



Generalizations of Amicable Numbers

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Abstract

In this paper we propose new generalizations of amicable numbers. We also give examples and prove properties of these new concepts.

1 Notation

One of the most remarkable sums in number theory is the aliquot sum $s(n)$ of a positive integer n . It is the sum of all proper divisors of n , that is, all divisors of n other than n itself. Thus

$$s(n) = \sum_{\substack{d|n, \\ d < n}} d.$$

As usual $\sigma(n)$ is the sum of all the divisors of n . Thus $\sigma(n) = s(n) + n$. We let $\zeta(s)$ denote the Riemann zeta function. Recall that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\text{Re}(s) > 1$. The abbreviations gcd and lcm stand for the greatest common divisor and the least common multiple.

2 Introduction

Two natural numbers m and n are said to be *amicable* if

$$\sigma(m) = \sigma(n) = m + n.$$

If $m = n$, then m is perfect. It is well known that the smallest perfect number is 6. The smallest pair of amicable numbers, (220, 284), was known to the Pythagoreans. Although more than 2000 years have passed, it is still unknown whether infinitely many amicable pairs exist. In 1913 Dickson [5] defined the natural numbers n_1, n_2, n_3 to form an amicable triple if

$$\sigma(n_1) = \sigma(n_2) = \sigma(n_3) = n_1 + n_2 + n_3$$

or equivalently

$$\left| \begin{array}{l} s(n_1) = n_2 + n_3 \\ s(n_2) = n_1 + n_3 \\ s(n_3) = n_1 + n_2 \end{array} \right. .$$

The smallest amicable triple, in the sense defined by Dickson, is (1980, 2016, 2556). The same generalization to the k -tuples was also proposed by Dickson. Afterwards, Yanney [19], using the same symbolism and terminology, defined the numbers n_1, n_2, n_3 to form an amicable triple if

$$2\sigma(n_1) = 2\sigma(n_2) = 2\sigma(n_3) = n_1 + n_2 + n_3$$

or equivalently

$$\left| \begin{array}{l} n_1 = s(n_2) + s(n_3) \\ n_2 = s(n_1) + s(n_3) \\ n_3 = s(n_1) + s(n_2) \end{array} \right. .$$

The smallest amicable triple according to Yanney's definition is (238, 255, 371). The same generalization to the k -tuples was also proposed by Yanney. Carmichael [3] defined a *multiply amicable pair* as two numbers m and n such that

$$\sigma(m) = \sigma(n) = t(m + n),$$

where t is a positive integer. The same generalization to the k -tuples was proposed by Mason [12]. In 1995 Cohen, Grettton and Hagsis [4] defined natural numbers m and n to

be multiamicable if the sum of the proper divisors of each is a multiple of the other. More precisely, the pair (m, n) is said to be (α, β) -amicable if

$$\sigma(m) - m = \alpha n \quad \text{and} \quad \sigma(n) - n = \beta m.$$

Another interesting generalization is due to Bishop, Bozarth, Kuss and Peet [2]. They defined a *feebly amicable k -tuple* as numbers $n_1, \dots, n_k$ such that

$$\frac{n_1}{\sigma(n_1)} + \dots + \frac{n_k}{\sigma(n_k)} = 1.$$

Clearly every amicable k -tuple satisfies the equation above. For $k = 2$, the smallest feebly amicable pair is $(4, 12)$. In Section 12, we will generalize the definition given in [2]. We say that an integer n is an amicable number if it belongs to an amicable pair, or equivalently

$$\sigma(\sigma(n) - n) = \sigma(n).$$

Let $A(x)$ denote the number of amicable numbers up to x . In 1955, Erdős [6] proved that the asymptotic density of amicable integers relative to the positive integers is 0. That is, the ratio of the number of amicable integers less than x to x tends to zero as x tends to infinity. Subsequently, several papers [7, 14, 15, 17] established progressively better upper bounds for $A(x)$ and the best result to date is due to Pomerance [16] with

$$A(x) \leq \frac{x}{e^{\sqrt{\log x}}}.$$

Many articles are devoted to generalizations of amicable numbers. We point out the papers [1, 8, 9, 10, 11, 13], but many other similar results can be found in the literature. Motivated by the aforementioned investigations, in this paper we propose an extension of the notion of amicable numbers.

