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Areas Between Cosines

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Abstract

We find the area between $\cos^n x$ and $\cos^n kx$ over the interval $[0, \pi]$ as k goes to infinity. We establish recursive formulas for these areas, and we show that these areas are related to the coefficients of two exponential generating functions involving $\arcsin x$.

1 Introduction

Figure 1 shows the region between $\cos^3 x$ and $\cos^3 11x$ over the interval $[0, \pi]$.

It is not hard to directly calculate the area of this region. We could write it as

$$\frac{6}{33}\left(\cot\frac{3\pi}{12} + 9\cot\frac{\pi}{12}\right) - \frac{5}{33}\left(\cot\frac{3\pi}{10} + 9\cot\frac{\pi}{10}\right) \approx 1.981887\ldots,$$

which has a pleasant symmetry, or we could write it as

$$\frac{1}{33}\left(144 + 54\sqrt{3} - 5\left(19 + 9\sqrt{5}\right)\sqrt{5 - 2\sqrt{5}}\right) \approx 1.981887\ldots,$$

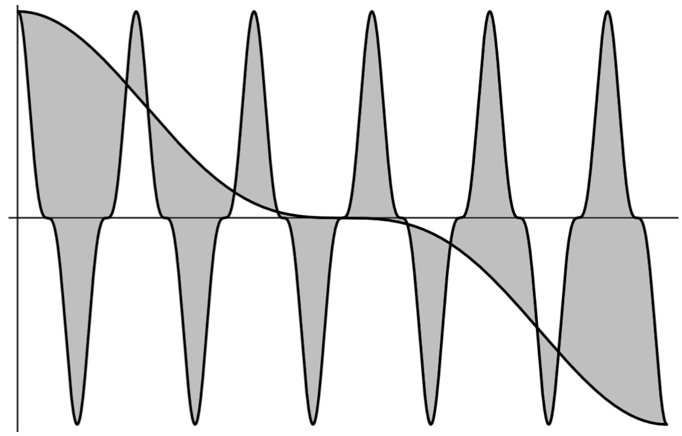

Figure 1: Region between $\cos^3 x$ and $\cos^3 11x$.

which has a purely algebraic expression. These are both interesting answers, but in this article we are more interested in what happens when we replace the 11 in $\cos^3 11x$ with k (giving us $\cos^3 kx$) and then find the area between $\cos^3 x$ and $\cos^3 kx$ as k goes to infinity. To be precise, we want to find

$$\lim_{k\to\infty} \int_0^\pi \left| \cos^3 x - \cos^3 kx \right| dx.$$

In this case, the limiting area turns out to be

$$\lim_{k\to\infty} \int_0^\pi \left| \cos^3 x - \cos^3 kx \right| dx = \frac{56}{9\pi} \approx 1.980594\ldots$$

which is rather surprising (and also fairly close numerically to the two expressions above).

Of course, there is no reason to restrict ourselves to just looking at the *third* power of cosine. With this in mind, we define A_n to be the limiting area (as $k \to \infty$) of the region between $\cos^n x$ and $\cos^n kx$ over the interval $[0, \pi]$. In other words, we define

$$A_n = \lim_{k\to\infty} \int_0^\pi \left| \cos^n x - \cos^n kx \right| dx. \tag{1}$$

In what follows, we will find formulas for A_n involving sums with binomial coefficients (Theorems 1 and 2). We then find recursive formulas for A_n involving just A_{n-2} (Theorem 3). Finally, in Theorem 4 we establish formulas for A_n that involve sums with double factorials, and we connect these formulas with two entries in the On-Line Encyclopedia of Integer Sequences (OEIS) [4]. These two entries are A296726 and A372324, and they are related to the exponential generating functions for $\arcsin x/(1-x)$ and $\arcsin^2 x/(2(1-x))$, respectively.

2 Area formulas

Since $\cos^n x$ looks quite different on the interval $[0, \pi]$ depending on the parity of n (as seen by comparing Figures 2 and 4), it is reasonable to separate our discussion of areas into the following two cases.

2.1 Odd n

To set the stage, we begin with a few values for the limiting area A_n when n is odd:

$$A_1 = \frac{8}{\pi} = \frac{8 \cdot 1}{(1)^2\pi}, \qquad A_3 = \frac{56}{9\pi} = \frac{8 \cdot 7}{(1 \cdot 3)^2\pi},$$

$$A_5 = \frac{1192}{225\pi} = \frac{8 \cdot 149}{(1 \cdot 3 \cdot 5)^2\pi}, \qquad A_7 = \frac{17228}{3675\pi} = \frac{8 \cdot 6483}{(1 \cdot 3 \cdot 5 \cdot 7)^2\pi}.$$

What can we say about these numbers $1, 7, 149, 6483$ that appear in the numerators of A_n for n odd? We will show that these are equal to every other term in the sequence A296726 in the OEIS, where we learn that they also appear as coefficients in the exponential generating function for $\arcsin x/(1 - x)$. See Theorem 4 for details.

We also note that the value $A_1 = 8/\pi$ was the answer to Problem 2191 which appeared in a recent issue [2] of *Mathematics Magazine*.

Here is our first theorem.

Theorem 1. *With A_n as defined above in Eq. (1), then*

$$\text{for } n \text{ odd}, \qquad A_n = \frac{8}{2^{n-1}\pi} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{1}{(n-2j)^2}. \tag{2}$$

To prove Theorem 1, we will need several technical results that we present and prove in Section 5. The proof of Theorem 1 is in Section 6

2.2 Even n

Next, we present a few values for the limiting area A_n when n is even:

$$A_2 = \frac{4}{\pi} = \frac{16 \cdot 1}{(2)^2\pi}, \qquad A_4 = \frac{4}{\pi} = \frac{16 \cdot 16}{(2 \cdot 4)^2\pi},$$

$$A_6 = \frac{34}{9\pi} = \frac{16 \cdot 544}{(2 \cdot 4 \cdot 6)^2\pi}, \qquad A_8 = \frac{32}{9\pi} = \frac{16 \cdot 32768}{(2 \cdot 4 \cdot 6 \cdot 8)^2\pi}.$$

What can we say about these numbers $1, 16, 544, 32768, \ldots$ that appear in the numerators of A_n for n even? We will show that these are equal to alternate terms in the sequence

A372324 in the OEIS, where we learn that they also appear as coefficients in the exponential generating function for $\arcsin^2 x/(2(1-x))$. See Theorem 4 for details.

For now, we have the following theorem.

Theorem 2. *With A_n as defined above in Eq. (1), then*

$$\text{for } n \equiv 2 \pmod 4, \qquad A_n = \frac{16}{2^n\pi}\sum_{j=0}^{(n-2)/4}\binom{n}{2j}\frac{1}{(n/2-2j)^2}, \tag{3}$$

and

$$\text{for } n \equiv 0 \pmod 4, \qquad A_n = \frac{16}{2^n\pi}\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}\frac{1}{(n/2-(2j+1))^2}. \tag{4}$$

Just as with Theorem 1, the proof of Theorem 2 will require some technical results that we present and prove in Section 5. The proof of Theorem 2 is in Section 7

3 Recursion formulas

Thanks to Theorems 1 and 2, we can produce the following rather simple recursion formula.

Theorem 3. *With A_n as defined above, then with $n \geq 3$ we have*

$$A_n = \frac{n-1}{n}A_{n-2} + \begin{cases} \dfrac{8}{n^2\pi}, & \text{if } n \text{ odd;} \\ \dfrac{16}{n^2\pi}, & \text{if } n \text{ even.} \end{cases}$$

Proof. We start with the easily-verified statement that

$$\binom{n}{j}\frac{1}{(n-2j)^2} - \frac{1}{n^2}\binom{n}{j} = \frac{4(n-1)}{n}\binom{n-2}{j-1}\frac{1}{((n-2)-2(j-1))^2}. \tag{5}$$

At this point, we will consider the three cases of n odd, $n \equiv 2 \pmod 4$, and $n \equiv 0 \pmod 4$.

- First, suppose n is odd. We sum both sides of Eq. (5) from $j = 1$ to $j = (n-1)/2$ to obtain

$$\sum_{j=1}^{(n-1)/2}\binom{n}{j}\frac{1}{(n-2j)^2} - \frac{1}{n^2}\binom{n}{j} = \frac{4(n-1)}{n}\sum_{j=1}^{(n-1)/2}\binom{n-2}{j-1}\frac{1}{((n-2)-2(j-1))^2}.$$

On the left, we can start that sum at $j = 0$ instead of $j = 1$ without changing the value. On the right, we re-index the sum by using $j' = j - 1$, so that j' starts at $j' = 0$ and ends at $j' = (n-3)/2$. After distributing the sum on the left, this gives us

$$\sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{1}{(n-2j)^2} - \frac{1}{n^2} \sum_{j=0}^{(n-1)/2} \binom{n}{j} = \frac{4(n-1)}{n} \sum_{j'=0}^{(n-3)/2} \binom{n-2}{j'} \frac{1}{((n-2)-2j')^2}.$$

Thanks to our value for A_n in Eq. (2) for n odd, we can re-write the above equation as

$$A_n \frac{2^{n-1}\pi}{8} - \frac{1}{n^2} \sum_{j=0}^{(n-1)/2} \binom{n}{j} = \frac{4(n-1)}{n} \cdot A_{n-2} \frac{2^{(n-2)-1}\pi}{8}.$$

Since n is odd, the sum on the left of the above equation is exactly half of the entire sum of the nth row of Pascal's triangle. The entire sum would be 2^n, so the sum in the above equation would be 2^{n-1}. So, after combining the 4 and the $2^{(n-2)-1}$ on the right, we have

$$A_n \frac{2^{n-1}\pi}{8} - \frac{1}{n^2} 2^{n-1} = \frac{(n-1)}{n} \cdot A_{n-2} \frac{2^{n-1}\pi}{8}.$$

If we now divide everything by $2^{n-1}\pi/8$ and re-arrange the terms, we obtain

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{8}{\pi n^2},$$

as desired (for n odd).

