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Formulas Involving Bernoulli and Stirling Numbers of Both Kinds

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Abstract

This paper is devoted to establishing several formulas relating Bernoulli numbers and Stirling numbers of both kinds. Some of these formulas are rediscoveries, presented with new proofs or from a fresh perspective, while others are entirely novel. Among the key results, we express Bernoulli numbers of the first kind in terms of Stirling numbers of the second kind, and another expresses Bernoulli numbers of the second kind in terms of Stirling numbers of the first kind. Additional formulas provide summation identities mixing Stirling numbers of both kinds with Bernoulli numbers (either of the first or second kind). Finally, the most original results transform certain linear combinations of Bernoulli numbers (first or second kind) into linear combinations of Stirling numbers (first or second kind).

1 Introduction and notation

Throughout this paper, we let $\mathbb{N}$ and $\mathbb{N}_0$ respectively denote the set of positive integers and the set of nonnegative integers. For $n \in \mathbb{N}_0$, we let $X^{\underline{n}}$ denote *the falling factorial of* X *to depth* n; that is,

$$X^{\underline{n}} := X(X-1)\cdots(X-n+1).$$

The Stirling numbers of the first kind $s(n, k)$ $(n, k \in \mathbb{N}_0,\ n \geq k)$ are then defined through the polynomial identity

$$X^{\underline{n}} = \sum_{k=0}^{n} s(n, k) X^k \qquad (\forall n \in \mathbb{N}_0), \tag{1}$$

while the Stirling numbers of the second kind $S(n, k)$ $(n, k \in \mathbb{N}_0,\ n \geq k)$ are defined through the inverse identity

$$X^n = \sum_{k=0}^{n} S(n, k) X^{\underline{k}} \qquad (\forall n \in \mathbb{N}_0). \tag{2}$$

For algebraic and combinatorial reasons, it is also considered that $s(n, k) = S(n, k) = 0$ for all $n, k \in \mathbb{N}_0$, with $n < k$. From (1) and (2), we immediately derive the following recurrence formulas allowing to compute, step by step, the numbers $s(n, k)$ and $S(n, k)$:

$$s(n, k) = s(n-1, k-1) - (n-1)\, s(n-1, k), \tag{3}$$
$$S(n, k) = S(n-1, k-1) + k\, S(n-1, k) \tag{4}$$

(valid for all $n, k \in \mathbb{N}$). By inserting (2) into (1) and vice versa, we also obtain the orthogonality relations

$$\sum_{m \leq i \leq n} s(n, i) S(i, m) = \delta_{nm}, \tag{5}$$
$$\sum_{m \leq i \leq n} S(n, i) s(i, m) = \delta_{nm} \tag{6}$$

(valid for all $n, m \in \mathbb{N}_0$ with $m \leq n$), where δ_{nm} denotes the Kronecker delta. For additional formulas and combinatorial interpretations regarding Stirling numbers of both kinds, we refer to [4, 11]. Further, by substituting in (1) X by $(-X)$ and rearranging, we get (for all $n \in \mathbb{N}_0$)

$$X(X+1)\cdots(X+n-1) = \sum_{k=0}^{n} (-1)^{n-k} s(n, k) X^k,$$

implying that the integers $(-1)^{n-k} s(n, k)$ $(n, k \in \mathbb{N}_0,\ n \geq k)$ are all nonnegative; that is, the sign of $s(n, k)$ is $(-1)^{n-k}$. Thus the above polynomial identity is equivalent to

$$X(X+1)\cdots(X+n-1) = \sum_{k=0}^{n} |s(n, k)|\, X^k \qquad (\forall n \in \mathbb{N}_0). \tag{7}$$

For $n \in \mathbb{N}$, dividing both sides of the above identity by X and reindexing gives

$$(X+1)(X+2)\cdots(X+n-1) = \sum_{k=0}^{n-1} |s(n, k+1)|\, X^k \qquad (\forall n \in \mathbb{N}). \tag{8}$$

Next, the Bernoulli polynomials $B_n(X)$ ($n \in \mathbb{N}_0$) can be defined by their exponential generating function

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(X)\frac{t^n}{n!} \tag{9}$$

and the Bernoulli numbers B_n ($n \in \mathbb{N}_0$) are the values of the Bernoulli polynomials at $X = 0$; that is, $B_n := B_n(0)$ ($\forall n \in \mathbb{N}_0$). To differentiate between the Bernoulli polynomials and the Bernoulli numbers, we always put the indeterminate X in evidence when referring to polynomials. For a comprehensive and modern treatment of Bernoulli polynomials and numbers, see [5]. Similarly, the Bernoulli numbers of the second kind B_n^* ($n \in \mathbb{N}_0$) are defined by their exponential generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{+\infty} B_n^* \frac{t^n}{n!}. \tag{10}$$

A simple and useful formula for computing B_n^* is given by [9, §3.2, p. 114]

$$B_n^* = \int_0^1 X^{\underline{n}}\, dX \qquad (\forall n \in \mathbb{N}_0). \tag{11}$$

To avoid ambiguity, we often refer to Bernoulli numbers as "Bernoulli numbers of the first kind" in this paper.