3 Lemmas

Lemma 1. *For a fixed integer $N > 0$, the set of integers l for which N does not divide $\sigma(l)$ has density 0.*

Proof. [6, Lemma 2] □

Lemma 2. *For each number $x \geq 1$, we have*

$$\sum_{n \leq x} \frac{\sigma(n)}{n} < \zeta(2)x.$$

Proof. Using the well-known identity

$$\frac{\sigma(n)}{n} = \sum_{u|n} \frac{1}{u} \quad (1)$$

we write

$$\sum_{n \leq x} \frac{\sigma(n)}{n} = \sum_{n \leq x} \sum_{u|n} \frac{1}{u} = \sum_{u \leq x} \frac{1}{u} \sum_{\substack{n \leq x \\ u|n}} 1 = \sum_{u \leq x} \frac{1}{u} \left\lfloor \frac{x}{u} \right\rfloor \leq x \sum_{u \leq x} \frac{1}{u^2} < \zeta(2)x.$$

□

Lemma 3. *Let $k \geq 2$ be integer. For each number $x \geq 1$, we have*

$$\sum_{n \leq x} \frac{\sigma^k(n)}{n^k} < (\zeta(2))^k \zeta(2k-1)x.$$

Proof. Using (1) and the well-known identity

$$\text{lcm}(n_1, \dots, n_k) \gcd\left(\frac{m}{n_1}, \dots, \frac{m}{n_k}\right) = m,$$

where

$$m = n_1 \cdots n_k$$

we deduce

$$\begin{aligned} \sum_{n \leq x} \frac{\sigma^k(n)}{n^k} &= \sum_{n \leq x} \sum_{n_1|n} \cdots \sum_{n_k|n} \frac{1}{n_1 \cdots n_k} = \sum_{n_1 \leq x} \cdots \sum_{n_k \leq x} \frac{1}{n_1 \cdots n_k} \sum_{\substack{n \leq x \\ \text{lcm}(n_1, \dots, n_k) | n}} 1 \\ &= \sum_{n_1 \leq x} \cdots \sum_{n_k \leq x} \frac{1}{n_1 \cdots n_k} \left\lfloor \frac{x}{\text{lcm}(n_1, \dots, n_k)} \right\rfloor \\ &\leq x \sum_{n_1 \leq x} \cdots \sum_{n_k \leq x} \frac{1}{n_1 \cdots n_k \text{lcm}(n_1, \dots, n_k)} \\ &= x \sum_{n_1 \leq x} \cdots \sum_{n_k \leq x} \frac{\gcd\left(\frac{m}{n_1}, \dots, \frac{m}{n_k}\right)}{n_1^2 \cdots n_k^2} < x \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{\gcd\left(\frac{m}{n_1}, \dots, \frac{m}{n_k}\right)}{n_1^2 \cdots n_k^2} \\ &= x \sum_{d=1}^{\infty} \sum_{\gcd\left(\frac{m}{n_1}, \dots, \frac{m}{n_k}\right)=d} \frac{d}{n_1^2 \cdots n_k^2} < x \sum_{d=1}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{d}{(dn'_1)^2 \cdots (dn'_k)^2} \\ &= x \sum_{d=1}^{\infty} \frac{1}{d^{2k-1}} \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{(n'_1)^2 \cdots (n'_k)^2} = (\zeta(2))^k \zeta(2k-1)x. \end{aligned}$$

□

4 $(\alpha_1, \dots, \alpha_k)$ -multiamicable k -tuples

We say that the natural numbers $n_1, \dots, n_k$ form an $(\alpha_1, \dots, \alpha_k)$ -multiamicable k -tuple if

$$\sigma(n_1) = \sigma(n_2) = \dots = \sigma(n_k) = \alpha_1 n_1 + \alpha_2 n_2 + \dots + \alpha_k n_k. \quad (2)$$