- Next, we consider $n \equiv 2 \pmod 4$. Looking back at Eq. (5), we replace j with $2j$, giving us

$$\binom{n}{2j} \frac{1}{(n-4j)^2} - \frac{1}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \binom{n-2}{2j-1} \frac{1}{((n-2)-2(2j-1))^2}.$$

We factor out 2^2 from the $(n-4j)^2$ in the denominator on the left, and likewise from the denominator on the right, giving us

$$\binom{n}{2j} \frac{1}{4(n/2-2j)^2} - \frac{1}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \binom{n-2}{2j-1} \frac{1}{4((n-2)/2-(2j-1))^2}.$$

We multiply through by 4 to get

$$\binom{n}{2j} \frac{1}{(n/2-2j)^2} - \frac{4}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \binom{n-2}{2j-1} \frac{1}{((n-2)/2-(2j-1))^2}. \tag{6}$$

We sum both sides of Eq. (6) from $j = 1$ to $j = (n-2)/4$ to obtain

$$\sum_{j=1}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2-2j)^2} - \frac{4}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \sum_{j=1}^{(n-2)/4} \binom{n-2}{2j-1} \frac{1}{((n-2)/2-(2j-1))^2}.$$

On the left, we can start that sum at $j = 0$ instead of $j = 1$ without changing the value. On the right, we re-index the sum by using $j' = j - 1$, so that j' starts at $j' = 0$ and ends at $j' = (n-6)/4$. This gives us

$$\sum_{j=0}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2-2j)^2} - \frac{4}{n^2}\binom{n}{2j} = \frac{4(n-1)}{n} \sum_{j'=0}^{(n-6)/4} \binom{n-2}{2j'+1} \frac{1}{((n-2)/2-(2j'+1))^2}.$$

Thanks to our value for A_n in Eq. (3) for $n \equiv 2 \pmod 4$, we can re-write the first sum in the above equation as $A_n \cdot 2^n\pi/16$. When we do so, it gives us

$$A_n \frac{2^n\pi}{16} - \frac{4}{n^2} \sum_{j=0}^{(n-2)/4} \binom{n}{2j} = \frac{4(n-1)}{n} \sum_{j'=0}^{(n-6)/4} \binom{n-2}{2j'+1} \frac{1}{((n-2)/2-(2j'+1))^2}.$$

Likewise, since $n \equiv 2 \pmod 4$, then $n - 2 \equiv 0 \pmod 4$, and so if we use Eq. (4) for A_{n-2} then we recognize that the sum on the right of the above equation is equal to $A_{n-2} \cdot (2^{n-2}\pi)/16$. This means we can now re-write the above equation as

$$A_n \frac{2^n\pi}{16} - \frac{4}{n^2} \sum_{j=0}^{(n-2)/4} \binom{n}{2j} = \frac{4(n-1)}{n} A_{n-2} \frac{2^{n-2}\pi}{16}.$$

Since $n \equiv 2 \pmod 4$, then the sum on the left of the above equation is exactly one quarter of the entire sum of the nth row of Pascal's triangle. The entire sum would be 2^n, so the sum in the above equation would be 2^{n-2}, giving us

$$A_n \frac{2^n\pi}{16} - \frac{4}{n^2} 2^{n-2} = \frac{4(n-1)}{n} A_{n-2} \frac{2^{n-2}\pi}{16}.$$

After combining the 4 and the 2^{n-2} on the left and on the right, we have

$$A_n \frac{2^n\pi}{16} - \frac{1}{n^2} 2^n = \frac{(n-1)}{n} A_{n-2} \frac{2^n\pi}{16}.$$

If we now divide everything by $2^n\pi/16$ and re-arrange the terms, we obtain

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{16}{n^2\pi},$$

as desired (for $n \equiv 2 \pmod 4$).

- Finally, we consider $n \equiv 0 \pmod 4$. Looking back once more at Eq. (5), we first factor out 2^2 from the $(n-2j)^2$ in the denominator on the left, and likewise from the denominator in the right. We also replace $n - 2$ with q in the expression on the right, leaving us with

$$\binom{n}{j} \frac{1}{4(n/2-j)^2} - \frac{1}{n^2}\binom{n}{j} = \frac{4(n-1)}{n} \binom{q}{j-1} \frac{1}{4(q/2-(j-1))^2}.$$

We now multiply through by 4, and replace j with $2j+1$, giving us

$$\binom{n}{2j+1}\frac{1}{(n/2-(2j+1))^2}-\frac{4}{n^2}\binom{n}{2j+1}=\frac{4(n-1)}{n}\binom{q}{2j}\frac{1}{(q/2-2j)^2}. \tag{7}$$

We sum both sides of Eq. (7) from $j=0$ to $j=(n-4)/4$ to obtain

$$\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}\frac{1}{(n/2-(2j+1))^2}-\frac{4}{n^2}\binom{n}{2j+1}=\frac{4(n-1)}{n}\sum_{j=0}^{(n-4)/4}\binom{q}{2j}\frac{1}{(q/2-2j)^2}.$$

Thanks to our value for A_n in Eq. (4) for $n \equiv 0 \pmod 4$, we can re-write the sum of the first expression on the left above as $A_n \cdot (2^n\pi)/16$. When we do so (after replacing $n-4$ with $q-2$ in the upper bound of the sum on the right) it give us

$$A_n\frac{2^n\pi}{16}-\frac{4}{n^2}\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}=\frac{4(n-1)}{n}\sum_{j=0}^{(q-2)/4}\binom{q}{2j}\frac{1}{(q/2-2j)^2}.$$

Since $n \equiv 0 \pmod 4$ and since $q=n-2$, then $q \equiv 2 \pmod 4$, and so if we use Eq. (3) for $A_q = A_{n-2}$ then we recognize that the sum on the right is equal to $A_{n-2}\cdot(2^{n-2}\pi)/16$. This means we can re-write the above equation as

$$A_n\frac{2^n\pi}{16}-\frac{4}{n^2}\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}=\frac{4(n-1)}{n}A_{n-2}\frac{2^{n-2}\pi}{16}.$$

Since $n \equiv 0 \pmod 4$, then the sum on the left of the above equation is exactly one quarter of the entire sum of the nth row of Pascal's triangle. The entire sum would be 2^n, so the sum in the above equation would be 2^{n-2}, giving us

$$A_n\frac{2^n\pi}{16}-\frac{4}{n^2}2^{n-2}=\frac{4(n-1)}{n}A_{n-2}\frac{2^{n-2}\pi}{16}.$$

After combining the 4 and the 2^{n-2} on the left and on the right, we have

$$A_n\frac{2^n\pi}{16}-\frac{1}{n^2}2^{n}=\frac{(n-1)}{n}A_{n-2}\frac{2^{n}\pi}{16}.$$

If we now divide everything by $2^n\pi/16$ and re-arrange the terms, we obtain

$$A_n=\frac{n-1}{n}A_{n-2}+\frac{16}{n^2\pi},$$

as desired (for $n \equiv 0 \pmod 4$).

Having covered all the cases for n, this completes the proof. □

4 Connections to the OEIS

As we mentioned earlier, the numbers that appear in Theorem 1 are related to the sequence A296726, and likewise those in Theorem 2 appear in sequence A372324. Here is the connection.

Theorem 4. *For A_n as defined above in Eq. (1), then*

$$\text{for } n \text{ odd,} \qquad A_n = \frac{8}{\pi} \cdot \frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}, \tag{8}$$

$$\text{and for } n \text{ even,} \qquad A_n = \frac{16}{\pi} \cdot \frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2}. \tag{9}$$

Furthermore, the numbers

$$n! \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \qquad \text{for } n \text{ odd,} \tag{10}$$

from Eq. (8) above, appear as every other entry in A296726, the terms from the exponential generating function for $\arcsin x/(1-x)$. *Likewise, the numbers*

$$n! \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2} \qquad \text{for } n \text{ even,} \tag{11}$$

from Eq. (9) above, appear as every other entry in A372324, the terms from exponential generating function for $\arcsin^2 x/(2(1-x))$.

We recall that the notation $n!$ refers to the usual factorial function, and the notation $n!!$ is the less-familiar *double factorial* function [3] although, to be honest, $n!!$ should be called an "every other factorial" instead. Here is the definition:

$$(2j)!! = (2j)(2j-2)(2j-4)\cdots 6 \cdot 4 \cdot 2,$$
$$(2j+1)!! = (2j+1)(2j-1)(2j-3)\cdots 5 \cdot 3 \cdot 1.$$

We also agree that $0!! = (-1)!! = 1$. The sequence of double factorials is A006882.

Proof of Theorem 4. We will need to consider the parity of n.

- First, suppose n is odd. We define A'_n to be the right-hand side of Eq. (8), so that

$$A'_n = \frac{8}{\pi} \cdot \frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}. \tag{12}$$

We know from Theorem 3 that for n odd, we have

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{8}{n^2 \pi}.$$

We now seek to prove that

$$A'_n = \frac{n-1}{n} A'_{n-2} + \frac{8}{n^2 \pi} \tag{13}$$

This, along with the fact that $A_1 = A'_1 = 8/\pi$, is all we will need to conclude that $A_n = A'_n$ for all n, thus proving the validity of Eq. (8).