We frequently use the linear operators I, τ_r ($r \in \mathbb{C}$), D, and Δ of $\mathbb{C}[X]$, which respectively denote the identity of $\mathbb{C}[X]$, the translation by r, the differential operator ($D := \frac{d}{dX}$), and the forward difference operator ($\Delta := \tau_1 - I$). Applying the operators D and Δ to certain types of polynomials yields useful formulas, such as

$$\Delta X^{\underline{n}} = nX^{\underline{n-1}}, \tag{12}$$

$$\Delta B_n(X) = nX^{n-1}, \tag{13}$$

$$DB_n(X) = nB_{n-1}(X) \tag{14}$$

(valid for all $n \in \mathbb{N}$). Iterating (14), we derive that $B_n^{(k)}(0) = n^{\underline{k}} B_{n-k}$ ($\forall n, k \in \mathbb{N}_0$, with $k \leq n$). Hence, by the Taylor formula, we get

$$B_n(X) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} X^k \qquad (\forall n \in \mathbb{N}_0). \tag{15}$$

From (13) and (14), we also derive that

$$\int_0^1 B_n(X)\, dX = 0 \qquad (\forall n \in \mathbb{N}). \tag{16}$$

Additionally, all these operators can be expressed both in terms of D and in terms of Δ. Indeed, the Taylor formula shows that $\tau_r = e^{rD}$ ($\forall r \in \mathbb{C}$), thus $\Delta = \tau_1 - I = e^D - I$, implying that $D = \log(I + \Delta)$. The relationships between different linear operators play a vital role in exploring new identities involving special types of polynomials and numbers, such as those studied here. Although the operators D and Δ do not have proper inverses, each has an inverse up to an additive constant. By abuse of notation, we let D^{-1} (or $\int$) and Δ^{-1} respectively denote the inverses of D and Δ (defined up to an additive constant). So, from Formulas (12) and (13), we immediately derive the following formulas which hold up to an additive constant and for all $n \in \mathbb{N}_0$:

$$\Delta^{-1} X^{\underline{n}} = \frac{X^{\underline{n+1}}}{n+1}, \tag{17}$$

$$\Delta^{-1} X^{n} = \frac{B_{n+1}(X)}{n+1}. \tag{18}$$

The Bernoulli numbers were first introduced by Jacob Bernoulli in the late 17^{th} century to establish a closed form for the sum of powers of consecutive natural numbers. Since then, they have become fundamental in various branches of mathematics, including number theory, mathematical analysis, and algebraic topology, among others.

The Bernoulli numbers of the second kind were introduced by James Gregory in 1670 and rediscovered several times by renowned mathematicians of the 19^{th} century (see [1]). However, it is worth noting that some authors define the Bernoulli numbers of the second kind as $\frac{B_n^*}{n!}$ rather than B_n^*. In this paper, we follow Roman's approach [9, pp. 113–114].

The goal of this paper is to establish and explore various formulas relating Bernoulli and Stirling numbers of both kinds. Some of these results are well-known identities that we revisit with new proofs or fresh perspectives, while others are entirely novel contributions. Specifically, Corollary 2 and Theorem 4 provide formulas expressing Bernoulli numbers of the first kind in terms of Bernoulli numbers of the second kind, and vice versa. Additionally, many of our results appear in complementary pairs, where each identity involving Bernoulli numbers of the first kind has a corresponding counterpart for Bernoulli numbers of the second kind. For instance, Corollary 7, Corollary 10, and Theorem 17 correspond respectively to Theorem 13, Corollary 15, and Theorem 19. Beyond these structural symmetries, the paper also introduces summation identities that mix Stirling numbers of both kinds with Bernoulli numbers (either of the first or second kind). Moreover, some of the most original results establish transformations of certain linear combinations of Bernoulli numbers (of either kind) into linear combinations of Stirling numbers (of either kind), thereby revealing deeper connections between these classical sequences.

2 The results and the proofs

2.1 Revisited results

The following results are already known in the literature, but we provide new proofs. We begin with the following theorem which provides, for a given $n \in \mathbb{N}_0$, the expression of the

Bernoulli polynomial $B_n(X)$ as a linear combination of the polynomials $X^{\underline{k}}$ $(0 \leq k \leq n)$. It is essentially a re-wording of the definition of Bernoulli numbers given in [7, §1.2, Formula (2.2)].