Of course, $\alpha_1, \dots, \alpha_k$ must be positive integers. If $k = 1$ and $\alpha_1 = 2$ then n_1 is a perfect number. When $\alpha_1 = \alpha_2 = \dots = \alpha_k$ then $n_1, \dots, n_k$ is a multiply amicable k -tuple according to Mason's definition. There is no extra interest in allowing the possibility that $n_1 = n_2 = \dots = n_k$, so we shall assume always that $n_1 < n_2 < \dots < n_k$ with the k -tuple $(\alpha_1, \dots, \alpha_k)$ ordered accordingly. Hence

$$(\alpha_1 + \dots + \alpha_k)n_1 < \sigma(n_j) < (\alpha_1 + \dots + \alpha_k)n_k \quad (3)$$

for each $j \in [1, k]$, which means that n_1 is $(\alpha_1 + \dots + \alpha_k)$ -abundant. Following the method in [5] and [12] we shall use the next proposition to find $(\alpha_1, \dots, \alpha_k)$ -multiamicable k -tuples.

Theorem 4. Suppose the natural numbers $N_1, \dots, N_k$, $\alpha_1, \dots, \alpha_k$ and a satisfy $(a, N_1) = \dots = (a, N_k) = 1$ and

$$\frac{\sigma(a)}{a} = \frac{\alpha_1 N_1 + \dots + \alpha_k N_k}{\sigma(N_1)} = \dots = \frac{\alpha_1 N_1 + \dots + \alpha_k N_k}{\sigma(N_k)}.$$

Then $aN_1, \dots, aN_k$ are an $(\alpha_1, \dots, \alpha_k)$ -multiamicable k -tuple.

Proof. This follows directly from the multiplicativity of σ . □

For $k = 2$ several of these pairs (aN_1, aN_2) that are (α_1, α_2) -multiamicable pairs are listed in Table 1.

α_1	α_2	N_1	N_2	$\sigma(a)/a$	a	(aN_1, aN_2)
1	2	$2^3 13$	$2^2 29$	$8/5$	$3 \cdot 5$	(1560, 1740)
1	2	$2^2 3^2 5 \cdot 41$	$2^5 3^5$	1	1	(7380, 7776)
1	2	$2^3 41$	$2^2 89$	$104/63$	$3^2 7$	(20664, 22428)
1	2	$17 \cdot 37$	683	$35/12$	$2^5 3^3$	(543456, 590112)
1	2	$17 \cdot 37$	683	$35/12$	$2^3 3^2 13$	(588744, 639288)
1	2	$7^2 11 \cdot 17$	$107 \cdot 113$	$65/24$	$2^3 3^2$	(659736, 870552)
1	2	$13 \cdot 89$	1259	$35/12$	$2^5 3^3$	(999648, 1087776)
2	1	$2^2 5 \cdot 107$	$2^5 71$	$13/9$	3^2	(19260, 20448)
2	1	$2^3 3^5 11$	$2^5 3^2 79$	1	1	(21384, 22752)
2	1	$2^2 3^3 29$	$2^3 3 \cdot 139$	$8/7$	7	(21924, 23352)
2	1	$17 \cdot 37 \cdot 59$	$179 \cdot 227$	$403/144$	$2^4 3^2$	(5343984, 5851152)
1	3	$3^3 5^3$	$3^2 5 \cdot 79$	$9/4$	$2^5 7$	(756000, 796320)
3	1	$11 \cdot 29$	$17 \cdot 19$	$32/9$	$2^2 3^3 5 \cdot 7$	(1205820, 1220940)
3	1	$7 \cdot 13^2$	11^3	$10/3$	$2^3 3^3 5$	(1277640, 1437480)
3	1	$3^2 19 \cdot 41$	$3^5 29$	$18/7$	$2^3 5 \cdot 7$	(1963080, 1973160)

Table 1

The sequence [A383239](#) in the OEIS [18] consists of the larger components of $(1, 2)$ -multiamicable pairs. The next result concerns the density of (α, β) -multiamicable pairs. Our argument is a modification of the one used by Cohen, Gretton, and Hagis in [4].