Now, going back to Eq. (12), we replace n with $n-2$ to give us

$$A'_{n-2} = \frac{8}{\pi} \cdot \frac{(n-2)!}{((n-2)!!)^2} \cdot \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}.$$

Next, we multiply both sides by $(n-1)/n$ and then add $8/(n^2\pi)$ to give us

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{n^2\pi} = \frac{8}{\pi} \left(\frac{1}{n^2} + \frac{n-1}{n} \cdot \frac{(n-2)!}{((n-2)!!)^2} \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right).$$

Since

$$\frac{n-1}{n} \cdot \frac{(n-2)!}{((n-2)!!)^2} = \frac{n(n-1)}{n^2} \cdot \frac{(n-2)!}{((n-2)!!)^2} = \frac{n!}{(n!!)^2}, \tag{14}$$

we can re-write the previous equation as

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{n^2\pi} = \frac{8}{\pi} \left(\frac{1}{n^2} + \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right).$$

Now, it is easy to verify that

$$\frac{1}{n^2} = \frac{n!}{(n!!)^2} \cdot \frac{(n-2)!!}{(n-1)!!} \cdot \frac{1}{n} \tag{15}$$

and if we substitute this into the previous equation then we get

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{n^2\pi} = \frac{8}{\pi} \left(\frac{n!}{(n!!)^2} \cdot \frac{(n-2)!!}{(n-1)!!} \cdot \frac{1}{n} + \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right),$$

and we add that first term on the right into the sum (as the $j = (n-1)/2$ term) to give us

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{n^2\pi} = \frac{8}{\pi} \left(\frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right).$$

Since the right-hand side of the above equation is the right-hand side of Eq. (12), we can re-write the above equation as

$$\frac{n-1}{n}A'_{n-2} + \frac{8}{n^2\pi} = A'_n,$$

thus establishing the validity of Eq. (13), as desired.

Hence, since both A_n and A'_n satisfy the same recursion from Eq. (13), and since they also start at the same value of $A_1 = A'_1 = 8/\pi$, then $A_n = A'_n$ for all odd values of n. This gives us the desired equality in Eq. (8) in the statement of our theorem.

Next, we will show that the numbers

$$n! \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}$$

from Eq. (10) really are the same as every other entry in A296726, which is the list of coefficients for the exponential generating function for $\arcsin x/(1-x)$. To show this, we begin with the series for $1/(1-x)$ which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots, \tag{16}$$

and for $\arcsin x$ which is

$$\arcsin x = x + \frac{1}{3!}x^3 + \frac{9}{5!}x^5 + \frac{225}{7!}x^7 + \frac{11025}{9!}x^9 + \cdots,$$

thanks to A177145. And furthermore, thanks to A001818, we can re-write those numerators in the above equation as follows:

$$\arcsin x = \frac{((-1)!!)^2}{1!}x + \frac{(1!!)^2}{3!}x^3 + \frac{(3!!)^2}{5!}x^5 + \frac{(5!!)^2}{7!}x^7 + \frac{(7!!)^2}{9!}x^9 + \cdots.$$

Hence, since the generating function for $\arcsin x/(1-x)$ will be the convolution of the generating functions for $\arcsin x$ and $1/(1-x)$, then the nth term in the *exponential* generating function for $\arcsin x/(1-x)$, for n odd, will be

$$n!\left(\frac{((-1)!!)^2}{1!} + \frac{(1!!)^2}{3!} + \frac{(3!!)^2}{5!} + \cdots + \frac{((n-2)!!)^2}{n!}\right),$$

which we can write as

$$n! \sum_{j=0}^{(n-1)/2} \frac{((2j-1)!!)^2}{(2j+1)!}.$$

Now, since $(2j+1)! = (2j-1)!!(2j)!!(2j+1)$, then the above expression becomes

$$n! \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1},$$

as seen in Eq. (10).

- Next, suppose n is even. We now define A''_n to be the right-hand side of Eq. (9), so that

$$A''_n = \frac{16}{\pi} \cdot \frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2}. \tag{17}$$

We know from Theorem 3 that for n even, we have

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{16}{n^2\pi}.$$

We now seek to prove that

$$A''_n = \frac{n-1}{n} A''_{n-2} + \frac{16}{n^2\pi} \tag{18}$$

This, along with the fact that $A_2 = A''_2 = 4/\pi$, is all we will need to conclude that $A_n = A''_n$ for all n, thus proving the validity of Eq. (9).

Now, going back to Eq. (17), we replace n with $n-2$ to give us

$$A''_{n-2} = \frac{16}{\pi} \cdot \frac{(n-2)!}{((n-2)!!)^2} \cdot \sum_{j=0}^{(n-4)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2}.$$

Next, we multiply both sides by $(n-1)/n$ and then add $16/(n^2\pi)$ to give us

$$\frac{n-1}{n} A''_{n-2} + \frac{16}{n^2\pi} = \frac{16}{\pi} \left(\frac{1}{n^2} + \frac{n-1}{n} \cdot \frac{(n-2)!}{((n-2)!!)^2} \cdot \sum_{j=0}^{(n-4)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2} \right).$$

Thanks to Eq. (14), the right-hand side simplifies to give us

$$\frac{n-1}{n} A''_{n-2} + \frac{16}{n^2\pi} = \frac{16}{\pi} \left(\frac{1}{n^2} + \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-4)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2} \right).$$

We now use Eq. (15) to replace the $1/n^2$ inside the parenthesis on the right to get

$$\frac{n-1}{n} A''_{n-2} + \frac{16}{n^2\pi} = \frac{16}{\pi} \left(\frac{n!}{(n!!)^2} \cdot \frac{(n-2)!!}{(n-1)!!} \cdot \frac{1}{n} + \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-4)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2} \right),$$

and we add that first term on the right into the sum (as the $j = (n-2)/2$ term) to give us

$$\frac{n-1}{n} A''_{n-2} + \frac{16}{n^2\pi} = \frac{16}{\pi} \left(\frac{n!}{(n!!)^2} \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2} \right).$$

Since the right-hand side of the above equation is the right-hand side of Eq. (17), we can re-write the above equation as

$$\frac{n-1}{n}A''_{n-2} + \frac{16}{n^2\pi} = A''_n,$$

thus establishing the validity of Eq. (18), as desired.

Hence, since both A_n and A''_n satisfy the same recursion from Eq. (18), and since they also start at the same value of $A_2 = A''_2 = 4/\pi$, then $A_n = A''_n$ for all even values of n. This gives us the desired equality (9) in the statement of our theorem.

Finally, we will show that the numbers

$$n! \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!}\frac{1}{2j+2}$$

from Eq. (11) really are the same as every other entry in A372324, which is the list of coefficients for the exponential generating function for $\arcsin^2 x/(2(1-x))$. To show this, we begin with the series

$$\frac{\arcsin x}{\sqrt{1-x^2}} = x + \frac{4}{3!}x^3 + \frac{64}{5!}x^5 + \frac{2304}{7!}x^7 + \frac{147456}{9!}x^9 + \cdots$$

as seen in A002454, which also tells us that we can re-write the numerators in the above equation as follows:

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \frac{(0!!)^2}{1!}x^1 + \frac{(2!!)^2}{3!}x^3 + \frac{(4!!)^2}{5!}x^5 + \frac{(6!!)^2}{7!}x^7 + \frac{(8!!)^2}{9!}x^9 + \cdots .$$

We now integrate both sides to get

$$\frac{1}{2}\arcsin^2 x = \frac{(0!!)^2}{2!}x^2 + \frac{(2!!)^2}{4!}x^4 + \frac{(4!!)^2}{6!}x^6 + \frac{(6!!)^2}{8!}x^8 + \frac{(8!!)^2}{10!}x^{10} + \cdots .$$

Hence, since the generating function for $\arcsin^2 x/(2(1-x))$ will be the convolution of Eq. (16) and the above equation, then the nth term in the *exponential* generating function for $\arcsin^2 x/(2(1-x))$, for n even, will be

$$n!\left(\frac{((0)!!)^2}{2!} + \frac{(2!!)^2}{4!} + \frac{(4!!)^2}{6!} + \cdots + \frac{((n-2)!!)^2}{n!}\right),$$

which we can write as

$$n! \sum_{j=0}^{(n-2)/2} \frac{((2j)!!)^2}{(2j+2)!}.$$

Now, since $(2j+2)! = (2j)!!(2j+1)!!(2j+2)$, then the above expression becomes

$$n! \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2},$$

as seen in Eq. (11).

□

We note that our proof of Theorem 4 established that the numbers in Eq. (10), valid only for n odd, appear as every other number in A296726 which has exponential generating function $\arcsin x/(1-x)$. To be precise, the numbers in Eq. (10), for n odd, are

$$1, 7, 149, 6483, 477801, \ldots,$$

and the numbers in A296726 coming from $\arcsin x/(1-x)$, for all n starting at $n = 0$, are

$$0, 1, 2, 7, 28, 149, 894, 6483, 51864, 477801, \ldots.$$

It is easy to show that the related function $\arcsin x/(1-x^2)$ has a exponential generating function with coefficients

$$0, 1, 0, 7, 0, 149, 0, 6483, 0, 477801, \ldots,$$

and the non-zero numbers in this sequence are exactly the numbers produced by Eq. (10).

