 1 \pmod{5}\\
n &\equiv 4 \pmod{8} & k &\equiv 1 \pmod{17}\\
n &\equiv 8 \pmod{16} & k &\equiv 1 \pmod{257}\\
n &\equiv 16 \pmod{32} & k &\equiv 1 \pmod{65537}\\
n &\equiv 32 \pmod{64} & k &\equiv 1 \pmod{641}\\
n &\equiv 0 \pmod{64} & k &\equiv -1 \pmod{6700417}
\end{aligned}$$

Next, we shift the covering shown above by one unit to ensure that $k + M_1 = k + 1$ is also a Sierpiński number. Note that we eliminate the last congruence in the covering above and replace it with two congruences instead.

$$\begin{aligned}
n &\equiv 0 \pmod{2} & k+1 &\equiv 2 \pmod{3}\\
n &\equiv 1 \pmod{4} & k+1 &\equiv 2 \pmod{5}\\
n &\equiv 3 \pmod{8} & k+1 &\equiv 2 \pmod{17}\\
n &\equiv 7 \pmod{16} & k+1 &\equiv 2 \pmod{257}\\
n &\equiv 15 \pmod{32} & k+1 &\equiv 2 \pmod{65537}\\
n &\equiv 31 \pmod{64} & k+1 &\equiv 2 \pmod{641}\\
n &\equiv 63 \pmod{128} & k+1 &\equiv 2 \pmod{274177}\\
n &\equiv 127 \pmod{128} & k+1 &\equiv -2 \pmod{6728004213107 21}
\end{aligned}$$

Notice that in order to keep the congruences for $k + 1$ compatible with those already established for k, we were not able to use the modulus 64 twice in the shifted covering. Instead, we use 64 only once, and then we employ the modulus 128 twice. We continue this process, shifting the covering again and discarding congruences that yield congruences for $k + M_2 = k + 3$ that are incompatible with those already established for k and $k + 1$.

$$\begin{aligned}
n &\equiv 1 \pmod{2} & k+3 &\equiv 4 \pmod{3}\\
n &\equiv 0 \pmod{4} & k+3 &\equiv 4 \pmod{5}\\
n &\equiv 2 \pmod{8} & k+3 &\equiv 4 \pmod{17}\\
n &\equiv 6 \pmod{16} & k+3 &\equiv 4 \pmod{257}\\
n &\equiv 14 \pmod{32} & k+3 &\equiv 4 \pmod{65537}
\end{aligned}$$

$$
\begin{aligned}
n &\equiv 30 \pmod{64} & k+3 &\equiv 4 \pmod{641}\\
n &\equiv 62 \pmod{128} & k+3 &\equiv 4 \pmod{274177}\\
n &\equiv 126 \pmod{256} & k+3 &\equiv 4 \pmod{59649589127497217}\\
n &\equiv 254 \pmod{256} & k+3 &\equiv -4 \pmod{5704689200685129054721}
\end{aligned}
$$

This is proceeding smoothly, but the reader may wonder how far this construction can be pushed given the status of the factorization of numbers of the form $2^{2^n}+1$. (See the Appendix for exactly how far we did push this.) We can be certain that we can construct a partial covering at each step due to the theorem of Bang [3] that guarantees a primitive prime divisor of $2^m - 1$ for all integers $m > 6$. However, Bang's result does not guarantee *two* primitive prime divisors, although the first three coverings shown above make use of the two primes dividing $2^\ell - 1$ (for $\ell = 6, 7, 8$) where the multiplicative order of 2 modulo these primes is 2^ℓ. Since we cannot count on the existence of two primitive prime divisors as we work through the values of κ in the set in (3), we take a different approach to constructing the set of required coverings.

3.1 Completing the construction: an example

Before we give the formal details for the general case, we illustrate our procedure with an example. To that end, let $r = 3$. In this section, we construct the four coverings needed to give an infinite arithmetic progression of values of k with the property that for each κ in the set $\{k, k+1, k+3, k+7\}$, the integer $\kappa \cdot 2^n + 1$ is composite for all natural numbers n. We now choose a positive integer s with the property that $\pi(s) \geq r + 3 = 6$. Since there are 6 primes less that 14, we choose $s = 14$. We start the construction of the four coverings necessary to produce the desired k using moduli that are powers of 2, so we identify primes p_i with the property that the multiplicative order of 2 modulo p_i is 2^i. That is, for each natural number $i \leq 14$, the prime p_i is a primitive prime divisor of $2^{2^i} - 1$. We list the primes p_1, p_2, …, p_{14} below.

$$
\begin{aligned}
p_1 &= 3 & p_8 &= 59649589127497217\\
p_2 &= 5 & p_9 &= 1238926361552897\\
p_3 &= 17 & p_{10} &= 2424833\\
p_4 &= 257 & p_{11} &= 45592577\\
p_5 &= 65537 & p_{12} &= 319489\\
p_6 &= 641 & p_{13} &= 114689\\
p_7 &= 274177 & p_{14} &= 2710954639361
\end{aligned}
$$

We recall the ordered triple notation for coverings, (a, m, p), where $a \pmod{m}$ is a congruence in the covering, and p is the prime associated with this congruence. We use this notation to give a partial covering corresponding to each integer in the set $\{k{+}M_0, k{+}M_1, k{+}M_2, k{+}M_3\}$.

$$\begin{aligned} k + M_t : \{&(2^0 - t, 2, p_1), (2^1 - t, 4, p_2), (2^2 - t, 8, p_3), (2^3 - t, 16, p_4), (2^4 - t, 32, p_5), \\ &(2^5 - t, 64, p_6), (2^6 - t, 128, p_7), (2^7 - t, 256, p_8), (2^8 - t, 512, p_9), (2^9 - t, 1024, p_{10}), \\ &(2^{10} - t, 2048, p_{11}), (2^{11} - t, 4096, p_{12}), (2^{12} - t, 8192, p_{13}), (2^{13} - t, 16384, p_{14})\} \end{aligned}$$