Theorem 5. *Let $M(x)$ denote the number of (α, β) -multiamicable pairs m, n with $m < n$ and $m \leq x$. Then $M(x) = o(x)$ as $x \rightarrow \infty$.*

Proof. According to (2) we have

$$\sigma(m) = \sigma(n) = \alpha m + \beta n. \quad (4)$$

Assume that K is a large positive integer. Let $M_1(x)$ denote the number of (α, β) -multiamicable pairs m, n with $m \leq x$ and $\sigma(m)/m \geq K$. From (3) we derive

$$\frac{\sigma(m)}{m} > \alpha + \beta. \quad (5)$$

Therefore the number of n 's that can correspond to a given value of m is less than

$$\frac{1}{2} \left(\frac{\sigma(m)}{m} - 1 \right) \left(\frac{\sigma(m)}{m} - 2 \right).$$

Now Lemma 2 and Lemma 3 give us

$$\begin{aligned} M_1(x) &\leq \frac{1}{2} \sum_{\substack{m \leq x \\ \sigma(m)/m \geq K}} \left(\frac{\sigma(m)}{m} - 1 \right) \left(\frac{\sigma(m)}{m} - 2 \right) \\ &\leq \frac{1}{2K} \sum_{m \leq x} \frac{\sigma(m)}{m} \left(\frac{\sigma(m)}{m} - 1 \right) \left(\frac{\sigma(m)}{m} - 2 \right) \\ &< \frac{1}{K} \sum_{m \leq x} \frac{\sigma(m)}{m} + \frac{3}{2K} \sum_{m \leq x} \frac{\sigma^2(m)}{m^2} + \frac{1}{2K} \sum_{m \leq x} \frac{\sigma^3(m)}{m^3} \\ &< \frac{1}{K} \zeta(2)x + \frac{3}{2K} (\zeta(2))^2 \zeta(3)x + \frac{1}{2K} (\zeta(2))^3 \zeta(5)x. \end{aligned} \quad (6)$$

Let $M_2(x)$ denote the number of (α, β) -multiamicable pairs m, n with $m \leq x$ and such that $\sigma(m)/m < K$. By (5) we get

$$\alpha + \beta < K.$$

Consequently the number of n 's that can correspond to a fixed value of m is less than $\frac{1}{2}(K-1)(K-2)$. Using this consideration, (4) and Lemma 1 we obtain

$$\begin{aligned} M_2(x) &\leq \frac{1}{2}(K-1)(K-2) \sum_{\substack{m \leq x \\ K^4 | \sigma(m)}} 1 + \frac{1}{2}(K-1)(K-2) \sum_{\substack{m \leq x \\ K^4 \nmid \sigma(m)}} 1 \\ &\leq \frac{1}{2}(K-1)(K-2) \sum_{\substack{m \leq x \\ \alpha m + \beta n \equiv 0 \pmod{K^4}}} 1 + o(x) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}(K-1)(K-2)\left(\frac{x}{K^3} + \mathcal{O}(K)\right) + o(x) \\
&\leq \frac{x}{K} + o(x).
\end{aligned} \tag{7}$$

Clearly

$$M(x) = M_1(x) + M_2(x). \tag{8}$$

Summarizing (6), (7), (8) and bearing in mind that K can be taken arbitrarily large we establish

$$M(x) = o(x) \quad \text{as } x \rightarrow \infty.$$

This completes the proof of Theorem 5. □

The following conjecture provides a natural conclusion to this section.

Conjecture 6. For any fixed positive integers $\alpha_1, \dots, \alpha_k$, there exist infinitely many $(\alpha_1, \dots, \alpha_k)$ -multiamicable k -tuples.

5 (α, β) -amicable pairs

Let α and β be positive integers. We say that the numbers m and n form an (α, β) -amicable pair if

$$\sigma(\alpha n) - \alpha n = m \quad \text{and} \quad \sigma(\beta m) - \beta m = n$$

or equivalently

$$m = s(\alpha n) \quad \text{and} \quad n = s(\beta m).$$

When $\alpha = \beta = 1$ then m and n are amicable. Several (α, β) -amicable pairs are listed in Table 2.

α	β	(α, β) -amicable pairs
1	2	(26, 46), (296, 586)
1	3	(3, 4), (15, 33), (5919, 7905)

Table 2

The sequence [A384411](#) in the OEIS [18] consists of (1, 2)-amicable pairs. The OEIS [18] sequence [A383483](#) lists the smaller elements of (1, 3)-amicable pairs.