We can say the same about the numbers in Eq. (11), which (as we showed in our proof) appear as every other number in A372324 which has exponential generating function $\arcsin^2 x/2(1-x)$. To be precise, the numbers in Eq. (11), for n even, are

$$0, 1, 16, 544, 32768, \ldots, \tag{19}$$

and the numbers in A372324 are from $\arcsin^2 x/2(1-x)$ and are

$$0, 0, 1, 3, 16, 80, 544, 3808, 32768, \ldots.$$

If we change the exponential generating function from $\arcsin^2 x/2(1-x)$ to $\arcsin^2 x/2(1-x^2)$, we obtain the numbers

$$0, 0, 1, 0, 16, 0, 544, 0, 32768, \ldots,$$

and this is a more pleasant representation of the numbers in Eq. (19) from Eq. (11).

5 Technical results

Before we can begin the proofs of Theorems 1 and 2, we will need some preliminary results.

5.1 Three applications of Lagrange's identity

Lemma 5. *Let N and q be positive integers. Then,*

$$\text{for } N \text{ even}, \qquad \sum_{\ell=1}^{N/2} \sin \frac{q\ell 2\pi}{N} = \begin{cases} 0, & \text{if } q \text{ even;} \\ \cot \dfrac{q\pi}{N}, & \text{if } q \text{ odd.} \end{cases}$$

Proof. We call upon Lagrange's trigonometric identity [1], which states that

$$\sum_{\ell=0}^{m} \sin \ell\theta = \frac{\cos \dfrac{\theta}{2} - \cos \left(m + \dfrac{1}{2} \right) \theta}{2 \sin \dfrac{\theta}{2}}. \tag{20}$$

Since we are assuming that N is even, we replace m with $N/2$ and we replace θ with $q2\pi/N$ in Eq. (20) to get

$$\sum_{\ell=0}^{N/2} \sin \frac{q\ell 2\pi}{N} = \frac{\cos \dfrac{q\pi}{N} - \cos \dfrac{(N+1)q\pi}{N}}{2 \sin \dfrac{q\pi}{N}}. \tag{21}$$

Now, $\cos \frac{(N+1)q\pi}{N}$ can be written as $\cos\left(q\pi + \frac{q\pi}{N}\right)$, and for q even then $q\pi$ is an even multiple of π and so $\cos\left(q\pi + \frac{q\pi}{N}\right)$ equals $\cos \frac{q\pi}{N}$. However, for q odd then $q\pi$ is an odd multiple of π and so $\cos\left(q\pi + \frac{q\pi}{N}\right)$ equals $-\cos \frac{q\pi}{N}$. When we plug these simplifications into the numerator of Eq. (21), we get either 0 or $2\cos \frac{q\pi}{N}$ in the numerator depending on whether q is even or odd, respectively, and this gives us our desired formula. □

Lemma 6. *Let N and q be positive integers. Then,*

$$\text{for } N \text{ odd}, \qquad \sum_{\ell=1}^{(N-1)/2} \sin \frac{q\ell 2\pi}{N} = \begin{cases} -\dfrac{1}{2} \tan \dfrac{q\pi}{2N}, & \text{if } q \text{ even;} \\ \dfrac{1}{2} \cot \dfrac{q\pi}{2N}, & \text{if } q \text{ odd.} \end{cases}$$

Proof. We call upon Lagrange's trigonometric identity (20), replacing m with $(N-1)/2$ and θ with $q2\pi/N$ to give us

$$\sum_{\ell=0}^{(N-1)/2} \sin \frac{q\ell 2\pi}{N} = \frac{\cos \dfrac{q\pi}{N} - \cos \dfrac{(N)q\pi}{N}}{2 \sin \dfrac{q\pi}{N}}. \tag{22}$$

Now, the last term in the above equation is simply $\cos q\pi$, which is either 1 or -1 depending on the parity of q. So, the right-hand side of Eq. (22) is either $(\cos q\pi/N - 1)/(2 \sin q\pi/N)$ or

$(\cos q\pi/N+1)/(2\sin q\pi/N)$ depending on whether q is even or odd, respectively. And, since $(\cos x-1)/(2\sin x)=(-1/2)\tan x/2$ and $(\cos x+1)/(2\sin x)=(1/2)\cot x/2$ by standard trig identities, we have our desired formula. $\square$

Lemma 7. *Let N and q be positive integers. Then,*

$$\text{for } N, q \text{ even,} \qquad \sum_{\ell=1}^{N/2}\sin\frac{q\ell\pi}{N} = \begin{cases} 0, & \text{if } q\equiv 0 \pmod 4; \\ \cot\dfrac{q\pi}{2N}, & \text{if } q\equiv 2 \pmod 4. \end{cases}$$

Proof. We call once more upon Lagrange's trigonometric identity (20). Since N is even, we will replace m with $N/2$ and we replace θ with $q\pi/N$ in Eq. (20) to get

$$\sum_{\ell=0}^{N/2}\sin\frac{q\ell\pi}{N} = \frac{\cos\dfrac{q\pi}{2N}-\cos\dfrac{(N+1)q\pi}{2N}}{2\sin\dfrac{q\pi}{2N}}. \tag{23}$$

Now, $\cos\frac{(N+1)q\pi}{2N}$ can be written as $\cos\left(\frac{q\pi}{2}+\frac{q\pi}{2N}\right)$, and for $q\equiv 0 \pmod 4$ then $\frac{q\pi}{2}$ is an even multiple of π and so $\cos\left(\frac{q\pi}{2}+\frac{q\pi}{2N}\right)$ simplifies to $\cos\frac{q\pi}{2N}$. However, for $q\equiv 2 \pmod 4$ then $\frac{q\pi}{2}$ is an odd multiple of π is odd and so $\cos\left(\frac{q\pi}{2}+\frac{q\pi}{2N}\right)$ simplifies to $-\cos\frac{q\pi}{2N}$ When we plug these simplifications into the right-hand side of Eq. (23), we get either 0 or $2\cos\frac{q\pi}{2N}$ in the numerator depending on whether q is equivalent to 0 or 2 (mod 4), respectively, and this gives us our desired formula. $\square$

5.2 A useful limit of cotangents

Lemma 8. *For x any real number,*

$$\lim_{k\to\infty}\left(\frac{1}{k}-1\right)\cot\frac{x}{k-1}+\left(\frac{1}{k}+1\right)\cot\frac{x}{k+1} \;=\; \frac{4}{x}.$$

Proof. We begin with the Taylor expansion for the cotangent, which gives us

$$\cot\theta \;=\; \frac{1}{\theta}-\frac{\theta}{3}-\frac{\theta^3}{45}+\cdots \;=\; \frac{1}{\theta}+\mathcal{O}(\theta).$$

If we apply this to our limit, we get

$$\left(\frac{1}{k}-1\right)\left(\frac{k-1}{x}+\mathcal{O}\left(\frac{x}{k-1}\right)\right) \;+\; \left(\frac{1}{k}+1\right)\left(\frac{k+1}{x}+\mathcal{O}\left(\frac{x}{k+1}\right)\right).$$

Since x is fixed, we can remove it from inside the $\mathcal{O}$. After expanding the above expression, we get

$$\left(\frac{1-k}{k}\right)\left(\frac{k-1}{x}\right)+\left(\frac{1-k}{k}\right)\cdot\mathcal{O}\left(\frac{1}{k-1}\right) \;+\; \left(\frac{1+k}{k}\right)\left(\frac{k+1}{x}\right)+\left(\frac{1+k}{k}\right)\cdot\mathcal{O}\left(\frac{1}{k+1}\right).$$

This simplifies nicely to

$$\left(\frac{(1-k)(k-1)+(1+k)(k+1)}{kx}\right)+\mathcal{O}\left(\frac{1}{k}\right).$$

We reduce this to get

$$\left(\frac{(1+k)^2-(1-k)^2}{kx}\right)+\mathcal{O}\left(\frac{1}{k}\right) \;=\; \left(\frac{4k}{kx}\right)+\mathcal{O}\left(\frac{1}{k}\right) \;=\; \frac{4}{x}+\mathcal{O}\left(\frac{1}{k}\right),$$

which, as $k\to\infty$, gives us our desired $4/x$. □

5.3 Four trig integrals

Lemma 9. *Let k and q be odd numbers. Then, if we define*

$$C_q = \lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)}\Big(\cos qkx-\cos qx\Big)\,dx, \tag{24}$$

we have that

$$C_q=\frac{4}{q^2\pi}.$$

Proof. First, we integrate the right-hand side of Eq. (24) to get

$$C_q = \lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\frac{1}{kq}\sin qkx-\frac{1}{q}\sin qx\,\Bigg|_{x=\ell 2\pi/(k+1)}^{x=\ell 2\pi/(k-1)}.$$

Taking out the $1/q$ and plugging in the endpoints, we get

$$C_q=\frac{1}{q}\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\left(\frac{1}{k}\sin\frac{qk\ell 2\pi}{k-1}-\sin\frac{q\ell 2\pi}{k-1}\right)-\left(\frac{1}{k}\sin\frac{qk\ell 2\pi}{k+1}-\sin\frac{q\ell 2\pi}{k+1}\right) \tag{25}$$