Each of these partial coverings is missing one residue class modulo $2^{14} = 16384$; these are listed below.

$$\begin{aligned} k + M_0 &: \quad 0 \pmod{16384} \\ k + M_1 &: \ -1 \pmod{16384} \\ k + M_2 &: \ -2 \pmod{16384} \\ k + M_3 &: \ -3 \pmod{16384} \end{aligned}$$

We pause to recall that in Section 2.1, we collapsed the congruences of the form $n \equiv 4k + 1 \pmod{12}$ into the single congruence $n \equiv 1 \pmod 4$. Note that we can reverse this process. That is, if a partial covering misses the congruence $n \equiv 1 \pmod 4$, we can fill this gap in our covering by using the congruences shown below.

$$\begin{aligned} n_0 &\equiv 1 \cdot 4 + 1 \pmod{20} \\ n_0 &\equiv 2 \cdot 4 + 1 \pmod{20} \\ n_0 &\equiv 3 \cdot 4 + 1 \pmod{20} \\ n_0 &\equiv 4 \cdot 4 + 1 \pmod{20} \\ n_0 &\equiv 5 \cdot 4 + 1 \pmod{20} \end{aligned}$$

With this idea in mind, we identify primes (denoted $q_0, \cdots, q_t$) such that $3 < q_0 < q_1 < q_2 < q_3 < s = 14$. That is, $q_0 = 5$, $q_1 = 7$, $q_2 = 11$, and $q_3 = 13$. These primes direct the completion of each of the four coverings.

To complete the covering for $k + M_0$, we cover the missing residue class $n_0 \equiv 0 \pmod{2^{14}}$ by covering the residue classes shown on the left below. However, we accomplish this by using the congruences shown on the right.

$$\begin{aligned} n_0 &\equiv 1 \cdot 2^{14} \pmod{5 \cdot 2^{14}} & \qquad n_0 &\equiv 1 \cdot 2^{14} \equiv \ \ 4 \pmod{5 \cdot 2^1} \\ n_0 &\equiv 2 \cdot 2^{14} \pmod{5 \cdot 2^{14}} & n_0 &\equiv 2 \cdot 2^{14} \equiv \ \ 8 \pmod{5 \cdot 2^2} \\ n_0 &\equiv 3 \cdot 2^{14} \pmod{5 \cdot 2^{14}} & n_0 &\equiv 3 \cdot 2^{14} \equiv 32 \pmod{5 \cdot 2^3} \\ n_0 &\equiv 4 \cdot 2^{14} \pmod{5 \cdot 2^{14}} & n_0 &\equiv 4 \cdot 2^{14} \equiv 16 \pmod{5 \cdot 2^4} \\ n_0 &\equiv 5 \cdot 2^{14} \pmod{5 \cdot 2^{14}} & n_0 &\equiv 5 \cdot 2^{14} \equiv \ \ 0 \pmod{5 \cdot 2^5} \end{aligned}$$

Next, we identify primes corresponding to the moduli in the congruences on the right; that is, for each i with $1 \le i \le 5$, we determine a prime, $p_i^{(0)}$, that is a primitive prime divisor of $2^{5\cdot 2^i}-1$. These primes are shown below.

$$p_1^{(0)}=11 \qquad p_4^{(0)}=4278255361$$
$$p_2^{(0)}=41 \qquad p_5^{(0)}=414721$$
$$p_3^{(0)}=61681$$

We now record the completed covering that yields the result that $k+M_0$ is a Sierpiński (or a Sierpiński-type) number.

$$k+M_0:\{(1,2,p_1),(2,4,p_2),(4,8,p_3),(8,16,p_4),(16,32,p_5),(32,64,p_6),(64,128,p_7),\\ (128,256,p_8),(256,512,p_9),(512,1024,p_{10}),(1024,2048,p_{11}),(2048,4096,p_{12}),\\ (4096,8192,p_{13}),(8192,16384,p_{14}),(4,10,p_1^{(0)}),(8,20,p_2^{(0)}),(32,40,p_3^{(0)}),\\ (16,80,p_4^{(0)}),(0,160,p_5^{(0)})\}$$

We continue on to complete the covering for $k+M_1$. Using $q_1=7$, we expand the missing residue class $n_1\equiv -1 \pmod{16384}$ to the 7 congruences shown on the left below, and we again replace these with the residue classes covered by the congruences on the right.

$$n_1\equiv 1\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 1\cdot 2^{14}-1\equiv 4-1 \pmod{7\cdot 2^{1}}$$
$$n_1\equiv 2\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 2\cdot 2^{14}-1\equiv 8-1 \pmod{7\cdot 2^{2}}$$
$$n_1\equiv 3\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 3\cdot 2^{14}-1\equiv 40-1 \pmod{7\cdot 2^{3}}$$
$$n_1\equiv 4\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 4\cdot 2^{14}-1\equiv 16-1 \pmod{7\cdot 2^{4}}$$
$$n_1\equiv 5\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 5\cdot 2^{14}-1\equiv 160-1 \pmod{7\cdot 2^{5}}$$
$$n_1\equiv 6\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 6\cdot 2^{14}-1\equiv 192-1 \pmod{7\cdot 2^{6}}$$
$$n_1\equiv 7\cdot 2^{14}-1 \pmod{7\cdot 2^{14}} \qquad n_1\equiv 7\cdot 2^{14}-1\equiv 0-1 \pmod{7\cdot 2^{7}}$$

The primes associated with the congruences are listed below.

$$p_1^{(1)}=43 \qquad p_5^{(1)}=5153$$
$$p_2^{(1)}=29 \qquad p_6^{(1)}=449$$
$$p_3^{(1)}=1489153 \qquad p_7^{(1)}=167773885276849215533569$$
$$p_4^{(1)}=15790321$$