6 $\text{PM}(p, q)$ -amicable k -tuples

The abbreviation PM stands for power mean. Let $k \geq 1$, $p \geq 1$ and $q \geq 1$ be integers. We say that the numbers $n_1, \dots, n_k$ form a $\text{PM}(p, q)$ -amicable k -tuple if

$$\sigma^p(n_1) + \sigma^p(n_2) + \dots + \sigma^p(n_k) = q(n_1 + n_2 + \dots + n_k)^p.$$

Obviously, when $k = q$, then every amicable k -tuple satisfies the above equation. Several $\text{PM}(p, q)$ -amicable k -tuples that are not amicable k -tuples are listed in Table 3.

k	p	q	$\text{PM}(p, q)$ -amicable k -tuples
2	1	2	(3, 20), (5, 12), (5, 70), (5, 88), (6, 28), (10, 20)
2	1	3	(6, 180), (10, 780), (24, 780), (26, 660), (34, 504)
2	2	1	(2, 3), (19, 33), (27, 77), (39, 161), (45, 133), (51, 69)
2	2	2	(1, 4), (1378, 9962), (1660, 4892), (1975, 10425)
3	1	2	(1, 2, 20), (1, 3, 18), (1, 4, 20), (1, 8, 20), (1, 10, 18)
3	1	3	(1, 76, 360), (2, 11, 240), (2, 41, 420), (6, 120, 180)
3	2	1	(1, 47, 185), (2, 11, 14), (2, 110, 371), (3, 302, 411)
3	2	2	(3, 36, 98), (5, 34, 135), (5, 40, 105), (10, 106, 406)
3	2	3	(14, 350, 1340), (22, 96, 1862), (31, 301, 1876)
3	2	4	(3, 39, 156), (11, 40, 294), (12, 14, 60), (17, 70, 210)
3	2	5	(6, 222, 1608), (15, 33, 168), (30, 66, 552)
3	3	1	(2, 10, 15), (4, 20, 39), (8, 40, 87), (9, 45, 63)
3	3	3	(56, 134, 710), (108, 268, 1724), (236, 404, 2510)

Table 3

The sequence [A383484](#) in the OEIS [18] consists of the larger components of $\text{PM}(2, 1)$ -amicable pairs. The OEIS [18] sequence [A385008](#) lists the larger elements of $\text{PM}(2, 2)$ -amicable pairs.

7 $\text{WPM}(p)$ -amicable k -tuples

The abbreviation WPM stands for weighted power mean. Let $k \geq 1$ and $p \geq 1$ be integers. We say that the numbers $n_1, \dots, n_k$ form an $\text{WPM}(p)$ -amicable k -tuple if

$$n_1 \sigma^p(n_1) + n_2 \sigma^p(n_2) + \dots + n_k \sigma^p(n_k) = (n_1 + n_2 + \dots + n_k)^{p+1}.$$

Clearly each amicable k -tuple satisfies the above equation. Several $\text{WPM}(p)$ -amicable k -tuples that are not amicable k -tuples are listed in Table 4.

k	p	WPM(p)-amicable k -tuples
2	1	(4, 6), (10, 16), (34, 68), (60, 81), (91, 273)
2	2	(7, 21), (105, 231), (1065, 2499)
3	1	(1, 21, 63), (1, 22, 44), (2, 38, 98), (4, 6, 34)
3	2	(12, 276, 412), (70, 210, 224), (87, 189, 264)

Table 4

The sequence [A383714](#) in the OEIS [18] consists of the larger components of WPM(2)-amicable pairs.

8 GM-amicable k -tuples

The abbreviation GM stands for geometric mean. Let $k \geq 1$ be integer. We say that the numbers $n_1, \dots, n_k$ form a GM-amicable k -tuple if

$$\sigma(n_1)\sigma(n_2) \cdots \sigma(n_k) = (n_1 + n_2 + \cdots + n_k)^k.$$

It is easy to see that every amicable k -tuple satisfies the above equation. Several GM-amicable k -tuples that are not amicable k -tuples are listed in Table 5.

k	GM-amicable k -tuples
2	(28, 84), (42, 102), (60, 276), (92, 160), (244, 624), (426, 582)
3	(1080, 1092, 1188), (10164, 10584, 11172), (10440, 10692, 11628)

Table 5

The OEIS [18] sequences [A383932](#) and [A386010](#) list the larger elements of GM-amicable pairs and GM-amicable triples, respectively.