Theorem 1. *For all $n \in \mathbb{N}_0$, we have*

$$B_n(X) = B_n + \sum_{k=1}^{n} \frac{n}{k} S(n-1, k-1) X^{\underline{k}}.$$

Proof. The identity of the theorem is clearly true for $n = 0$. Next, let $n \in \mathbb{N}$ be fixed. According to (13), (2), and (12), we have

$$\begin{aligned}
\Delta B_n(X) &= n X^{n-1} \\
&= n \sum_{k=0}^{n-1} S(n-1, k) X^{\underline{k}} \\
&= n \sum_{k=0}^{n-1} \frac{S(n-1, k)}{k+1} \Delta X^{\underline{k+1}} \\
&= \Delta \left(n \sum_{k=0}^{n-1} \frac{S(n-1, k)}{k+1} X^{\underline{k+1}} \right) \\
&= \Delta \left(n \sum_{k=1}^{n} \frac{S(n-1, k-1)}{k} X^{\underline{k}} \right).
\end{aligned}$$

Consequently, there exists a complex constant c satisfying the polynomial identity

$$B_n(X) = n \sum_{k=1}^{n} \frac{S(n-1, k-1)}{k} X^{\underline{k}} + \mathrm{c}.$$

By taking in the last identity $X = 0$, we get $\mathrm{c} = B_n(0) = B_n$, confirming the required identity of the theorem. □

From Theorem 1, we derive the following corollary which provides, for a given $n \in \mathbb{N}$, an expression of the Bernoulli number of the first kind B_n in terms of the Bernoulli numbers of the second kind B_k^* $(1 \leq k \leq n)$ and Stirling numbers of the second kind.

Corollary 2. *For all $n \in \mathbb{N}$, we have*

$$B_n = -\sum_{k=1}^{n} \frac{n}{k} S(n-1, k-1) B_k^*.$$

Proof. It suffices to integrate from 0 to 1 the polynomial identity of Theorem 1 and use Formulas (16) and (11). □

Now, we invert the formulas in Theorem 1 and Corollary 2; that is, we express the polynomial $X^{\underline{n}}$ ($n \in \mathbb{N}_0$) as a linear combination of Bernoulli polynomials, and we express the Bernoulli number of the second kind B_n^* ($n \in \mathbb{N}_0$) in terms of Bernoulli and Stirling numbers of the first kind. To do so, we need the following lemma.

Lemma 3. *Let $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ be two complex sequences. Then the following two identities are equivalent.*

$$u_n = \sum_{k=0}^{n} s(n,k) v_k \qquad (\forall n \in \mathbb{N}_0), \tag{I}$$

$$v_n = \sum_{k=0}^{n} S(n,k) u_k \qquad (\forall n \in \mathbb{N}_0). \tag{II}$$

Proof. Use the orthogonality relations (5) and (6) (see e.g., [4, §3.6, p. 144] or [8, §2, Problem 21, p. 90] for the details). □

By applying Lemma 3 for the formulas in Theorem 1 and Corollary 2, we obtain the following results.

Theorem 4. *For all $n \in \mathbb{N}$, we have*

$$B_n^* = -\sum_{k=1}^{n} \frac{n}{k} s(n-1, k-1) B_k.$$

Proof. First, notice that the formula in Corollary 2 can be rewritten as

$$\frac{B_{n+1}}{n+1} = \sum_{k=0}^{n} S(n,k) \left(-\frac{B_{k+1}^*}{k+1}\right) \qquad (\forall n \in \mathbb{N}_0),$$

and then apply Lemma 3 to invert this last formula. □

Theorem 5. *For all $n \in \mathbb{N}_0$, we have*

$$X^{\underline{n}} = B_n^* + \sum_{k=1}^{n} \frac{n}{k} s(n-1, k-1) B_k(X).$$

Proof. First, notice that the identity in Theorem 1 can be rewritten as

$$\frac{B_{n+1}(X) - B_{n+1}}{n+1} = \sum_{k=0}^{n} S(n,k) \left(\frac{X^{\underline{k+1}}}{k+1}\right) \qquad (\forall n \in \mathbb{N}_0),$$

and then apply Lemma 3 to invert this last identity and use Theorem 4. □

Remark 6. If we define the Bernoulli polynomials of the second kind $B_n^*(X)$ $(n \in \mathbb{N}_0)$ by

$$B_n^*(X) := \int_X^{X+1} t^{\underline{n}}\, dt,$$

then by integrating both sides of the formula in Theorem 5 from X to $X+1$, we obtain

$$B_n^*(X) = B_n^* + \sum_{k=1}^{n} \frac{n}{k} s(n-1, k-1) X^k,$$

a result already noted by Roman [9, p. 115].