We thus have the covering from which we obtain the conclusion that $k+M_1$ is a Sierpiński(-type) number.

$$k+M_1:\{(0,2,p_1),(1,4,p_2),(3,8,p_3),(7,16,p_4),(15,32,p_5),(31,64,p_6),(63,128,p_7),\\ (127,256,p_8),(255,512,p_9),(511,1024,p_{10}),(1023,2048,p_{11}),(2047,4096,p_{12}),\\ (4095,8192,p_{13}),(8191,16384,p_{14}),(3,14,p_1^{(1)}),(7,28,p_2^{(1)}),(39,56,p_3^{(1)}),\\ (15,112,p_4^{(1)}),(159,224,p_5^{(1)}),(191,448,p_6^{(1)}),(895,896,p_7^{(1)})\}$$

For $k+M_2$, we use $q_2 = 11$.

$$
\begin{aligned}
n_2 &\equiv 1\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 16-2 \pmod{11\cdot 2^{1}}\\
n_2 &\equiv 2\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 32-2 \pmod{11\cdot 2^{2}}\\
n_2 &\equiv 3\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 48-2 \pmod{11\cdot 2^{3}}\\
n_2 &\equiv 4\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 64-2 \pmod{11\cdot 2^{4}}\\
n_2 &\equiv 5\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 256-2 \pmod{11\cdot 2^{5}}\\
n_2 &\equiv 6\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 448-2 \pmod{11\cdot 2^{6}}\\
n_2 &\equiv 7\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 640-2 \pmod{11\cdot 2^{7}}\\
n_2 &\equiv 8\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 1536-2 \pmod{11\cdot 2^{8}}\\
n_2 &\equiv 9\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 1024-2 \pmod{11\cdot 2^{9}}\\
n_2 &\equiv 10\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 6144-2 \pmod{11\cdot 2^{10}}\\
n_2 &\equiv 11\cdot 2^{14}-2 \pmod{11\cdot 2^{14}} & n_2 &\equiv 0-2 \pmod{11\cdot 2^{11}}
\end{aligned}
$$

The first seven primes associated with the congruences are listed below.

$$
\begin{aligned}
p_1^{(2)} &= 683 & p_4^{(2)} &= 229153\\
p_2^{(2)} &= 397 & p_5^{(2)} &= 5304641\\
p_3^{(2)} &= 353 & p_6^{(2)} &= 1409\\
p_7^{(2)} &= 60299259845689822028046342401
\end{aligned}
$$

We ended the list above with only seven primes because of computational issues. The primes shown have the property that the order of 2 modulo $p_i^{(2)}$ is $2^i \cdot 11$. The existence of the primes $p_8^{(2)}, p_9^{(2)}, \cdots, p_{11}^{(2)}$ is certain due to Bang's [3] theorem. However, instead of using those particular primes, in this setting, we continue the process using the primes shown below. These are all primes with the order of 2 dividing $11 \cdot 2^6$, including unused primitive prime divisors of $2^{11\cdot 2^i} - 1$, for $i \le 6$.

$$
\begin{aligned}
q_1^{(2)} &= 683 & q_7^{(2)} &= 119782433\\
q_2^{(2)} &= 397 & q_8^{(2)} &= 43872038849\\
q_3^{(2)} &= 2113 & q_9^{(2)} &= 5304641\\
q_4^{(2)} &= 353 & q_{10}^{(2)} &= 1409\\
q_5^{(2)} &= 2931542417 & q_{11}^{(2)} &= 1258753\\
q_6^{(2)} &= 229153 & &
\end{aligned}
$$

Using these primes in place of the primes $p_1^{(2)}, p_2^{(2)}, \cdots, p_{11}^{(2)}$ alters the congruences in the covering slightly, as shown below.

$n_2 \equiv 16 - 2 \pmod{11 \cdot 2^1}$	$n_2 \equiv 16 - 2 \pmod{11 \cdot 2^1}$
$n_2 \equiv 32 - 2 \pmod{11 \cdot 2^2}$	$n_2 \equiv 32 - 2 \pmod{11 \cdot 2^2}$
$n_2 \equiv 48 - 2 \pmod{11 \cdot 2^3}$	$n_2 \equiv 4 - 2 \pmod{11 \cdot 2^2}$
$n_2 \equiv 64 - 2 \pmod{11 \cdot 2^4}$	$n_2 \equiv 64 - 2 \pmod{11 \cdot 2^3}$
$n_2 \equiv 256 - 2 \pmod{11 \cdot 2^5}$	$n_2 \equiv 80 - 2 \pmod{11 \cdot 2^3}$
$n_2 \equiv 448 - 2 \pmod{11 \cdot 2^6}$	$n_2 \equiv 96 - 2 \pmod{11 \cdot 2^4}$
$n_2 \equiv 640 - 2 \pmod{11 \cdot 2^7}$	$n_2 \equiv 112 - 2 \pmod{11 \cdot 2^4}$
$n_2 \equiv 1536 - 2 \pmod{11 \cdot 2^8}$	$n_2 \equiv 128 - 2 \pmod{11 \cdot 2^4}$
$n_2 \equiv 1024 - 2 \pmod{11 \cdot 2^9}$	$n_2 \equiv 320 - 2 \pmod{11 \cdot 2^5}$
$n_2 \equiv 6144 - 2 \pmod{11 \cdot 2^{10}}$	$n_2 \equiv 512 - 2 \pmod{11 \cdot 2^6}$
$n_2 \equiv 0 - 2 \pmod{11 \cdot 2^{11}}$	$n_2 \equiv 0 - 2 \pmod{11 \cdot 2^6}$

The final form of the covering that tells us that $k + M_2$ is Sierpiński(-type) is exhibited now.