9 WGM-amicable k -tuples

The abbreviation WGM stands for weighted geometric mean. Let $k \geq 1$ be integer. We say that the numbers $n_1, \dots, n_k$ form a WGM-amicable k -tuple if

$$\sigma(n_1)^{n_1}\sigma(n_2)^{n_2} \cdots \sigma(n_k)^{n_k} = (n_1 + n_2 + \cdots + n_k)^{n_1+n_2+\cdots+n_k}.$$

It is clear that each amicable k -tuple satisfies the above equation. Although we cannot set an example, we cannot rule out the existence of WGM-amicable k -tuples that are not amicable k -tuples. The sequence [A385186](#) in the OEIS [18] consists of the larger components of WGM-amicable pairs.

10 IWGM-amicable k -tuples

The abbreviation IWGM stands for inverse weighted geometric mean. The positive integers $n_1, \dots, n_k$ form an IWGM-amicable k -tuple if

$$n_1^{\sigma(n_1)} n_2^{\sigma(n_2)} \dots n_k^{\sigma(n_k)} = (n_1 n_2 \dots n_k)^{n_1 + n_2 + \dots + n_k}.$$

Clearly each amicable k -tuple satisfies the above equation. Several IWGM-amicable pairs that are not amicable pairs are listed in Table 6.

IWGM-amicable pairs	
(1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (1, 13), (1, 17), (1, 19), (1, 23), (1, 29), (1, 31), (1, 37)	

Table 6

The OEIS [18] sequence [A385492](#) lists the larger elements of IWGM-amicable pairs.

11 HM(p, q)-amicable k -tuples

The abbreviation HM stands for harmonic mean. Let $k \geq 1$, $p \geq 1$ and $q \geq 1$ be integers. We say that the numbers $n_1, \dots, n_k$ form a HM(p, q)-amicable k -tuple if

$$\left(\frac{1}{\sigma^p(n_1)} + \frac{1}{\sigma^p(n_2)} + \dots + \frac{1}{\sigma^p(n_k)} \right) (n_1 + n_2 + \dots + n_k)^p = q.$$

Obviously, if $k = q$, then every amicable k -tuple satisfies the above equation. Several HM(p, q)-amicable k -tuples that are not amicable k -tuples are listed in Table 7.

k	p	q	HM(p, q)-amicable k -tuples
2	1	2	(20, 28), (24, 56), (30, 66), (40, 90), (56, 88), (92, 132)
2	1	3	(3, 6), (10, 32), (12, 60), (15, 33), (24, 116), (33, 57)
2	2	1	(120, 168), (1272, 1320), (2160, 3792), (3672, 4968)
2	2	4	(435, 717), (447, 513), (2001, 2607), (2001, 2607)
3	1	3	(840, 1020, 1380), (1008, 1260, 1638), (2016, 2232, 2772)
3	2	4	(480, 480, 1056), (1400, 1400, 2160), (3936, 3936, 6240)

Table 7

The sequence [A384814](#) in the OEIS [18] consists of the larger components of HM(1, 2)-amicable pairs. The OEIS [18] sequence [A383964](#) lists the larger elements of HM(2, 1)-amicable pairs. The sequence [A385155](#) in the OEIS [18] consists of the larger members of HM(1, 3)-amicable triples.

12 WHM(p)-amicable k -tuples

The abbreviation WHM stands for weighted harmonic mean. Let $k \geq 1$ and $p \geq 1$ be integers. We say that the numbers $n_1, \dots, n_k$ form a WHM(p)-amicable k -tuple if

$$\left(\frac{n_1^p}{\sigma^p(n_1)} + \frac{n_2^p}{\sigma^p(n_2)} + \dots + \frac{n_k^p}{\sigma^p(n_k)} \right) (n_1 + n_2 + \dots + n_k)^p = n_1^p + n_2^p + \dots + n_k^p.$$

Clearly each amicable k -tuple satisfies the above equation. It is easy to see that WHM(1)-amicable k -tuple coincides with feebly amicable k -tuple defined in [2]. Several WHM(p)-amicable k -tuples that are not amicable k -tuples are listed in Table 8.

k	p	WHM(p)-amicable k -tuples
3	1	(72, 360, 504), (84, 120, 840), (84, 672, 840), (96, 660, 840)
3	2	(117, 117, 4680)

Table 8

The OEIS [18] sequences [A384487](#) and [A385749](#) list the larger elements of WHM(1)- and WHM(2)-amicable triples, respectively.