From Theorem 1, we also derive a formula expressing the Bernoulli numbers in terms of Stirling numbers of both kinds, a result previously pointed out by Quaintance and Gould [6, §15, p. 217, Formula (15.41)] and, more recently, by Cereceda [3]. We have the following corollary:

Corollary 7. *For all $n, k \in \mathbb{N}$, with $n \geq k$, we have*

$$\sum_{k \leq i \leq n} \frac{S(n-1, i-1)s(i,k)}{i} = \frac{1}{n}\binom{n}{k} B_{n-k}.$$

Proof. Let $n \in \mathbb{N}$ be fixed. According to Theorem 1 and Formula (1), we have that

$$\begin{aligned}
B_n(X) &= B_n + \sum_{i=1}^{n} \frac{n}{i} S(n-1, i-1) X^{\underline{i}} \\
&= B_n + \sum_{i=1}^{n} \frac{n}{i} S(n-1, i-1) \sum_{k=1}^{i} s(i,k) X^k \\
&= B_n + n \sum_{\substack{1 \leq i \leq n \\ 1 \leq k \leq i}} \frac{S(n-1, i-1)s(i,k)}{i} X^k \\
&= B_n + n \sum_{\substack{1 \leq k \leq n \\ k \leq i \leq n}} \frac{S(n-1, i-1)s(i,k)}{i} X^k \\
&= B_n + n \sum_{k=1}^{n} \left(\sum_{k \leq i \leq n} \frac{S(n-1, i-1)s(i,k)}{i} \right) X^k.
\end{aligned}$$

By identifying this with (15), we derive that for all $k \in \mathbb{N}$, with $k \leq n$, we have

$$n \sum_{k \leq i \leq n} \frac{S(n-1, i-1)s(i,k)}{i} = \binom{n}{k} B_{n-k},$$

confirming the formula of the corollary. □

Remark 8. Since $s(n,k) = S(n,k) = 0$ for all $n,k \in \mathbb{N}_0$ with $n < k$, the summation condition "$k \leq i \leq n$" appearing in the formula of Corollary 7 can be replaced by the simpler condition "$i \in \mathbb{N}$".

From Corollary 7, we immediately derive the following well-known result, which expresses the Bernoulli numbers in terms of Stirling numbers of the second kind. This result, in particular, served as the starting point for Carlitz [2] in providing an interesting alternative proof of the famous Von Staudt-Clausen theorem.

Corollary 9. *For all $n \in \mathbb{N}_0$, we have*

$$B_n = \sum_{i=0}^{n} (-1)^i \frac{i!}{i+1} S(n,i).$$

Proof. It suffices to apply Corollary 7 for (n,k) replaced by $(n+1,1)$ $(n \in \mathbb{N}_0)$ and note that $s(m,1) = (-1)^{m-1}(m-1)!$ $(\forall m \in \mathbb{N})$. □

Further, by relying on the recurrence relation (4) and the orthogonality relation (6), we derive from Corollary 7 the following corollary, which was previously established by Quaintance and Gould [6, §15, p. 215, Formula (15.35)] and by Cereceda [3].

Corollary 10. *For all $n,k \in \mathbb{N}$, with $n \geq k$, we have*

$$\sum_{k \leq i \leq n} \frac{S(n,i)s(i,k)}{i} = \frac{1}{n}\binom{n}{k} B_{n-k} + \delta_{n-1,k}.$$

Proof. Let $n,k \in \mathbb{N}$ be fixed, with $n \geq k$. By the recurrence relation (4), we have for all $i \in \{k, k+1, \dots, n\}$: $S(n,i) = S(n-1,i-1) + iS(n-1,i)$; that is,

$$S(n-1,i-1) = S(n,i) - iS(n-1,i).$$

By inserting this into the formula of Corollary 7 (for all $i \in \mathbb{N}$, with $k \leq i \leq n$), we get after simplifying and rearranging

$$\begin{aligned}\sum_{k \leq i \leq n} \frac{S(n,i)s(i,k)}{i} &= \frac{1}{n}\binom{n}{k} B_{n-k} + \sum_{k \leq i \leq n} S(n-1,i)s(i,k) \\ &= \frac{1}{n}\binom{n}{k} B_{n-k} + \delta_{n-1,k}\end{aligned}$$

(according to the orthogonality relation (6)). The corollary is proved. □

We complete these revisited results with the following proposition, which provides an expression for the Bernoulli numbers of the second kind in terms of Stirling numbers of the first kind. This result was previously pointed out by Roman and Rota [10, §9, p. 145].