$$\begin{aligned} k + M_2 : \{&(1, 2, p_1), (0, 4, p_2), (2, 8, p_3), (6, 16, p_4), (14, 32, p_5), (30, 64, p_6), (62, 128, p_7), \\ &(126, 256, p_8), (254, 512, p_9), (510, 1024, p_{10}), (1022, 2048, p_{11}), (2046, 4096, p_{12}), \\ &(4094, 8192, p_{13}), (8190, 16384, p_{14}), (14, 22, q_1^{(2)}), (30, 44, q_2^{(2)}), (2, 44, q_3^{(2)}), \\ &(62, 88, q_4^{(2)}), (78, 88, q_5^{(2)}), (94, 176, q_6^{(2)}), (110, 176, q_7^{(2)}), (126, 176, q_8^{(2)}), \\ &(318, 352, q_9^{(2)}), (510, 704, q_{10}^{(2)}), (702, 704, q_{11}^{(2)})\} \end{aligned}$$

Finally, to complete this side quest, we construct the missing part of the covering for $k + M_3$. Since $q_3 = 13$, we expand the missing congruence class $n \equiv -3 \pmod{16384}$ to the congruences shown on the left below. On the right we see the congruences we use to cover the same equivalence classes.

$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^1}$
$n_3 \equiv 2 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^2}$
$n_3 \equiv 3 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^3}$
$n_3 \equiv 4 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^4}$
$n_3 \equiv 5 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^5}$
$n_3 \equiv 6 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^6}$
$n_3 \equiv 7 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^7}$
$n_3 \equiv 8 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^8}$
$n_3 \equiv 9 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}}$	$n_3 \equiv 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^9}$

$$
\begin{array}{ll}
n_3 \equiv 10 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}} & n_3 \equiv \ \ 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{10}} \\
n_3 \equiv 11 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}} & n_3 \equiv \ \ 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{11}} \\
n_3 \equiv 12 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}} & n_3 \equiv \ \ 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{12}} \\
n_3 \equiv 13 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{14}} & n_3 \equiv \ \ 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{13}}
\end{array}
$$

Again, computational issues make it more time-consuming than helpful to compute a primitive prime divisor of $2^{13 \cdot 2^i} - 1$ for $7 < i \leq 13$. Moreover, there are also several unused primes for which the order of 2 is a multiple of 13 that divides $1664 = 13 \cdot 2^7$. Hence, we use the primes shown below. Note that we decided to forgo the use of a 29-digit prime and an 18-digit prime that are prime divisors of $2^{1664} - 1$ in favor of smaller primes.

$$
\begin{array}{ll}
q_1^{(3)} = 2731 & q_8^{(3)} = 18558466369 \\
q_2^{(3)} = 53 & q_9^{(3)} = 23877647873 \\
q_3^{(3)} = 157 & q_{10}^{(3)} = 21316654212673 \\
q_4^{(3)} = 1613 & q_{11}^{(3)} = 4940417 \\
q_5^{(3)} = 858001 & q_{12}^{(3)} = 11342687617 \\
q_6^{(3)} = 308761441 & q_{13}^{(3)} = 3329 \\
q_7^{(3)} = 928513 &
\end{array}
$$

These primes prompt the shifting of the congruence classes for the covering as shown below.

$$
\begin{array}{ll}
n_3 \equiv \ \ 1 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{1}} & n_3 \equiv \ \ 1 \cdot 2^{14} - 3 \equiv \ \ 4 - 3 \pmod{13 \cdot 2^{1}} \\
n_3 \equiv \ \ 2 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{2}} & n_3 \equiv \ \ 2 \cdot 2^{14} - 3 \equiv \ \ 8 - 3 \pmod{13 \cdot 2^{2}} \\
n_3 \equiv \ \ 3 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{3}} & n_3 \equiv \ \ 3 \cdot 2^{14} - 3 \equiv 12 - 3 \pmod{13 \cdot 2^{2}} \\
n_3 \equiv \ \ 4 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{4}} & n_3 \equiv \ \ 4 \cdot 2^{14} - 3 \equiv 16 - 3 \pmod{13 \cdot 2^{2}} \\
n_3 \equiv \ \ 5 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{5}} & n_3 \equiv \ \ 5 \cdot 2^{14} - 3 \equiv 72 - 3 \pmod{13 \cdot 2^{3}} \\
n_3 \equiv \ \ 6 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{6}} & n_3 \equiv \ \ 6 \cdot 2^{14} - 3 \equiv 24 - 3 \pmod{13 \cdot 2^{3}} \\
n_3 \equiv \ \ 7 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{7}} & n_3 \equiv \ \ 7 \cdot 2^{14} - 3 \equiv 288 - 3 \pmod{13 \cdot 2^{5}} \\
n_3 \equiv \ \ 8 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{8}} & n_3 \equiv \ \ 8 \cdot 2^{14} - 3 \equiv 32 - 3 \pmod{13 \cdot 2^{5}} \\
n_3 \equiv \ \ 9 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{9}} & n_3 \equiv \ \ 9 \cdot 2^{14} - 3 \equiv 192 - 3 \pmod{13 \cdot 2^{5}} \\
n_3 \equiv 10 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{10}} & n_3 \equiv 10 \cdot 2^{14} - 3 \equiv 352 - 3 \pmod{13 \cdot 2^{5}} \\
n_3 \equiv 11 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{11}} & n_3 \equiv 11 \cdot 2^{14} - 3 \equiv 512 - 3 \pmod{13 \cdot 2^{6}} \\
n_3 \equiv 12 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{12}} & n_3 \equiv 12 \cdot 2^{14} - 3 \equiv 256 - 3 \pmod{13 \cdot 2^{6}} \\
n_3 \equiv 13 \cdot 2^{14} - 3 \pmod{13 \cdot 2^{13}} & n_3 \equiv 13 \cdot 2^{14} - 3 \equiv \ \ 0 - 3 \pmod{13 \cdot 2^{7}}
\end{array}
$$

The final covering that completes this example is listed below.