Now, if we write

$$\frac{qk\ell 2\pi}{k-1}=\frac{q(k-1+1)\ell 2\pi}{k-1}=q\ell 2\pi+\frac{q\ell 2\pi}{k-1}$$

then we see that

$$\sin\frac{qk\ell 2\pi}{k-1}=\sin\frac{q\ell 2\pi}{k-1}. \tag{26}$$

Likewise, if we write

$$\frac{qk\ell 2\pi}{k+1}=\frac{q(k+1-1)\ell 2\pi}{k-1}=q\ell 2\pi-\frac{q\ell 2\pi}{k-1}$$

then we see that

$$\sin \frac{qk\ell 2\pi}{k+1} = -\sin \frac{q\ell 2\pi}{k-1}. \tag{27}$$

By substituting Eqs. (26) and (27) into the right-hand side of Eq. (25), we have that

$$C_q = \frac{1}{q} \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} \left(\frac{1}{k} - 1\right) \sin \frac{q\ell 2\pi}{k-1} + \left(\frac{1}{k} + 1\right) \sin \frac{q\ell 2\pi}{k+1}.$$

We now distribute the sum, and change the upper limit of the second summation from $(k-1)/2$ to $(k+1)/2$, which fortunately does not change the value of the sum, to get

$$C_q = \frac{1}{q} \lim_{k\to\infty} \left(\frac{1}{k} - 1\right) \sum_{\ell=1}^{(k-1)/2} \sin \frac{q\ell 2\pi}{k-1} + \left(\frac{1}{k} + 1\right) \sum_{\ell=1}^{(k+1)/2} \sin \frac{q\ell 2\pi}{k+1}.$$

At this point, since k is odd, then both $k-1$ and $k+1$ are even and so we can apply Lemma 5 (with q odd) to rewrite the above equation as

$$C_q = \frac{1}{q} \lim_{k\to\infty} \left(\frac{1}{k} - 1\right) \cot \frac{q\pi}{k-1} + \left(\frac{1}{k} + 1\right) \cot \frac{q\pi}{k+1}.$$

We can now apply Lemma 8 with $x = q\pi$ to the above equation to get that

$$C_q = \frac{1}{q}\frac{4}{q\pi} = \frac{4}{q^2\pi},$$

as desired. □

Lemma 10. *For k odd and q even, if we define*

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} \int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} f_{q,k}(x)\, dx - \int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} f_{q,k}(x)\, dx \tag{28}$$

with

$$f_{q,k}(x) = \cos qx - \cos qkx,$$

then we have that

$$C_q = \begin{cases} 0, & \text{if } q \equiv 0 \pmod 4; \\ \dfrac{16}{q^2\pi}, & \text{if } q \equiv 2 \pmod 4. \end{cases}$$

Proof. If we let $F_{q,k}(x)$ be the anti-derivative of $f_{q,k}(x) = \cos qx - \cos qkx$, then Eq. (28) becomes

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} F_{q,k}(x)\Big|_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} + F_{q,k}(x)\Big|_{\ell\pi/(k-1)}^{\ell\pi/(k+1)} \tag{29}$$

where we replaced $-F_{q,k}$ with $F_{q,k}$ and reversed the limits in the second integral. We note that almost every term in the above expression for C_q will appear twice when we plug in the endpoints and write out the sum, with the exception of $F_{q,k}(0)$ and $F_{q,k}(\pi/2)$ which will each appear once. However, since an easy calculation gives us that

$$F_{q,k}(x) = \frac{1}{q}\sin qx - \frac{1}{qk}\sin qkx, \tag{30}$$

then $F_{q,k}(0) = 0$ and since q is even then $F_{q,k}(\pi/2) = 0$ as well.

So, if we plug in the endpoints, write out the sum, and replace the $F_{q,k}(0)$ term with $F_{q,k}(\pi/2)$, then Eq. (29) becomes

$$C_q = 2\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2} F_{q,k}\left(\frac{\ell\pi}{k+1}\right) - F_{q,k}\left(\frac{\ell\pi}{k-1}\right).$$

Replacing $F_{q,k}$ with the expression in Eq. (30) and taking out the $1/q$ gives us

$$C_q = \frac{2}{q}\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\left(\sin\frac{q\ell\pi}{k+1} - \frac{1}{k}\sin\frac{qk\ell\pi}{k+1}\right) - \left(\sin\frac{q\ell\pi}{k-1} - \frac{1}{k}\sin\frac{qk\ell\pi}{k-1}\right). \tag{31}$$

Now, if we write

$$\frac{qk\ell\pi}{k+1} = \frac{q(k+1-1)\ell\pi}{k+1} = q\ell\pi - \frac{q\ell\pi}{j+1} \tag{32}$$

and if we remember that q is even, then we see that

$$\sin\frac{qk\ell\pi}{k+1} = -\sin\frac{q\ell\pi}{k+1}. \tag{33}$$

Likewise, if we write

$$\frac{qk\ell\pi}{k-1} = \frac{q(k-1+1)\ell\pi}{k-1} = q\ell\pi + \frac{q\ell\pi}{k-1} \tag{34}$$

and again recall that q is even, then we see that

$$\sin\frac{qk\ell\pi}{k-1} = \sin\frac{q\ell\pi}{k-1}. \tag{35}$$

By substituting Eqs. (33) and (35) into the right-hand side of Eq. (31), we have that

$$C_q = \frac{2}{q}\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\left(1+\frac{1}{k}\right)\sin\frac{q\ell\pi}{k+1} - \left(1-\frac{1}{k}\right)\sin\frac{q\ell\pi}{k-1}.$$

We now distribute the sum, factor through the negative in the second expression, and change the upper limit of the first summation from $(k-1)/2$ to $(k+1)/2$, which fortunately does not change the value of the sum, to get

$$C_q = \frac{2}{q}\lim_{k\to\infty}\left(\frac{1}{k}+1\right)\sum_{\ell=1}^{(k+1)/2}\sin\frac{q\ell\pi}{k+1} + \left(\frac{1}{k}-1\right)\sum_{\ell=1}^{(k-1)/2}\sin\frac{q\ell\pi}{k-1}.$$

At this point, since k is odd, then both $k+1$ and $k-1$ are even and so we can apply Lemma 7 (with q even). If $q/2$ is even, then Lemma 7 tells us that both the above sums are zero and so $C_q = 0$ in this case. If $q/2$ is odd, we apply Lemma 7 to tell us that

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\frac{1}{k} + 1\right) \cot \frac{q\pi}{2(k+1)} + \left(\frac{1}{k} - 1\right) \cot \frac{q\pi}{2(k-1)} \qquad \text{for } q/2 \text{ odd.}$$

We can now apply Lemma 8 with $x = q\pi/2$ to the above equation to get that

$$C_q = \frac{2}{q}\frac{4}{q\pi/2} = \frac{16}{q^2\pi} \qquad \text{for } q/2 \text{ odd,}$$

as desired. □

Lemma 11. *For k even and q odd, if we define*

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{k/2} \int_{(\ell-1)2\pi/(k-1)}^{\ell 2\pi/(k+1)} f_{q,k}(x)\, dx - \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} f_{q,k}(x)\, dx \tag{36}$$

such that

$$f_{q,k}(x) = \cos qx - \cos qkx,$$

then we have

$$C_q = \frac{8}{q^2\pi}.$$

Proof. If we let $F_{q,k}(x)$ be the anti-derivative of $f_{q,k}(x) = \cos qx - \cos qkx$, then Eq. (36) becomes

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{k/2} F_{q,k}(x)\Big|_{(\ell-1)2\pi/(k-1)}^{\ell 2\pi/(k+1)} + F_{q,k}(x)\Big|_{\ell 2\pi/(k-1)}^{\ell 2\pi/(k+1)} \tag{37}$$

where we replaced $-F_{q,k}$ with $F_{q,k}$ and reversed the limits in the second integral. We note that almost every term in the above expression for C_q will appear twice when we plug in the endpoints and write out the sum, with the exception of $F_{q,k}(0)$ and $F_{q,k}(k\pi/(k-1))$ which will each appear once. However, from Eq. (30) we note that $F_{q,k}(0) = 0$, and as for $F_{q,k}(k\pi/(k-1))$, we note that we can use the same technique as in equations (34) and (35), this time with k in place of ℓ, to give us

$$F_{q,k}(k\pi/(k-1)) = \frac{1}{q}\sin\frac{qk\pi}{k-1} - \frac{1}{qk}\sin\frac{qk\pi}{k-1}.$$

Since k goes to infinity, we see that $F_{q,k}(k\pi/(k-1))$ will vanish.