Proposition 11. *For all $n \in \mathbb{N}_0$, we have*

$$B_n^* = \sum_{i=0}^{n} \frac{s(n,i)}{i+1}.$$

Proof. This follows by integrating Formula (1) from $X = 0$ to $X = 1$ and using Formula (11). □

Remark 12. The formula in Proposition 11 can be considered as an analog of the one in Corollary 9. Later, we will generalize it by establishing a closed form for the general sum $\sum_{i=0}^{n} \frac{s(n,i)}{i+r}$ ($r \in \mathbb{N}$, $n \in \mathbb{N}_0$); see Theorem 19.

2.2 New results

The following results are original contributions. First, we establish analogs of the formulas in Corollaries 7 and 10, where the kinds of Stirling numbers are permuted. In other words, we will find closed forms for the two sums

$$\sum_{k \le i \le n} \frac{s(n-1,i-1)S(i,k)}{i} \quad \text{and} \quad \sum_{k \le i \le n} \frac{s(n,i)S(i,k)}{i}$$

($n, k \in \mathbb{N}$, $n \ge k$). Interestingly, the resulting closed forms depend on Bernoulli numbers of the second kind (see Theorem 13 and Corollary 15 below).

Theorem 13. *For all $n, k \in \mathbb{N}$, with $n \ge k$, we have*

$$\sum_{k \le i \le n} \frac{s(n-1,i-1)S(i,k)}{i} = \frac{1}{n}\binom{n}{k} B_{n-k}^*.$$

Proof. Let $n \in \mathbb{N}$ be fixed. According to Formula (1), we have

$$X^{\underline{n-1}} = \sum_{i=0}^{n-1} s(n-1,i) X^i.$$

Integrating with respect to X, we get (up to an additive constant)

$$\begin{aligned}
\int X^{\underline{n-1}}\, dX &= \sum_{i=0}^{n-1} \frac{s(n-1,i)}{i+1} X^{i+1} \\
&= \sum_{i=0}^{n-1} \frac{s(n-1,i)}{i+1} \sum_{k=1}^{i+1} S(i+1,k) X^{\underline{k}} \qquad \text{(by Formula (2))} \\
&= \sum_{k=1}^{n} \left(\sum_{k-1 \le i \le n-1} \frac{s(n-1,i)S(i+1,k)}{i+1} \right) X^{\underline{k}}.
\end{aligned}$$

That is, up to an additive constant, we have

$$\int X^{\underline{n-1}}\,dX = \sum_{k=1}^{n}\left(\sum_{k\leq i\leq n}\frac{s(n-1,i-1)S(i,k)}{i}\right)X^{\underline{k}}. \tag{19}$$

On the other hand, we have (according to Formula (10))

$$\frac{1}{\log(1+t)} = \frac{1}{t}\sum_{k=0}^{+\infty}B_k^*\frac{t^k}{k!} = \frac{1}{t}+\sum_{k=1}^{+\infty}B_k^*\frac{t^{k-1}}{k!} = \frac{1}{t}+\sum_{k=0}^{+\infty}B_{k+1}^*\frac{t^k}{(k+1)!}.$$

By applying this to the operator Δ and use the fact that $\log(I+\Delta)=D$, we get the identity of operators

$$D^{-1} = \Delta^{-1}+\sum_{k=0}^{+\infty}B_{k+1}^*\frac{\Delta^k}{(k+1)!}.$$

Then, by applying this last identity of operators to the polynomial $X^{\underline{n-1}}$ (which is of degree $(n-1)$), we get (according to (17) and (12))

$$\begin{aligned}\int X^{\underline{n-1}}\,dX &= \frac{X^{\underline{n}}}{n}+\sum_{k=0}^{n-1}B_{k+1}^*(n-1)^{\underline{k}}\frac{X^{\underline{n-1-k}}}{(k+1)!}\\ &= \frac{X^{\underline{n}}}{n}+\sum_{k=0}^{n-1}\frac{1}{n}\binom{n}{k+1}B_{k+1}^*X^{\underline{n-1-k}}\\ &= \frac{X^{\underline{n}}}{n}+\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}B_k^*X^{\underline{n-k}}\\ &= \sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}B_k^*X^{\underline{n-k}};\end{aligned}$$

that is, (up to an additive constant)

$$\int X^{\underline{n-1}}\,dX = \sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}B_{n-k}^*X^{\underline{k}}. \tag{20}$$

Finally, by identifying the coefficients of $X^{\underline{k}}$ $(1\leq k\leq n)$ in the right-hand sides of (19) and (20), we derive that for all $k\in\mathbb{N}$, with $k\leq n$, we have

$$\sum_{k\leq i\leq n}\frac{s(n-1,i-1)S(i,k)}{i} = \frac{1}{n}\binom{n}{k}B_{n-k}^*,$$

as required. □

Remark 14. Proposition 11 can also be derived by applying Theorem 13 to the pair $(n+1, 1)$ instead of (n, k), noting that $S(m, 1) = 1$ for all $m \in \mathbb{N}$. From this perspective, Theorem 13 can be seen as a generalization of Proposition 11.