$$k+M_3: \{(0,2,p_1),(3,4,p_2),(1,8,p_3),(5,16,p_4),(13,32,p_5),(29,64,p_6),(61,128,p_7),$$
$$(125,256,p_8),(253,512,p_9),(509,1024,p_{10}),(1021,2048,p_{11}),$$
$$(2045,4096,p_{12}),(4093,8192,p_{13}),(8189,16384,p_{14}),(1,26,q_1^{(3)}),$$
$$(5,52,q_2^{(3)}),(9,52,q_3^{(3)}),(13,52,q_4^{(3)}),(69,104,q_5^{(3)}),(21,104,q_6^{(3)}),$$
$$(77,208,q_7^{(3)}),(29,208,q_8^{(3)}),(189,208,q_9^{(3)}),(349,416,q_{10}^{(3)}),(405,532,q_{11}^{(3)}),$$
$$(297,532,q_{12}^{(3)}),(189,1064,q_{13}^{(3)})\}$$

4 Back to the general construction

Now we present the details of the construction of a covering in the general setting. To that end, let r be a fixed positive integer, and now we choose a positive integer s so that $\pi(s) > r+3$, where $\pi(s)$ denotes the number of primes up to s. To show that $\kappa = k+M_t$ has the Sierpiński property (where $0 \le t < r$), we first form the partial covering shown below. Note that p_i is a primitive prime divisor of $2^{2^i}-1$.

$$\begin{array}{ll}
n_t \equiv 2^0 - t \pmod{2} & k+M_t \equiv 2^t \pmod{3} \\
n_t \equiv 2^1 - t \pmod{4} & k+M_t \equiv 2^t \pmod{5} \\
n_t \equiv 2^2 - t \pmod{8} & k+M_t \equiv 2^t \pmod{17} \\
n_t \equiv 2^3 - t \pmod{16} & k+M_t \equiv 2^t \pmod{257} \\
n_t \equiv 2^4 - t \pmod{32} & k+M_t \equiv 2^t \pmod{65537} \\
\vdots & \vdots \\
n_t \equiv 2^i - t \pmod{2^{i+1}} & k+M_t \equiv 2^t \pmod{p_i} \\
\vdots & \vdots \\
n_t \equiv 2^{s-1} - t \pmod{2^s} & k+M_t \equiv 2^t \pmod{p_s}
\end{array}$$

We observe that for all nonnegative integers s up to t, the partial covering relies on using the same primes: $p_1 = 3$, $p_2 = 5$, $p_3 = 17$, …, p_r. Notice that the congruences for $k+M_s$ modulo these primes imply that $k \equiv 1 \pmod{p_i}$ for $i = 1, 2, \ldots, r$ for each value of s.

The partial covering accounts for all natural numbers modulo 2^r, except for one residue class; $n \equiv 2^r - s \pmod{2^r}$ is left out. Notice that if q is a natural number, we can use the congruences below to cover the missing residue class.

$$n_s \equiv 1 \cdot 2^r - s \pmod{q \cdot 2^r}$$
$$n_s \equiv 2 \cdot 2^r - s \pmod{q \cdot 2^r}$$

$$n_s \equiv 3 \cdot 2^r - s \pmod{q \cdot 2^r}$$
$$\vdots$$
$$n_s \equiv q \cdot 2^r - s \pmod{q \cdot 2^r}$$

We now use the choice of r to determine the value of q to use in the congruences above. Since $\pi(r) > t+3$, there are primes $q_0, q_1, \ldots, q_t$ such that $2 < q_0 < q_1 < \cdots < q_t < r$. To complete the covering for $\kappa = k + M_s$, we employ the prime $q = q_s$. Moreover, we replace the congruences shown above with the following congruences since these congruences cover *more* than the ones shown above.

$$n_s \equiv 1 \cdot 2^r - s \pmod{q \cdot 2^1}$$
$$n_s \equiv 2 \cdot 2^r - s \pmod{q \cdot 2^2}$$
$$n_s \equiv 3 \cdot 2^r - s \pmod{q \cdot 2^3}$$
$$\vdots$$
$$n_s \equiv q \cdot 2^r - s \pmod{q \cdot 2^r}$$

To complete the proof, we now choose primes $p_1^{(s)}, p_2^{(s)}, \ldots, p_q^{(s)}$ such that $p_j^{(s)}$ is a primitive prime divisor of $2^{q \cdot 2^j} - 1$. Thus, the multiplicative order of 2 modulo $p_j^{(s)}$ is $q \cdot 2^j$. We now complete the list of congruences for k as shown below.

$$
\begin{array}{ll}
n_s \equiv 2^0 - s \pmod{2} & k \equiv 1 \pmod{3} \\
n_s \equiv 2^1 - s \pmod{4} & k \equiv 1 \pmod{5} \\
n_s \equiv 2^2 - s \pmod{8} & k \equiv 1 \pmod{17} \\
n_s \equiv 2^3 - s \pmod{16} & k \equiv 1 \pmod{257} \\
n_s \equiv 2^4 - s \pmod{32} & k \equiv 1 \pmod{65537} \\
\vdots & \vdots \\
n_s \equiv 2^{r-1} - s \pmod{2^r} & k \equiv -2^s \pmod{p_r} \\
n_s \equiv 1 \cdot 2^r - s \pmod{q \cdot 2^1} & k \equiv 1 - 2^s - 2^{s-1 \cdot 2^r} \pmod{p_1^{(s)}} \\
n_s \equiv 2 \cdot 2^r - s \pmod{q \cdot 2^2} & k \equiv 1 - 2^s - 2^{s-2 \cdot 2^r} \pmod{p_2^{(s)}} \\
n_s \equiv 3 \cdot 2^r - s \pmod{q \cdot 2^3} & k \equiv 1 - 2^s - 2^{s-3 \cdot 2^r} \pmod{p_3^{(s)}} \\
\vdots & \vdots \\
n_s \equiv q \cdot 2^r - s \pmod{q \cdot 2^r} & k \equiv 1 - 2^s - 2^{s-q \cdot 2^r} \pmod{p_q^{(s)}}
\end{array}
$$

The congruences for k modulo the primes $p_1^{(s)}, p_2^{(s)}, \ldots p_q^{(s)}$, come from the fact that if $k \equiv 1 - 2^s - 2^{s-j \cdot 2^r} \pmod{p_j^{(s)}}$ and $n \equiv j \cdot 2^r - s \pmod{q \cdot 2^j}$, then $p_j^{(s)}$ divides $\big(k + M_s\big) \cdot 2^n + 1$. This completes the covering and the construction of k to ensure that $k + M_s$ is a Sierpiński number, where $0 \le s \le t$ is arbitrary. Thus, we have proven Theorem 1.

