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On a Family of Solutions to Arithmetic Differential Equations Involving the Collatz Map

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Abstract

The arithmetic derivative is a nonlinear derivation on the positive integers which forms a natural analog of the conventional derivative. While exploring solutions to arithmetic differential equations, we stumbled across a curious pattern in the positive integers for which the arithmetic derivative and the Collatz map commute. Here we report on these empirical findings, and prove several analytical results on the form of such numbers. Among these findings is the existence of a family of semiprime numbers which are mapped by the Collatz function to another semiprime having a sum of prime factors which is half of the original semiprime's. We show that this family of semiprimes solves the commutation problem and that the sum of their reciprocals converges.

1 Introduction

The arithmetic derivative is a distinctive analogue to the conventional derivative from calculus. Unlike its continuous counterpart, the arithmetic derivative is tailored specifically for positive integers and encodes information about a number's prime factorization, offering a novel perspective on the structural properties of numbers. It was introduced initially by Shelly in 1911 [15] and subsequently refined by various mathematicians, including Barbeau [1], who extended it to the rationals, showing that the arithmetic derivative satisfies an analog of the familiar quotient rule. Later, this operation was extended to the set of irrational numbers which can be written as the product of primes raised to rational powers by Ufnarovski and Ahlander [17]. Central to the arithmetic derivative is its adherence to a product rule akin to that of traditional calculus, facilitating computations that reveal relationships between integers. This operation has garnered attention not only for its theoretical elegance but also for its connections to conjectures in number theory, such as the Goldbach and twin prime conjectures [17], and these links hint at deeper insights into the distribution and behavior of prime numbers.

Meanwhile, the Collatz conjecture has been studied at least as far back as 1937, and it remains a surprisingly challenging open problem. It has been attacked from many angles, including studies of the number of steps needed to reach 1 (the total stopping time) [12], continuous extensions [4], and the determination of bounds on the size of a nontrivial cycle [8]. In 1972, Conway showed that a natural generalization of the Collatz conjecture was undecidable [5] and later built on this work to construct a method for universal computation known as FRACTRAN [6]. Interestingly, the Collatz iterates were also related to Benford's Law [11, 13]. More recently, it was shown by Tao that almost all orbits of the Collatz map attain almost bounded values in the sense of logarithmic density [16], and Barina has confirmed that positive integers as high as 2^{68} obey the conjecture [2]. Despite decades of scrutiny and extensive computational exploration, the Collatz conjecture remains unproven, captivating mathematicians with its deceptively simple nature.

The original motivation for this work was to investigate numbers for which the arithmetic derivative and compositions of the Collatz map commute. It can be shown that arithmetic functions which satisfy this property for some integer n in a cycle Ω necessarily send n into a cycle of length dividing $|\Omega|$. Indeed, if $D(C^{|\Omega|}(n)) = C^{|\Omega|}(D(n))$ and $C^{|\Omega|}(n) = n$, then $D(n) = C^{|\Omega|}(D(n))$, so that $D(n)$ generates a cycle. Note that the choice of D was arbitrary here, and the arithmetic derivative was chosen merely out of curiosity while exploring concrete examples. In the course of our exploration, we find that for $|\Omega| = 1$, these numbers satisfy a puzzling property. Indeed, all but one example are congruent to 9 modulo 10 and can be written as the product of two distinct primes, one which ends in 1, while the other ends in 9. The exception is the number 606938385, which is the product of the three distinct primes $3, 5$, and 40462559.

In this work, we introduce a class of semiprime numbers which we call *compatible* (A376275 in the OEIS [9]), showing that they satisfy the arithmetic differential equation $D\left(\frac{3n+1}{2}\right) = \frac{D(n)}{2}$, thereby solving the commutation problem $D(C(n)) = C(D(n))$. Naturally, we ask whether there are infinitely many such numbers. Just as in Brun's work on twin primes [3], we show that the sum of the reciprocals of the compatible semiprimes converges, so that an easy answer to this question does not seem likely. However, we show that all known solutions to the commutation problem are semiprimes of this form except for the case 606938385, which instead solves the arithmetic differential equation $D\left(\frac{3n+1}{2}\right) = \frac{3D(n)+1}{2}$. Moreover, we explain the observation that all solutions (again, except for 606938385) are congruent to 9 modulo 10 by showing that this is the case for all compatible semiprimes. This result does not rule out the possibility that further solutions which break this congruence exist, but merely explains what is seen empirically.

The rest of this paper is organized as follows. In Section 2, we review the arithmetic derivative, re-deriving several known results, and we define the Collatz map. In Section 3, we present several analytical results about the numbers for which these maps commute. We show that all such numbers are odd, and motivated by our empirical results, attempt to characterize the distinct primes appearing in these numbers by showing that they cannot have the forms of several classes of prime numbers. We then give a bound on the difference between these primes, showing that it grows at least linearly with the larger prime. We also produce a class of semiprime numbers which belong to this sequence of numbers and show that all known elements of the sequence belong to this class, with the exception of 606938385. In Section 4, we show that the sum of the reciprocals of the numbers in this class converges. Finally, in Section 5, we give concluding remarks and put forth several conjectures, including the assertion that there are infinitely many numbers for which the arithmetic derivative and Collatz map commute.