So, returning to Eq. (37), if we plug in the endpoints and write out the sum, eliminating the term $F_{q,k}(0)$ and doubling the term $F_{q,k}(k\pi/(k-1))$ (this is legitimate since both terms will vanish as $k\to\infty$), we have

$$C_q = 2\lim_{k\to\infty} \sum_{\ell=1}^{k/2} F_{q,k}\left(\frac{\ell 2\pi}{k+1}\right) - F_{q,k}\left(\frac{\ell 2\pi}{k-1}\right).$$

Replacing $F_{q,k}$ with the expression in Eq. (30) and taking out the $1/q$ gives us

$$C_q = \frac{2}{q} \lim_{k\to\infty} \sum_{\ell=1}^{k/2} \left(\sin\frac{q\ell 2\pi}{k+1} - \frac{1}{k}\sin\frac{qk\ell 2\pi}{k+1} \right) - \left(\sin\frac{q\ell 2\pi}{k-1} - \frac{1}{k}\sin\frac{qk\ell 2\pi}{k-1} \right). \tag{38}$$

Using the same techniques as seen in Eqs. (32), (33), (34), and (35), we can write

$$\sin\frac{qk\ell 2\pi}{k+1} = -\sin\frac{q\ell 2\pi}{k+1} \qquad \text{and} \qquad \sin\frac{qk\ell 2\pi}{k-1} = \sin\frac{q\ell 2\pi}{k-1}.$$

By substituting these two equations into the right-hand side of Eq. (38), we have that

$$C_q = \frac{2}{q} \lim_{k\to\infty} \sum_{\ell=1}^{k/2} \left(1+\frac{1}{k}\right)\sin\frac{q\ell 2\pi}{k+1} - \left(1-\frac{1}{k}\right)\sin\frac{q\ell 2\pi}{k-1}.$$

We now distribute the sum, and we factor through the negative in the second expression, to get

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \sum_{\ell=1}^{k/2} \sin\frac{q\ell 2\pi}{k+1} + \left(\frac{1}{k}-1\right) \sum_{\ell=1}^{k/2} \sin\frac{q\ell 2\pi}{k-1} \right).$$

We can change the upper limit of the last sum from $k/2$ to $(k-2)/2$, because when $\ell = k/2$ that last term becomes $\sin q\pi k/(k-1)$ which approaches $\sin q\pi = 0$ as $k \to \infty$. This gives us

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \sum_{\ell=1}^{k/2} \sin\frac{q\ell 2\pi}{k+1} + \left(\frac{1}{k}-1\right) \sum_{\ell=1}^{(k-2)/2} \sin\frac{q\ell 2\pi}{k-1} \right).$$

At this point, since k is even, then both $k+1$ and $k-1$ are odd and so we can apply Lemma 6 to both of the above sums (the first with $N = k+1$ and the second with $N = k-1$), to give us

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \frac{1}{2}\cot\frac{q\pi}{2(k+1)} + \left(\frac{1}{k}-1\right) \frac{1}{2}\cot\frac{q\pi}{2(k-1)} \right).$$

We can now factor out $1/2$ and apply Lemma 8 with $x = q\pi/2$ to the above equation to get that

$$C_q = \frac{1}{q}\left(\frac{4}{q\pi/2}\right) = \frac{8}{q^2\pi},$$

as desired. □

Lemma 12. *For k and q both even, if we define*

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{k/2} \int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} f_{q,k}(x)\,dx - \int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} f_{q,k}(x)\,dx \tag{39}$$

with

$$f_{q,k}(x) = \cos qx - \cos qkx,$$

then we have that

$$C_q = \begin{cases} 0, & \text{if } q \equiv 0 \pmod 4; \\ \dfrac{16}{q^2\pi}, & \text{if } q \equiv 2 \pmod 4. \end{cases}$$

Proof. We proceed as in the proof of Lemma 10. If we (again) let $F_{q,k}(x)$ be the anti-derivative of $f_{q,k}(x) = \cos qx - \cos qkx$, then Eq. (39) becomes

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{k/2} F_{q,k}(x)\Big|_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} + F_{q,k}(x)\Big|_{\ell\pi/(k-1)}^{\ell\pi/(k+1)} \tag{40}$$

where we replaced $-F_{q,k}$ with $F_{q,k}$ and reversed the limits in the second integral.

We note that almost every term in the above expression for C_q will appear twice when we plug in the endpoints and write out the sum, with the exception of $F_{q,k}(0)$ and $F_{q,k}(k\pi/(2(k-1)))$ which will each appear once. From Eq. (30) we have that

$$F_{q,k}(x) = \frac{1}{q}\sin qx - \frac{1}{qk}\sin qkx,$$

which tells us that $F_{q,k}(0) = 0$. As for $F_{q,k}(k\pi/(2(k-1)))$, we have that

$$F_{q,k}(k\pi/(2(k-1))) = \frac{1}{q}\sin\frac{qk\pi}{2(k-1)} - \frac{1}{qk}\sin\frac{qk^2\pi}{2(k-1)}.$$

As $k \to \infty$, the first term approaches $(1/q)\sin(q\pi/2)$ which is zero since q is even, and the second term approaches zero thanks to the $1/(qk)$ in front.

So, if we plug in the endpoints in Eq. (40), write out the sum, and eliminate the terms $F_{q,k}(0)$ and $F_{q,k}(k\pi/(2(k-1)))$, then we have

$$C_q = 2\lim_{k\to\infty} \sum_{\ell=1}^{k/2} F_{q,k}\left(\frac{\ell\pi}{k+1}\right) - F_{q,k}\left(\frac{\ell\pi}{k-1}\right).$$

If we replace $F_{q,k}$ with the expression in Eq. (30) and take out the $1/q$, we get

$$C_q = \frac{2}{q}\lim_{k\to\infty} \sum_{\ell=1}^{k/2} \left(\sin\frac{q\ell\pi}{k+1} - \frac{1}{k}\sin\frac{qk\ell\pi}{k+1}\right) - \left(\sin\frac{q\ell\pi}{k-1} - \frac{1}{k}\sin\frac{qk\ell\pi}{k-1}\right).$$

Since q is even, we can substitute equations (33) and (35) into the above expression to give us

$$C_q = \frac{2}{q}\lim_{k\to\infty} \sum_{\ell=1}^{k/2} \left(1 + \frac{1}{k}\right)\sin\frac{q\ell\pi}{k+1} - \left(1 - \frac{1}{k}\right)\sin\frac{q\ell\pi}{k-1}.$$

We now distribute the sum, and we factor through the negative in the second expression, to give us

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \sum_{\ell=1}^{k/2} \sin\frac{q\ell\pi}{k+1} + \left(\frac{1}{k}-1\right) \sum_{\ell=1}^{k/2} \sin\frac{q\ell\pi}{k-1} \right). \tag{41}$$

Let us now consider the last term in the last sum of the above equation, namely,

$$\left(\frac{1}{k}-1\right) \sin\frac{q(k/2)\pi}{k-1}.$$

As $k \to \infty$, this term approaches

$$\Big(-1\Big) \sin\frac{q\pi}{2},$$

and since q is even then this equals zero. Hence, we can safely eliminate the $\ell = k/2$ term from the second sum. Furthermore, using once again that q is even, we can write the above Eq. (41) as

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \sum_{\ell=1}^{k/2} \sin\frac{(q/2)\ell 2\pi}{k+1} + \left(\frac{1}{k}-1\right) \sum_{\ell=1}^{(k-2)/2} \sin\frac{(q/2)\ell 2\pi}{k-1} \right).$$

We now wish to call upon Lemma 6 for the two sums above (the first with $N = k+1$ and the second with $N = k-1$), but we will have to consider the two cases for $q/2$ even and $q/2$ odd. If $q/2$ is even, then Lemma 6 tells us that the above expression is

$$\frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \left(\frac{-1}{2} \tan\frac{(q/2)\pi}{2(k+1)}\right) + \left(\frac{1}{k}-1\right) \left(\frac{-1}{2} \tan\frac{(q/2)\pi}{2(k-1)}\right) \right),$$

and as $k \to \infty$ this is clearly zero. If $q/2$ is odd, then from Lemma 6 we have

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\left(\frac{1}{k}+1\right) \left(\frac{1}{2} \cot\frac{(q/2)\pi}{2(k+1)}\right) + \left(\frac{1}{k}-1\right) \left(\frac{1}{2} \cot\frac{(q/2)\pi}{2(k-1)}\right) \right).$$

We now apply Lemma 8 with $x = (q/4)\pi$ to get

$$C_q = \frac{2}{q} \cdot \frac{1}{2} \cdot \frac{4}{(q/4)\pi} = \frac{16}{q^2\pi} \qquad \text{for } q/2 \text{ odd},$$

as desired. □

6 Proof of Theorem 1

Proof of Theorem 1. We recall that in Theorem 1 we are considering the area between $\cos^n x$ and $\cos^n kx$ where the exponent n is an odd number. We also need to consider the parity of the coefficient k.

- First, we suppose k is odd. As seen in Figure 2 with $n = 3$ and $k = 11$, there is odd symmetry across the midpoint $x = \pi/2$ and so each region "below" $\cos^3 x$ (in color) has an equivalent area "above" $\cos^3 x$ (in a matching color).

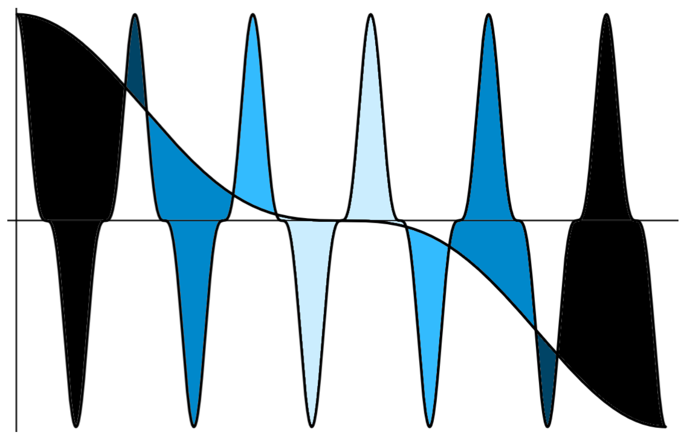

Figure 2: Region between $\cos^3 x$ and $\cos^3 11x$.