This type of construction of a covering appeared first in the proof in [12] that for any arbitrary positive integer r, there are infinitely many k such that k^r is a Sierpiński number. (In fact, the authors prove the stronger result that for an arbitrary positive integer r, there are infinitely many k such that k, k^2, k^3, ..., k^r are all Sierpiński numbers.) It is notable that until this paper, this particular method for constructing a covering has not been a useful tool in the search for Sierpiński or Riesel numbers in any other integer sequences.

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6 Appendix

In the beginning of Section 3, we construct the desired Sierpiński numbers using coverings with the property that all moduli are powers of 2. Bang's theorem only guarantees the existence of a single new prime p dividing $2^{2^m} - 1$ for each $m > 2$, so we do not know whether we can use this type of construction in general. However, since there are many known factors of Fermat numbers we can find coverings where all of the moduli are powers of two for $t \leq 27$. Note that not all of the large primes are listed; especially large primes are denoted by Pxx, where xx indicates the number of digits in the prime.

$$
\begin{array}{lll}
n_0 \equiv 2^0 - 0 \pmod{2^1} & k + 2^0 - 1 \equiv 2^0 \pmod{p_0} & \\
n_0 \equiv 2^1 - 0 \pmod{2^2} & k + 2^0 - 1 \equiv 2^0 \pmod{p_1} & \\
n_0 \equiv 2^2 - 0 \pmod{2^3} & k + 2^0 - 1 \equiv 2^0 \pmod{p_2} & \\
n_0 \equiv 2^3 - 0 \pmod{2^4} & k + 2^0 - 1 \equiv 2^0 \pmod{p_3} & \\
n_0 \equiv 2^4 - 0 \pmod{2^5} & k + 2^0 - 1 \equiv 2^0 \pmod{p_4} & \\
n_0 \equiv 2^5 - 0 \pmod{2^6} & k + 2^0 - 1 \equiv 2^0 \pmod{p_5} & \\
n_0 \equiv 2^6 - 0 \pmod{2^6} & k + 2^0 - 1 \equiv -2^0 \pmod{q_0} & \\
n_1 \equiv 2^i - 1 \pmod{2^{i+1}} & k + 2^1 - 1 \equiv 2^1 \pmod{p_i} & 0 \leq i \leq 5 \\
n_1 \equiv 2^6 - 1 \pmod{2^7} & k + 2^1 - 1 \equiv 2^1 \pmod{p_6} & \\
n_1 \equiv 2^7 - 1 \pmod{2^7} & k + 2^1 - 1 \equiv -2^1 \pmod{q_1} & \\
n_2 \equiv 2^i - 2 \pmod{2^{i+1}} & k + 2^2 - 1 \equiv 2^2 \pmod{p_i} & 0 \leq i \leq 6 \\
n_2 \equiv 2^7 - 2 \pmod{2^8} & k + 2^2 - 1 \equiv 2^2 \pmod{p_7} & \\
n_2 \equiv 2^8 - 2 \pmod{2^8} & k + 2^2 - 1 \equiv -2^2 \pmod{q_2} & \\
n_3 \equiv 2^i - 3 \pmod{2^{i+1}} & k + 2^3 - 1 \equiv 2^3 \pmod{p_i} & 0 \leq i \leq 7
\end{array}
$$