Corollary 15. *For all $n, k \in \mathbb{N}$, with $n \geq k$, we have*

$$\sum_{k \leq i \leq n} \frac{s(n,i)S(i,k)}{i} = (-1)^{n-k} \frac{(n-1)!}{k!} \sum_{\ell=0}^{n-k} \frac{(-1)^{\ell}}{\ell!} B_{\ell}^{*}.$$

Proof. Let $n, k \in \mathbb{N}$ be fixed such that $n \geq k$. Since the formula of the corollary is clearly true for $n = k$, we may assume that $n \geq k+1$. According to Theorem 13, we have for all integer $m \geq k+1$

$$\sum_{k \leq i \leq m} \frac{s(m-1, i-1)S(i,k)}{i} = \frac{1}{m} \binom{m}{k} B_{m-k}^{*}.$$

But (according to the recurrence relation (3)), we have for all integers i, m such that $k \leq i \leq m$

$$s(m-1, i-1) = s(m,i) + (m-1)s(m-1,i).$$

By inserting this into the previous formula, we get (for all integers $m \geq k+1$)

$$\sum_{k \leq i \leq m} \frac{s(m,i)S(i,k)}{i} + (m-1) \sum_{k \leq i \leq m-1} \frac{s(m-1,i)S(i,k)}{i} = \frac{1}{m} \binom{m}{k} B_{m-k}^{*}$$

(since $s(m-1, m) = 0$). Then, by multiplying by $\frac{(-1)^m}{(m-1)!}$, we get

$$\frac{(-1)^m}{(m-1)!} \sum_{k \leq i \leq m} \frac{s(m,i)S(i,k)}{i} - \frac{(-1)^{m-1}}{(m-2)!} \sum_{k \leq i \leq m-1} \frac{s(m-1,i)S(i,k)}{i} = \frac{(-1)^m}{k!(m-k)!} B_{m-k}^{*}$$

(which is valid for all integers $m \geq k+1$). By summing both sides of this last identity from $m = k+1$ to $m = n$ and noticing that the sum on the left-hand side is telescopic, we derive the identity

$$\frac{(-1)^n}{(n-1)!} \sum_{k \leq i \leq n} \frac{s(n,i)S(i,k)}{i} - \frac{(-1)^k}{k!} = \frac{1}{k!} \sum_{m=k+1}^{n} \frac{(-1)^m}{(m-k)!} B_{m-k}^{*},$$

hence

$$\frac{(-1)^n}{(n-1)!} \sum_{k \leq i \leq n} \frac{s(n,i)S(i,k)}{i} = \frac{1}{k!} \sum_{m=k}^{n} \frac{(-1)^m}{(m-k)!} B_{m-k}^{*} = \frac{1}{k!} \sum_{\ell=0}^{n-k} \frac{(-1)^{\ell+k}}{\ell!} B_{\ell}^{*},$$

implying the required formula of the corollary. □

Remark 16. In Corollary 15, the quantity

$$(-1)^{n-k}\sum_{\ell=0}^{n-k}\frac{(-1)^{\ell}}{\ell!}B_{\ell}^{*}$$

represents the coefficient of t^{n-k} in the power series $\frac{t}{(1+t)\log(1+t)}$. In fact, we can reprove Corollary 15 by applying the analytic function $t \mapsto \frac{t}{(1+t)\log(1+t)}$ to the operator Δ, in a manner similar to the proof of Theorem 13.

We now proceed to generalize the formula in Corollary 9 by finding a closed form for the general sum

$$\sum_{k=0}^{n}(-1)^k\frac{k!}{k+r}S(n,k) \quad (r \in \mathbb{N}, n \in \mathbb{N}_0).$$

Theorem 17. *Let $r \in \mathbb{N}$. Then for all $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n}(-1)^k\frac{k!}{k+r}S(n,k) = \frac{1}{(r-1)!}\sum_{k=0}^{r-1}\left|s(r,k+1)\right|B_{n+k}.$$

Proof. Let $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$ be fixed. By multiplying side by side the two polynomial identities

$$\sum_{k=0}^{n}S(n,k)X^{\underline{k}} = X^n$$

and

$$(X+1)(X+2)\cdots(X+r-1) = \sum_{k=0}^{r-1}\left|s(r,k+1)\right|X^k$$

(which are nothing else (2) and (8)), we get

$$\sum_{k=0}^{n}S(n,k)(X+r-1)^{\underline{k+r-1}} = \sum_{k=0}^{r-1}\left|s(r,k+1)\right|X^{n+k}.$$