In other words, we can just find the areas "above" $\cos^n x$ on the interval $[0, \pi]$ and then double them. To do so, we first need to find the intersection points. Since n is odd, then to find the the solutions to $\cos^n x = \cos^n kx$ we take the nth root of both sides and rewrite it to get $\cos x - \cos kx = 0$, and we then use a trig identity to write that as

$$\sin \frac{(k+1)x}{2} \cdot \sin \frac{(k-1)x}{2} \;=\; 0.$$

This has solutions $x = \ell \cdot 2\pi/(k+1)$ and $x = \ell \cdot 2\pi/(k-1)$ for ℓ any integer, and we note that we can order these as follows:

$$0 < \frac{1 \cdot 2\pi}{k+1} < \frac{1 \cdot 2\pi}{k-1} < \frac{2 \cdot 2\pi}{k+1} < \frac{2 \cdot 2\pi}{k-1} < \cdots$$

$$\cdots < \frac{\ell \cdot 2\pi}{k+1} < \frac{\ell \cdot 2\pi}{k-1} < \frac{(\ell+1) \cdot 2\pi}{k+1} < \frac{(\ell+1) \cdot 2\pi}{k-1} < \cdots$$

$$\cdots < \frac{(k-1)/2 \cdot 2\pi}{k+1} < \frac{(k-1)/2 \cdot 2\pi}{k-1} = \pi,$$

and in particular we have that

$$\frac{\ell \cdot 2\pi}{k-1} < \frac{(\ell+1)\cdot 2\pi}{k+1} \qquad \text{so long as } \ell < (k-1)/2. \tag{42}$$

With these intersection points, we have the following formula for the total area which takes just the "upper" regions and doubles them:

$$2\sum_{\ell=1}^{(k-1)/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \Big(\cos^n kx - \cos^n x\Big)\, dx. \tag{43}$$

We now use the power-reduction formula for cosine to an odd power n,

$$\cos^n \theta = \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \cos(n-2j)\theta, \tag{44}$$

and when we substitute this into Eq. (43), twice, we get the following expression for the area:

$$2\sum_{\ell=1}^{(k-1)/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \Big(\cos(n-2j)kx - \cos(n-2j)x\Big)\, dx. \tag{45}$$

Of course, we want the limit of the expression in (45) as k goes to infinity, so when we do this, and re-arrange the sums and integrals and such, we get

$$A_n = \frac{4}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \Big(\cos(n-2j)kx - \cos(n-2j)x\Big)\, dx. \tag{46}$$

We now recognize the limit in the right-hand side of Eq. (46) as being the same as in Lemma 9, but with the q in that lemma replaced by $n-2j$. In other words, we now have that

$$\begin{aligned} A_n &= \frac{4}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{4}{(n-2j)^2 \pi} \\ &= \frac{8}{\pi} \cdot \frac{1}{2^{n-1}} \cdot \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{1}{(n-2j)^2}, \end{aligned}$$

as desired.

- Next, we suppose k is even. In contrast to the previous case for k odd, we see in Figure 3 with $n=3$ and $k=10$ that when k is even we do *not* have odd symmetry across the midpoint $x=\pi/2$. Furthermore, the last region on the right is actually cut in half.

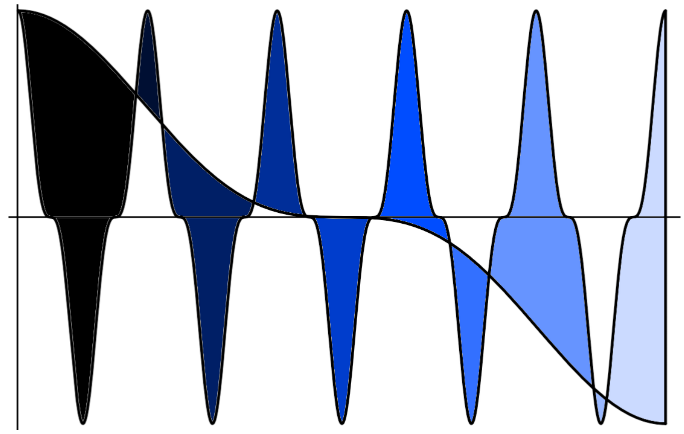

Figure 3: Region between $\cos^3 x$ and $\cos^3 10x$.

Putting all that aside for the moment, we will now find the endpoints for our integrals. They are mostly the same as the intersection points but with a slight change towards the end of the list. We have

$$0 < \frac{1 \cdot 2\pi}{k+1} < \frac{1 \cdot 2\pi}{k-1} < \frac{2 \cdot 2\pi}{k+1} < \frac{2 \cdot 2\pi}{k-1} < \cdots$$
$$\cdots < \frac{\ell \cdot 2\pi}{k+1} < \frac{\ell \cdot 2\pi}{k-1} < \frac{(\ell+1) \cdot 2\pi}{k+1} < \frac{(\ell+1) \cdot 2\pi}{k-1} < \cdots < \frac{k/2 \cdot 2\pi}{k+1} < \pi,$$

and we note that the last endpoint (namely, π) does not fit the pattern. This is because the last region from Figure 3 is cut in half.

None the less, we do still have the inequality from Eq. (42), and so with all this in mind, we now set up our integrals for the total area. If we set $f_{n,k}(x) = \cos^n x - \cos^n kx$, then the total area in Figure 3, moving from left to right, is the integral of $f_{n,k}(x)$ over $[0, 1 \cdot 2\pi/(k+1)]$, followed by the integral of $-f_{n,k}(x)$ over $[1 \cdot 2\pi/(k+1), 1 \cdot 2\pi/(k-1)]$, followed by the integral of $f_{n,k}(x)$ over $[1 \cdot 2\pi/(k-1), 2 \cdot 2\pi/(k+1)]$, and so on. The last region should be from $(k/2)2\pi/(k+1)$ to π, but for convenience we will consider this as being from $(k/2)2\pi/(k+1)$ to $(k/2)2\pi/(k-1)$ and then subtract away that "extra" area from π to $(k/2)2\pi/(k-1)$.

In total, then, we have

$$\sum_{\ell=1}^{k/2} \left(\int_{(\ell-1)2\pi/(k-1)}^{\ell 2\pi/(k+1)} f_{n,k}(x)\, dx + \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} -f_{n,k}(x)\, dx \right) - \int_{\pi}^{k\pi/(k-1)} -f_{n,k}(x)\, dx,$$

but of course we actually want the *limit* of the above expression as k goes to infinity. Since the last term in the above expression is the integral of a bounded function over an interval of width $\pi/(k-1)$, then as k goes to infinity this last term will vanish. So, we have

$$A_n = \lim_{k\to\infty} \sum_{\ell=1}^{k/2} \left(\int_{(\ell-1)2\pi/(k-1)}^{\ell 2\pi/(k+1)} f_{n,k}(x)\,dx + \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} -f_{n,k}(x)\,dx \right). \tag{47}$$

We now use our power-reduction formula from Eq. (44) to re-write $f_{n,k}(x)$ as

$$f_{n,k}(x) = \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \Big(\cos(n-2j)x - \cos(n-2j)kx \Big).$$

When we substitute this into Eq. (47) twice, and distribute the outer sum, we have the following equation for A_n:

$$\begin{aligned} A_n = \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \lim_{k\to\infty} \Bigg(\sum_{\ell=1}^{k/2} \int_{(\ell-1)2\pi/(k-1)}^{\ell 2\pi/(k+1)} \Big(\cos(n-2j)x - \cos(n-2j)kx \Big)\,dx \\ - \sum_{\ell=1}^{k/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \Big(\cos(n-2j)x - \cos(n-2j)kx \Big)\,dx \Bigg). \end{aligned} \tag{48}$$

At this point, we recognize the limit on the right-hand side of Eq. (48) as being the same as in Lemma 11 with k even, but with the odd value of q in that lemma replaced by the odd number $n-2j$. In other words, we now have that

$$\begin{aligned} A_n &= \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{8}{(n-2j)^2\pi} \\ &= \frac{8}{\pi} \cdot \frac{1}{2^{n-1}} \cdot \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{1}{(n-2j)^2}, \end{aligned}$$

as desired.

□

7 Proof of Theorem 2

Proof of Theorem 2. We recall that for this theorem, we are considering the area between $\cos^n$ and $\cos^n kx$ where the exponent n is an even number. The proofs are slightly different, depending on the parity of the coefficient k.

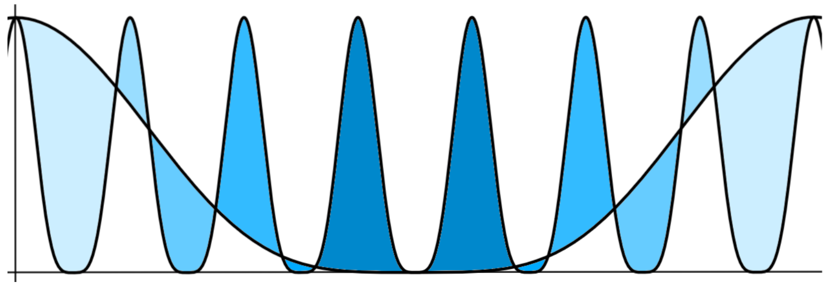

Figure 4: Region between $\cos^4 x$ and $\cos^4 7x$.

- First, suppose k is odd. As seen in Figure 4 with $n = 4$ and $k = 7$, there is even symmetry across the midpoint $x = \pi/2$ and so each region on the left of $x = \pi/2$ (in color) has an equivalent area on the right of $x = \pi/2$ (in a matching color).