13 Cross-harmonious pairs

Let a and b be positive integers. We say that a and b form a *cross-harmonious pair* if

$$\frac{b}{\sigma(a)} + \frac{a}{\sigma(b)} = 1.$$

Clearly, every amicable pair is cross-harmonious. Several cross-harmonious pairs that are not amicable pairs are listed in Table 9.

Cross-harmonious pairs
(12, 14), (12, 20), (30, 54), (32, 42), (48, 62), (70, 88), (70, 108), (78, 126), (88, 114)

Table 9

The sequence [A384706](#) in the OEIS [18] consists of the larger components of cross-harmonious pairs.

14 MP(p, q)-amicable k -tuples

The abbreviation MP stands for mean power. Let $k \geq 1$, $p \geq 2$ and $q \geq 1$ be integers. We say that the numbers $n_1, \dots, n_k$ form an MP(p, q)-amicable k -tuple if

$$\sigma^p(n_1) + \sigma^p(n_2) + \dots + \sigma^p(n_k) = q(n_1^p + n_2^p + \dots + n_k^p).$$

Several MP(p, q)-amicable k -tuples are listed in Table 10.

k	p	q	MP(p, q)-amicable k -tuples
2	2	2	(1, 2), (13, 21), (13, 27), (17, 175), (45, 123), (1069, 2133), (1093, 2187)
2	2	4	(6, 28), (12, 14), (48, 62), (112, 124), (135, 208), (160, 189), (192, 254)
3	2	2	(2, 4, 51), (3, 40, 71), (5, 12, 23), (7, 116, 303), (11, 20, 55), (12, 215, 333)
3	2	3	(1, 81, 148), (10, 94, 164), (14, 20, 34), (20, 82, 116), (26, 70, 418)
4	3	3	(1, 4, 5, 9), (32, 34, 202, 245), (52, 108, 135, 233), (55, 58, 230, 281)
4	3	4	(2, 49, 56, 118), (7, 35, 51, 75), (10, 168, 207, 307), (37, 74, 232, 253)
4	3	5	(11, 25, 95, 148), (15, 59, 128, 129), (35, 170, 186, 237), (44, 125, 139, 266)
4	3	6	(6, 80, 85, 135), (20, 22, 34, 92), (41, 66, 123, 190), (41, 70, 107, 190)
4	3	7	(3, 181, 198, 212), (11, 21, 24, 25), (14, 30, 94, 102), (23, 81, 112, 135)
4	3	8	(13, 91, 116, 176), (30, 74, 82, 112), (48, 62, 112, 124), (48, 62, 160, 189)
4	3	9	(18, 55, 178, 220), (28, 231, 273, 320), (36, 43, 44, 78), (44, 46, 96, 258)

Table 10

The OEIS [18] sequence [A384255](#) lists the larger members of MP(2, 2)-amicable pairs.

Question 7. *Is there an MP(2, 2)-amicable pair (m, n) such that $\sigma(m) = \sigma(n)$? More precisely, we raise the question of the existence of a number pair (m, n) such that*

$$\sigma^2(m) = \sigma^2(n) = m^2 + n^2.$$

15 Sigma-quadratic triples

The positive integers $a \leq b$ and c form a *sigma-quadratic triple* if

$$\sigma^2(a) = \sigma^2(b) = a^2 + b^2 + c^2.$$

Several sigma-quadratic triples are listed in Table 11.

Sigma-Quadratic triples
(2, 2, 1), (40, 40, 70), (40, 58, 56), (164, 194, 148), (196, 196, 287), (224, 224, 392), (1120, 1120, 2576), (3040, 4024, 5632), (44200, 49300, 96680), (687184, 703312, 965780)

Table 11

The sequence [A385356](#) in the OEIS [18] consists of the first components of sigma-quadratic triples.