Then, by applying the operator Δ^{-1} to the two sides of this last polynomial identity and taking into account (17) and (18), we get

$$\sum_{k=0}^{n}\frac{S(n,k)}{k+r}(X+r-1)^{\underline{k+r}} = \sum_{k=0}^{r-1}\left|s(r,k+1)\right|\frac{B_{n+k+1}(X)}{n+k+1} + \mathrm{c}_{n,r}$$

(where $\mathrm{c}_{n,r}$ is a constant depending only on n and r). By dividing by X and observing that

$$\begin{aligned}\frac{(X+r-1)^{\underline{k+r}}}{X} &= (X+r-1)(X+r-2)\cdots(X+1)\cdot(X-1)(X-2)\cdots(X-k)\\ &= (X+r-1)^{\underline{r-1}}\cdot(X-1)^{\underline{k}}\end{aligned}$$

(for all $k \in \mathbb{N}_0$, $k \le n$), we get

$$\sum_{k=0}^{n} \frac{S(n,k)}{k+r}(X+r-1)^{\underline{r-1}}(X-1)^{\underline{k}} = \frac{\sum_{k=0}^{r-1} |s(r,k+1)| \frac{B_{n+k+1}(X)}{n+k+1} + c_{n,r}}{X}.$$

Finally, by letting $X \to 0$ and observing that the polynomial

$$\sum_{k=0}^{r-1} |s(r,k+1)| \frac{B_{n+k+1}(X)}{n+k+1} + c_{n,r}$$

vanishes at $X = 0$ (according to the previous formula), we get

$$\begin{aligned}\sum_{k=0}^{n} \frac{S(n,k)}{k+r}(r-1)^{\underline{r-1}}(-1)^{\underline{k}} &= \lim_{X\to 0} \frac{\sum_{k=0}^{r-1} |s(r,k+1)| \frac{B_{n+k+1}(X)}{n+k+1} + c_{n,r}}{X} \\ &= \frac{d}{dX}\left(\sum_{k=0}^{r-1} |s(r,k+1)| \frac{B_{n+k+1}(X)}{n+k+1} + c_{n,r}\right)(0) \\ &= \sum_{k=0}^{r-1} |s(r,k+1)| \, B_{n+k} \qquad \text{(according to (14)).}\end{aligned}$$

It remains to note that $(r-1)^{\underline{r-1}} = (r-1)!$ and $(-1)^{\underline{k}} = (-1)^k k!$ to conclude with the required formula of the theorem. □

Examples 18.

- Taking $r = 1$ in Theorem 17 yields the formula of Corollary 9.
- By taking $r = 2$ in Theorem 17 and taking into account the fact that $|s(2,1)| = |s(2,2)| = 1$, we obtain the remarkable formula

$$\sum_{k=0}^{n} (-1)^k \frac{k!}{k+2} S(n,k) = B_n + B_{n+1} \tag{21}$$

  (valid for all $n \in \mathbb{N}_0$).

- By taking $r = 3$ in Theorem 17 and taking into account the facts that $|s(3,1)| = 2$, $|s(3,2)| = 3$, and $|s(3,3)| = 1$, we obtain the formula

$$\sum_{k=0}^{n} (-1)^k \frac{k!}{k+3} S(n,k) = B_n + \frac{3}{2} B_{n+1} + \frac{1}{2} B_{n+2} \tag{22}$$

  (valid for all $n \in \mathbb{N}_0$).

We finally present a generalization of the formula in Corollary 11 by finding a closed form for the general sum

$$\sum_{k=0}^{n} \frac{s(n,k)}{k+r} \quad (r \in \mathbb{N}, n \in \mathbb{N}_0).$$

Theorem 19. *Let $r \in \mathbb{N}$. Then for all $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} \frac{s(n,k)}{k+r} = \sum_{k=0}^{r-1} \lambda_{r,n,k} B_{n+k}^{*},$$

where

$$\lambda_{r,n,k} := \sum_{\ell=0}^{r-1-k} \binom{r-1}{\ell} S(r-1-\ell, k) n^{\ell}$$

(for all $k \in \{0, 1, \dots, r-1\}$).