$$
\begin{array}{llr}
n_3 \equiv 2^8 - 3 \pmod{2^9} & k + 2^3 - 1 \equiv 2^3 \pmod{p_8} & \\
n_3 \equiv 2^9 - 3 \pmod{2^9} & k + 2^3 - 1 \equiv -2^3 \pmod{q_3} & \\
n_4 \equiv 2^i - 4 \pmod{2^{i+1}} & k + 2^4 - 1 \equiv 2^4 \pmod{p_i} & 0 \le i \le 8 \\
n_4 \equiv 2^9 - 4 \pmod{2^{10}} & k + 2^4 - 1 \equiv 2^4 \pmod{p_9} & \\
n_4 \equiv 2^{10} - 4 \pmod{2^{10}} & k + 2^4 - 1 \equiv -2^4 \pmod{q_4} & \\
n_5 \equiv 2^i - 5 \pmod{2^{i+1}} & k + 2^5 - 1 \equiv 2^5 \pmod{p_i} & 0 \le i \le 9 \\
n_5 \equiv 2^{10} - 5 \pmod{2^{10}} & k + 2^5 - 1 \equiv -2^5 \pmod{q_5} & \\
n_6 \equiv 2^i - 6 \pmod{2^{i+1}} & k + 2^6 - 1 \equiv 2^6 \pmod{p_i} & 0 \le i \le 9 \\
n_6 \equiv 2^{10} - 6 \pmod{2^{11}} & k + 2^6 - 1 \equiv 2^6 \pmod{p_{10}} & \\
n_6 \equiv 2^{11} - 6 \pmod{2^{11}} & k + 2^6 - 1 \equiv -2^6 \pmod{q_6} & \\
n_7 \equiv 2^i - 7 \pmod{2^{i+1}} & k + 2^7 - 1 \equiv 2^7 \pmod{p_i} & 0 \le i \le 10 \\
n_7 \equiv 2^{11} - 7 \pmod{2^{11}} & k + 2^7 - 1 \equiv -2^7 \pmod{q_7} & \\
n_8 \equiv 2^i - 8 \pmod{2^{i+1}} & k + 2^8 - 1 \equiv 2^8 \pmod{p_i} & 0 \le i \le 10 \\
n_8 \equiv 2^{11} - 8 \pmod{2^{11}} & k + 2^8 - 1 \equiv -2^8 \pmod{q_8} & \\
n_9 \equiv 2^i - 9 \pmod{2^{i+1}} & k + 2^9 - 1 \equiv 2^9 \pmod{p_i} & 0 \le i \le 10 \\
n_9 \equiv 2^{11} - 9 \pmod{2^{12}} & k + 2^9 - 1 \equiv 2^9 \pmod{p_{11}} & \\
n_9 \equiv 2^{12} - 9 \pmod{2^{12}} & k + 2^9 - 1 \equiv -2^9 \pmod{q_9} & \\
n_{10} \equiv 2^i - 10 \pmod{2^{i+1}} & k + 2^{10} - 1 \equiv 2^{10} \pmod{p_i} & 0 \le i \le 11 \\
n_{10} \equiv 2^{12} - 10 \pmod{2^{12}} & k + 2^{10} - 1 \equiv -2^{10} \pmod{q_{10}} & \\
n_{11} \equiv 2^i - 11 \pmod{2^{i+1}} & k + 2^{11} - 1 \equiv 2^{11} \pmod{p_i} & 0 \le i \le 11 \\
n_{11} \equiv 2^{12} - 11 \pmod{2^{12}} & k + 2^{11} - 1 \equiv -2^{11} \pmod{q_{11}} & \\
n_{12} \equiv 2^i - 12 \pmod{2^{i+1}} & k + 2^{12} - 1 \equiv 2^{12} \pmod{p_i} & 0 \le i \le 11 \\
n_{12} \equiv 2^{12} - 12 \pmod{2^{12}} & k + 2^{12} - 1 \equiv -2^{12} \pmod{q_{12}} & \\
n_{13} \equiv 2^i - 13 \pmod{2^{i+1}} & k + 2^{13} - 1 \equiv 2^{13} \pmod{p_i} & 0 \le i \le 11 \\
n_{13} \equiv 2^{12} - 13 \pmod{2^{13}} & k + 2^{13} - 1 \equiv 2^{13} \pmod{p_{12}} & \\
n_{10} \equiv 2^{13} - 13 \pmod{2^{13}} & k + 2^{13} - 1 \equiv -2^{13} \pmod{q_{13}} & \\
n_{14} \equiv 2^i - 14 \pmod{2^{i+1}} & k + 2^{14} - 1 \equiv 2^{14} \pmod{p_i} & 0 \le i \le 12 \\
n_{14} \equiv 2^{13} - 14 \pmod{2^{13}} & k + 2^{14} - 1 \equiv -2^{14} \pmod{q_{14}} & \\
n_{15} \equiv 2^i - 15 \pmod{2^{i+1}} & k + 2^{15} - 1 \equiv 2^{15} \pmod{p_i} & 0 \le i \le 12 \\
n_{15} \equiv 2^{13} - 15 \pmod{2^{13}} & k + 2^{15} - 1 \equiv -2^{15} \pmod{q_{15}} & \\
n_{16} \equiv 2^i - 16 \pmod{2^{i+1}} & k + 2^{16} - 1 \equiv 2^{16} \pmod{p_i} & 0 \le i \le 12 \\
n_{16} \equiv 2^{13} - 16 \pmod{2^{13}} & k + 2^{16} - 1 \equiv -2^{16} \pmod{q_{16}} & \\
n_{17} \equiv 2^i - 17 \pmod{2^{i+1}} & k + 2^{17} - 1 \equiv 2^{17} \pmod{p_i} & 0 \le i \le 12 \\
n_{17} \equiv 2^{13} - 17 \pmod{2^{13}} & k + 2^{17} - 1 \equiv -2^{17} \pmod{q_{17}} &
\end{array}
$$