16 Sigma-quadratic quadruples

The positive integers $a \leq b \leq c$ and d form a *sigma-quadratic quadruple* if

$$\sigma^2(a) = \sigma^2(b) = \sigma^2(c) = a^2 + b^2 + c^2 + d^2.$$

Several sigma-quadratic quadruples are listed in Table 12.

Sigma-Quadratic quadruples
(4, 4, 4, 1), (204, 220, 220, 340), (3480, 3672, 4296, 8520), (4796, 4988, 5276, 3108), (5532, 6812, 6812, 6628), (7152, 9272, 9584, 10816), (33468, 35380, 43676, 42700)

Table 12

The OEIS [18] sequence [A385531](#) lists the first elements of sigma-quadratic quadruples.

17 Sigma-cubic triples

The positive integers a , b , and c form a *sigma-cubic triple* if

$$\sigma^3(a) = a^3 + b^3 + c^3.$$

Several sigma-cubic triples are listed in Table 13.

Sigma-Cubic triples
(5, 3, 4), (6, 8, 10), (53, 12, 19), (58, 59, 69), (102, 26, 208), (102, 117, 195), (118, 116, 138), (152, 172, 264), (168, 336, 408), (197, 27, 46), (214, 173, 267)

Table 13

The sequence [A385325](#) in the OEIS [18] consists of the first members of sigma-cubic triples.

18 Sigma-cubic quadruples

The positive integers $a \leq b$, c and d form a *sigma-cubic quadruple* if

$$\sigma^3(a) = \sigma^3(b) = a^3 + b^3 + c^3 + d^3.$$

Several sigma-cubic quadruples are listed in Table 14.

Sigma-Cubic quadruples
(153, 153, 105, 165), (216, 216, 168, 576), (255, 321, 84, 312), (324, 324, 271, 804), (672, 910, 147, 1925), (735, 1243, 215, 615)

Table 14

The OEIS [18] sequence [A385397](#) lists the first components of sigma-cubic quadruples.

19 Sigma-cubic quintuples

The positive integers $a \leq b \leq c$ and $d \leq e$ form a *sigma-cubic quintuple* if

$$\sigma^3(a) = \sigma^3(b) = \sigma^3(c) = a^3 + b^3 + c^3 + d^3 + e^3.$$

Several sigma-cubic quintuples are listed in Table 15.

Sigma-Cubic quintuples
(30, 55, 55, 11, 23), (62, 62, 69, 4, 43), (90, 153, 153, 135, 135), (174, 190, 323, 5, 94), (238, 321, 321, 77, 81), (357, 385, 385, 233, 266), (390, 476, 598, 470, 814)

Table 15

The OEIS [18] sequence [A386378](#) consists of the first elements of sigma-cubic quintuples.

20 $P(i, j)$ -amicable pairs

The abbreviation P stands for prime. We say that the prime numbers p and q , where $p < q$, form a $P(i, j)$ -amicable pair if

$$\sigma(p + i) = \sigma(q + j) = p + q.$$

It is easy to see that when $i + j = 0$ and (p, q) as a $P(i, j)$ -amicable pair then $(p + i, q + j)$ is an amicable pair. Several $P(i, j)$ -amicable pairs are listed in Table 16.

i	j	$P(i, j)$ -amicable pairs
1	1	(23, 37), (34673, 34687), (55373, 65587), (2056961, 2089951), (5174363, 8161477)
-1	1	(5021, 5563), (185369, 203431), (308621, 389923), (879713, 901423)
-1	-1	(853, 1163), (4513, 7583), (9109, 17099), (44917, 65963), (46183, 48857)
1	2	(179, 367), (263, 457), (557, 691), (332273, 341647), (401309, 909091)
1	3	(19, 23), (29, 43), (137, 151), (223, 281), (727, 953), (22651, 22709)
3	1	(11, 13), (41, 43), (107, 109), (149, 151), (257, 331), (881, 883), (2141, 2143)
3	3	(2377, 3671), (7069, 8807), (7949, 9907), (44701, 52067), (71761, 85487)

Table 16

The sequences [A385586](#), [A385718](#), [A385739](#) and [A385740](#) in the OEIS [18] list the larger elements of $P(1, 1)$ -, $P(1, 2)$ -, $P(-1, 1)$ - and $P(-1, -1)$ -amicable pairs, respectively.

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