Proof. Fix $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Starting from the polynomial identity in (1)

$$\sum_{k=0}^{n} s(n,k) X^k = X^{\underline{n}},$$

which we multiply by X^{r-1} and then integrate from 0 to 1, we get

$$\sum_{k=0}^{n} \frac{s(n,k)}{k+r} = \int_0^1 X^{\underline{n}} \cdot X^{r-1} \, dX.$$

So, we are led to show the identity

$$\int_0^1 X^{\underline{n}} \cdot X^{r-1} \, dX = \sum_{k=0}^{r-1} \lambda_{r,n,k} B_{n+k}^{*},$$

where the $\lambda_{r,n,k}$ are defined in the statement of the theorem. Using successively the binomial formula and the polynomial identity (2), we have that

$$\begin{aligned}
\int_0^1 X^{\underline{n}} \cdot X^{r-1} \, dX &= \int_0^1 X^{\underline{n}} (X - n + n)^{r-1} \, dX \\
&= \int_0^1 X^{\underline{n}} \left(\sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (X-n)^{r-1-\ell} n^{\ell} \right) dX \\
&= \sum_{\ell=0}^{r-1} \left(\binom{r-1}{\ell} n^{\ell} \int_0^1 X^{\underline{n}} (X-n)^{r-1-\ell} \, dX \right) \\
&= \sum_{\ell=0}^{r-1} \left(\binom{r-1}{\ell} n^{\ell} \int_0^1 X^{\underline{n}} \left(\sum_{k=0}^{r-1-\ell} S(r-1-\ell, k) (X-n)^{\underline{k}} \right) dX \right) \\
&= \sum_{\ell=0}^{r-1} \sum_{k=0}^{r-1-\ell} \binom{r-1}{\ell} n^{\ell} S(r-1-\ell, k) \int_0^1 X^{\underline{n}} (X-n)^{\underline{k}} \, dX.
\end{aligned}$$

But for all $k \in \{0, 1, \ldots, r-1-\ell\}$, we have that

$$X^{\underline{n}}(X-n)^{\underline{k}} = X^{\underline{n+k}},$$

so

$$\int_0^1 X^{\underline{n}}(X-n)^{\underline{k}}\,dX = \int_0^1 X^{\underline{n+k}}\,dX = B^*_{n+k}$$

(according to (11)). Thus

$$\begin{aligned}\int_0^1 X^{\underline{n}} \cdot X^{r-1}\,dX &= \sum_{\ell=0}^{r-1}\sum_{k=0}^{r-1-\ell}\binom{r-1}{\ell} n^\ell S(r-1-\ell,k) B^*_{n+k} \\ &= \sum_{k=0}^{r-1}\left(\sum_{\ell=0}^{r-1-k}\binom{r-1}{\ell} S(r-1-\ell,k) n^\ell\right) B^*_{n+k} \\ &= \sum_{k=0}^{r-1}\lambda_{r,n,k} B^*_{n+k},\end{aligned}$$

as required. This completes the proof. □

Examples 20.

- Taking $r = 1$ in Theorem 19 yields the formula of Corollary 11.
- By taking $r = 2$ in Theorem 19, we obtain the formula

  $$\sum_{k=0}^{n}\frac{s(n,k)}{k+2} = nB^*_n + B^*_{n+1}$$

  (valid for all $n \in \mathbb{N}_0$).
- By taking $r = 3$ in Theorem 19, we obtain the formula

  $$\sum_{k=0}^{n}\frac{s(n,k)}{k+3} = n^2 B^*_n + (2n+1)B^*_{n+1} + B^*_{n+2}$$

  (valid for all $n \in \mathbb{N}_0$).

Remark 21. The formula in Theorem 19 can be considered an analog of the formula in Theorem 17; however, in the formula of Theorem 19, the coefficients $\lambda_{r,n,k}$ of the linear combination of the Bernoulli numbers of the second kind depend on n (in contrast to the coefficients of the linear combination of the Bernoulli numbers of the first kind in the formula of Theorem 17).

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References

[1] I. V. Blagouchine, A note on some recent results for the Bernoulli numbers of the second kind, *J. Integer Sequences* **20** (2017), Article 17.3.8.

[2] L. Carlitz, Bernoulli numbers, *Fibonacci Quart.* **6** (1968), 71–85.

[3] J. L. Cereceda, A note on the coefficients of power sums of arithmetic progressions, *Miskolc Math. Notes* **23** (2022), 551–557.

[4] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Springer, 2012.

[5] O. Kouba, Lecture notes, Bernoulli polynomials and applications, preprint, 2016. Available at https://arxiv.org/abs/1309.7560v2.

[6] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers. The Unpublished Notes of H. W. Gould*, World Scientific Publishing, 2016.

[7] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, 1973.

[8] J. Riordan, *Combinatorial Identities*, John Wiley and Sons Inc., 1968.

[9] S. M. Roman, *The Umbral Calculus*, Academic Press, 1984.

[10] S. M. Roman and G.-C. Rota, The umbral calculus, *Adv. Math.* **27** (1978), 95–188.

[11] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, 2^{nd} ed., Cambridge University Press, 2012.

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