  In other words, we can just find the areas from 0 to $\pi/2$ and double them. To do so, we first need to find the intersection points. If we set

$$\cos^n x = \cos^n kx \tag{49}$$

  and take the nth root of both sides, then since n is even we will get

$$\cos x = \pm \cos kx,$$

  which becomes two equations,

$$\cos x - \cos kx = 0 \qquad \text{and} \qquad \cos x + \cos kx = 0.$$

  Using two familiar trig identities, these become

$$\sin \frac{(k+1)x}{2} \sin \frac{(k-1)x}{2} = 0 \qquad \text{and} \qquad \cos \frac{(k+1)x}{2} \cos \frac{(k-1)x}{2} = 0$$

  The first equation has solutions $x = 0$, and also $x = 2\pi/(k+1)$ and $x = 2\pi/(k-1)$, and also $x = 4\pi/(k+1)$ and $x = 4\pi/(k-1)$, and so on. The second equation has solutions $x = \pi/(k+1)$ and $x = \pi/(k-1)$, and also $x = 3\pi/(k+1)$ and $x = 3\pi/(k-1)$, and so on. Hence, the complete list of solutions to Eq. (49) in the interval $[0, \pi/2]$, written out in order, is

$$0 < \frac{\pi}{k+1} < \frac{\pi}{k-1} < \frac{2\pi}{k+1} < \frac{2\pi}{k-1} < \frac{3\pi}{k+1} < \frac{3\pi}{k-1} < \cdots$$
$$\cdots < \frac{(k-1)/2 \cdot \pi}{k+1} < \frac{(k-1)/2 \cdot \pi}{k-1} = \frac{\pi}{2}.$$

With these intersection points, we have the following formula for the total area (for k any fixed odd number) which takes just the regions on the left of $x = \pi/2$ and doubles them:

$$2\sum_{\ell=1}^{(k-1)/2}\int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} f_{n,k}(x)\,dx + \int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} -f_{n,k}(x)\,dx, \tag{50}$$

where $f_{n,k}(x) = \cos^n x - \cos^n kx$.

We now use the power-reduction formula for cosine to an even power n,

$$\cos^n\theta = \frac{1}{2^n}\binom{n}{n/2} + \frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\cos(n-2j)\theta,$$

to give us that

$$f_{n,k}(x) = \frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\Big(\cos(n-2j)x - \cos(n-2j)kx\Big).$$

When we substitute this into Eq. (50) twice, and distribute the outer sum, we get the following expression for the area:

$$2\sum_{\ell=1}^{(k-1)/2}\int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)}\frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\Big(\cos(n-2j)x - \cos(n-2j)kx\Big)\,dx$$
$$-2\sum_{\ell=1}^{(k-1)/2}\int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)}\frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\Big(\cos(n-2j)x - \cos(n-2j)kx\Big)\,dx. \tag{51}$$

Of course, we want the limit of the expression in (51) as k goes to infinity, so when we do this, and re-arrange the sums and integrals and such, we get

$$A_n = \frac{4}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\lim_{k\to\infty}\Bigg(\sum_{\ell=1}^{(k-1)/2}\int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)}\Big(\cos(n-2j)x - \cos(n-2j)kx\Big)\,dx$$
$$-\sum_{\ell=1}^{(k-1)/2}\int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)}\Big(\cos(n-2j)x - \cos(n-2j)kx\Big)\,dx\Bigg) \tag{52}$$

We now recognize the limit in the right-hand side of Eq. (52) as being the same as in Lemma 10. In other words, we now have that

$$A_n = \frac{4}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}C_{n-2j} \tag{53}$$

where C_{n-2j} from Lemma 10 is defined as

$$C_{n-2j} = \begin{cases} 0, & \text{if } n-2j \equiv 0 \pmod 4; \\ \dfrac{16}{(n-2j)^2\pi}, & \text{if } n-2j \equiv 2 \pmod 4. \end{cases}$$

We now consider the case when $n \equiv 2 \pmod 4$. In this case, if we write out the terms in Eq. (53) and use our definition of C_{n-2j} from above, we have only the terms with j even (as that is when $n-2j \equiv 2 \pmod 4$), giving us

$$A_n = \frac{4}{2^n}\left(\binom{n}{0}\frac{16}{(n)^2\pi} + \binom{n}{2}\frac{16}{(n-4)^2\pi} + \binom{n}{4}\frac{16}{(n-8)^2\pi} + \cdots + \binom{n}{(n/2)-1}\frac{16}{(2)^2\pi}\right)$$

We now factor out $16/(2^2\pi)$ from each term, giving us

$$A_n = \frac{4}{2^n}\frac{16}{2^2\pi}\left(\binom{n}{0}\frac{1}{(n/2)^2} + \binom{n}{2}\frac{1}{(n/2-2)^2} + \binom{n}{4}\frac{1}{(n/2-4)^2} + \cdots + \binom{n}{(n/2)-1}\frac{1}{(1)^2}\right)$$

We re-index the above sum, and simplify the coefficients on the left, to get

$$A_n = \frac{16}{\pi} \cdot \frac{1}{2^n} \cdot \sum_{j=0}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2-2j)^2},$$

as desired (for $n \equiv 2 \pmod 4$).

Finally, for $n \equiv 0 \pmod 4$, we again write out the terms in Eq. (53) and use our definition of C_{n-2j} from above. This time, the only non-zero contributions come from j odd (as this is when $n-2j \equiv 2 \pmod 4$), giving us

$$A_n = \frac{4}{2^n}\left(\binom{n}{1}\frac{16}{(n-2)^2\pi} + \binom{n}{3}\frac{16}{(n-6)^2\pi} + \cdots + \binom{n}{(n/2)-1}\frac{16}{(2)^2\pi}\right)$$

We again factor out $16/(2^2\pi)$ from each term, giving us

$$A_n = \frac{4}{2^n}\frac{16}{2^2\pi}\left(\binom{n}{1}\frac{1}{(n/2-1)^2} + \binom{n}{3}\frac{1}{(n/2-3)^2} + \cdots + \binom{n}{(n/2)-1}\frac{1}{(1)^2\pi}\right)$$

We re-index the above sum, and simplify the coefficients on the left, to get

$$A_n = \frac{16}{\pi} \cdot \frac{1}{2^n} \cdot \sum_{j=0}^{(n-4)/4} \binom{n}{2j+1} \frac{1}{(n/2-(2j+1))^2},$$

as desired (for $n \equiv 0 \pmod 4$).

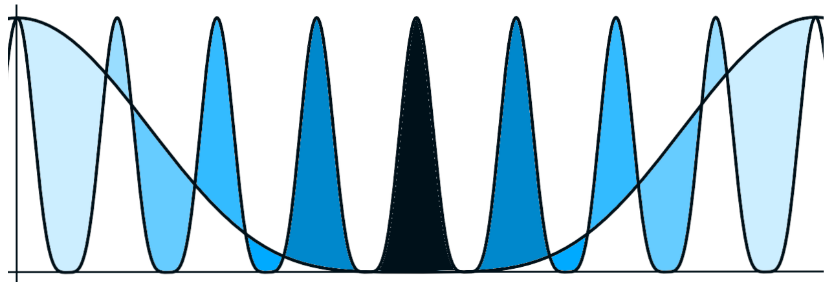

Figure 5: Region between $\cos^4 x$ and $\cos^4 8x$.

- Next, we suppose k is even. As seen in Figure 5 with $n = 4$ and $k = 8$, we again have even symmetry across the midpoint $x = \pi/2$ and so each region on the left of $x = \pi/2$ (in color) has an equivalent area on the right of $x = \pi/2$ (in a matching color). The difference from the earlier part when k was odd is that here, with k even, the "center" spike needs to be considered as a special case.

  Proceeding as before, the intersection points are mostly the same, except that here we will go just a bit beyond the midpoint of $x = \pi/2$ because that center spike (in black) extends just a bit beyond our interval of $[0, \pi/2]$. As k goes to ∞, this extra little bit of area just beyond $x = \pi/2$ will be negligible and so can be ignored. With this in mind, we will integrate over the following sub-intervals:

$$0 < \frac{\pi}{k+1} < \frac{\pi}{k-1} < \frac{2\pi}{k+1} < \frac{2\pi}{k-1} < \cdots < \frac{k/2 \cdot \pi}{k+1} < \frac{k/2 \cdot \pi}{k-1}.$$

  With these intersection points, we have the following formula for the total area.

$$2\sum_{\ell=1}^{k/2} \int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} f_{n,k}(x)\, dx + \int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} -f_{n,k}(x)\, dx, \tag{54}$$

  where, as before, $f_{n,k}(x) = \cos^n x - \cos^n kx$.

  The only difference between Eq. (54) above, and Eq. (50) earlier in the proof (for the case when k odd) is that the upper limit for the sum in Eq. (54) is $k/2$ instead of $(k-1)/2$. So, following the same steps as before, we can jump ahead to what would be our version of Eq. (52), which would be

$$A_n = \frac{4}{2^n} \sum_{j=0}^{(n/2)-1} \binom{n}{j} \lim_{k\to\infty} \left(\sum_{\ell=1}^{k/2} \int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} \Big(\cos(n-2j)x - \cos(n-2j)kx \Big)\, dx \right.$$
$$\left. - \sum_{\ell=1}^{k/2} \int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} \Big(\cos(n-2j)x - \cos(n-2j)kx \Big)\, dx \right).$$

At this point, we recognize the limit in the right-hand side of the above equation as being the same as in Lemma 12. In other words, we now have that

$$A_n = \frac{4}{2^n} \sum_{j=0}^{(n/2)-1} \binom{n}{j} C_{n-2j}$$

where C_{n-2j} from Lemma 12 is defined as

$$C_{n-2j} = \begin{cases} 0, & \text{if } n-2j \equiv 0 \pmod 4; \\ \dfrac{16}{(n-2j)^2\pi}, & \text{if } n-2j \equiv 2 \pmod 4. \end{cases}$$

This is identical to Eq. (53) and so we can arrive at the same conclusion here (with k even) as we did before (with k odd).

□

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