$$
\begin{array}{lll}
n_{18} \equiv \ \ 2^{i} \ - 18 \pmod{2^{i+1}} & k + 2^{18} - 1 \equiv \ \ \ 2^{18} \pmod{p_i} & 0 \le i \le 12 \\
n_{18} \equiv \ \ 2^{13} - 18 \pmod{2^{14}} & k + 2^{18} - 1 \equiv \ \ \ 2^{18} \pmod{p_{13}} & \\
n_{18} \equiv \ \ 2^{14} - 18 \pmod{2^{14}} & k + 2^{18} - 1 \equiv -2^{18} \pmod{q_{18}} & \\
n_{19} \equiv \ \ 2^{i} \ - 19 \pmod{2^{i+1}} & k + 2^{19} - 1 \equiv \ \ \ 2^{19} \pmod{p_i} & 0 \le i \le 13 \\
n_{19} \equiv \ \ 2^{14} - 19 \pmod{2^{14}} & k + 2^{19} - 1 \equiv -2^{19} \pmod{q_{18}} & \\
n_{20} \equiv \ \ 2^{i} \ - 20 \pmod{2^{i+1}} & k + 2^{20} - 1 \equiv \ \ \ 2^{20} \pmod{p_i} & 0 \le i \le 13 \\
n_{20} \equiv \ \ 2^{14} - 20 \pmod{2^{14}} & k + 2^{20} - 1 \equiv -2^{20} \pmod{q_{20}} & \\
n_{21} \equiv \ \ 2^{i} \ - 21 \pmod{2^{i+1}} & k + 2^{21} - 1 \equiv \ \ \ 2^{21} \pmod{p_i} & 0 \le i \le 13 \\
n_{21} \equiv \ \ 2^{14} - 21 \pmod{2^{15}} & k + 2^{21} - 1 \equiv \ \ \ 2^{21} \pmod{p_{14}} & \\
n_{21} \equiv \ \ 2^{15} - 21 \pmod{2^{16}} & k + 2^{21} - 1 \equiv \ \ \ 2^{21} \pmod{p_{15}} & \\
n_{21} \equiv \ \ 2^{16} - 21 \pmod{2^{16}} & k + 2^{21} - 1 \equiv -2^{21} \pmod{q_{21}} & \\
n_{22} \equiv \ \ 2^{i} \ - 22 \pmod{2^{i+1}} & k + 2^{22} - 1 \equiv \ \ \ 2^{22} \pmod{p_i} & 0 \le i \le 15 \\
n_{22} \equiv \ \ 2^{16} - 22 \pmod{2^{16}} & k + 2^{22} - 1 \equiv -2^{22} \pmod{q_{22}} & \\
n_{23} \equiv \ \ 2^{i} \ - 23 \pmod{2^{i+1}} & k + 2^{23} - 1 \equiv \ \ \ 2^{23} \pmod{p_i} & 0 \le i \le 15 \\
n_{23} \equiv \ \ 2^{16} - 23 \pmod{2^{17}} & k + 2^{23} - 1 \equiv \ \ \ 2^{23} \pmod{p_{16}} & \\
n_{23} \equiv \ \ 2^{17} - 23 \pmod{2^{17}} & k + 2^{23} - 1 \equiv -2^{23} \pmod{q_{23}} & \\
n_{24} \equiv \ \ 2^{i} \ - 24 \pmod{2^{i+1}} & k + 2^{24} - 1 \equiv \ \ \ 2^{24} \pmod{p_i} & 0 \le i \le 16 \\
n_{24} \equiv \ \ 2^{17} - 24 \pmod{2^{18}} & k + 2^{24} - 1 \equiv \ \ \ 2^{24} \pmod{p_{17}} & \\
n_{24} \equiv \ \ 2^{18} - 24 \pmod{2^{18}} & k + 2^{24} - 1 \equiv -2^{24} \pmod{q_{24}} & \\
n_{25} \equiv \ \ 2^{i} \ - 25 \pmod{2^{i+1}} & k + 2^{25} - 1 \equiv \ \ \ 2^{25} \pmod{p_i} & 0 \le i \le 17 \\
n_{25} \equiv \ \ 2^{18} - 25 \pmod{2^{19}} & k + 2^{25} - 1 \equiv \ \ \ 2^{25} \pmod{p_{18}} & \\
n_{25} \equiv \ \ 2^{19} - 25 \pmod{2^{19}} & k + 2^{25} - 1 \equiv -2^{25} \pmod{q_{25}} & \\
n_{26} \equiv \ \ 2^{i} \ - 26 \pmod{2^{i+1}} & k + 2^{26} - 1 \equiv \ \ \ 2^{26} \pmod{p_i} & 0 \le i \le 18 \\
n_{26} \equiv \ \ 2^{19} - 26 \pmod{2^{20}} & k + 2^{26} - 1 \equiv \ \ \ 2^{26} \pmod{p_{19}} & \\
n_{26} \equiv \ \ 2^{20} - 26 \pmod{2^{20}} & k + 2^{26} - 1 \equiv -2^{26} \pmod{q_{26}} & \\
n_{27} \equiv \ \ 2^{i} \ - 27 \pmod{2^{i+1}} & k + 2^{27} - 1 \equiv \ \ \ 2^{27} \pmod{p_i} & 0 \le i \le 19 \\
n_{27} \equiv \ \ 2^{20} - 27 \pmod{2^{20}} & k + 2^{27} - 1 \equiv -2^{27} \pmod{q_{27}} &
\end{array}
$$

$$
\begin{aligned}
p_0 &= 3 \\
p_1 &= 5 \\
p_2 &= 17 \\
p_3 &= 257
\end{aligned}
$$

$$
\begin{aligned}
p_4 &= 65537\\
p_5 &= 641\\
p_6 &= 274177\\
p_7 &= 59649589127497217\\
p_8 &= 1238926361552897\\
p_9 &= 2424833\\
p_{10} &= 45592577\\
p_{11} &= 319489\\
p_{12} &= 114689\\
p_{13} &= 2710954639361\\
p_{14} &= 116928085873074369829035993834596371340386703423373313\\
p_{15} &= 1214251009\\
p_{16} &= 825753601\\
p_{17} &= 31065037602817\\
p_{18} &= 13631489\\
p_{19} &= 70525124609\\
q_1 &= 67280421310721\\
q_2 &= 5704689200685129054721\\
q_3 &= 93461639715357977769163558199606896584051237541638188580280321\\
q_4 &= 7455602825647884208337395736200454918783366342657\\
q_5 &= 741640062627530801524787141901937474059940781097519023905821316144-\\
&\quad 15759504705008092818711693940737\\
q_5 &= P99\\
q_6 &= 6487031809\\
q_7 &= 4659775785220018543264560743076778192897\\
q_8 &= P252\\
q_9 &= 974849\\
q_{10} &= 167988556341760475137\\
q_{11} &= 3560841906445833920513\\
q_{12} &= P564\\
q_{13} &= 26017793\\
q_{14} &= 63766529\\
q_{15} &= 190274191361\\
q_{16} &= 1256132134125569\\
q_{17} &= 568630647535356955169033410940867804839360742060818433
\end{aligned}
$$

$q_{18} = 2663848877152141313$
$q_{19} = 3603109844542291969$
$q_{20} = 31954602082055164322067251 3$
$q_{21} = 2327042503868417$
$q_{22} = 16876881702951697238302412701 6961$
$q_{23} = 18898175797502131842003 7633$
$q_{24} = 77510610998025225893589670583928869226935804 23169$
$q_{25} = 81274690703860512587777$
$q_{26} = 646730219521$
$q_{27} = 3759005551413375428652444608049 9713$

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