2 Background

The arithmetic derivative is a natural analog of the conventional derivative from calculus, at least algebraically. We define the arithmetic derivative to be a non-linear derivation $D : \mathbb{N} \to \mathbb{N}$ on the set of natural numbers with the property that $D(1) = D(0) = 0$ and $D(p) = 1$ for all primes p. Explicitly, we define D so that

$$D(mn) = D(m)n + mD(n) \tag{1}$$

for every $m, n \in \mathbb{N}$. This requirement already demands $D(1) = D(1)+D(1)$, so that $D(1) = 0$ by construction. Moreover, if $n = pm$ for some $m \in \mathbb{N}$ and some prime p, it follows from (1) that $D(n) = D(p)m + pD(m)$. If m is composite, we may proceed inductively until the only derivatives appearing on the right side of the equality are derivatives of primes. Thus, the arithmetic derivative (and indeed every arithmetic function that satisfies (1)) is completely determined by its action on prime numbers. The choice of $D(p) = 1$ serves to treat all prime numbers with equal weighting. Another consequence of (1) is that the arithmetic derivative satisfies a power rule. Indeed, we have

$$D(n^k) = kn^{k-1}D(n),$$

and so it follows that if $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$ is the prime factorization of n, then

$$D(n) = n \sum_{j=1}^{k} \frac{\nu_j}{p_j} D(p_j) = n \sum_{j=1}^{k} \frac{\nu_j}{p_j}. \tag{2}$$

Notice the logarithmic derivative flavor of (2). If we divide by n and define the logarithmic arithmetic derivative $\mathrm{ld}(n) := D(n)/n$, then (2) becomes

$$\mathrm{ld}(n) = \sum_{j=1}^{k} \frac{\nu_j}{p_j}.$$

Interestingly, the arithmetic derivative can be linked to both the Goldbach and twin primes conjectures rather easily. Indeed, if $p, p+2$ are twin primes, then $D(2p) = D(2)p + 2D(p) = p + 2$. Now applying the arithmetic derivative again, we find that $D^2(2p) = 1$. Thus, if the twin prime conjecture holds, then there are infinitely many n for which $D^2(n) = 1$. As for the Goldbach conjecture, observe that if $2n = p + q$ for some p, q prime, then $D(pq) = p + q = 2n$. Thus, if the Goldbach conjecture holds, then for every even integer $2n$, there is another integer k such that $D(k) = 2n$.

We make use of inequalities for the arithmetic derivative derived by Barbeau [1] and then strengthened by Dahl, Olsson, and Loiko [7]. In particular, we make use of the following lemma featuring the prime omega function (the number of prime factors in n).

Lemma 1. *Let Ω denote the prime omega function and let p be the least prime in n. Then*

$$\Omega(n)n^{\frac{\Omega(n)-1}{\Omega(n)}} \leq D(n) \leq \frac{n\log_p(n)}{p}.$$

Moreover, equality holds if and only if n is a prime power.

Proof. Let $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$. Then

$$D(n) = n \sum_{j=1}^{k} \frac{\nu_j}{p_j}.$$

Let p be the least prime of $p_1, \ldots, p_k$. Then

$$D(n) \leq n \sum_{j=1}^{k} \frac{\nu_j}{p} \leq n \sum_{j=1}^{k} \frac{\nu_j \log_p(p_j)}{p} = \frac{n\log_p(n)}{p}$$

and equality holds whenever $p_1 = \cdots = p_k = p$. For the lower bound, note that $\frac{1}{\Omega(n)}D(n) = \frac{n}{\Omega(n)}\sum_{j=1}^{k}\frac{\nu_j}{p_j}$ is an arithmetic mean and apply the AM-GM inequality. This produces

$$\frac{D(n)}{\Omega(n)} \geq n\left(\prod_{j=1}^{k} \frac{1}{p_j^{\nu_j}}\right)^{1/\Omega(n)} = n^{\frac{\Omega(n)-1}{\Omega(n)}},$$

and equality holds in the AM-GM inequality whenever $p_1 = \cdots = p_k$. □

The Collatz mapping is the arithmetic function defined by

$$C(n) = \begin{cases} 3n+1, & \text{if } n \text{ odd;} \\ \frac{n}{2}, & \text{if } n \text{ even.} \end{cases}$$

A longstanding open problem in mathematics is the so-called Collatz conjecture, which states that the discrete dynamical system with trajectories defined by C always converges to the $\{4, 2, 1\}$ cycle. Equivalently, it says that for all $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that

$C^k(n) = 1$. This seemingly simple problem has gone unproven for more than eighty years, and those who study it deeply often comment that the problem is completely intractable and outside the scope of the mathematical tools available today. While this may be the case, it can be interesting to explore certain aspects of this problem in an attempt to gain even the smallest insight. We forgo a study of what is known about the Collatz conjecture and instead refer the reader to the excellent survey by Lagarias [12]. We, however, point out that when n is odd, the next iterate in the trajectory of n is even. It is therefore common to redefine the Collatz map as

$$C(n) = \begin{cases} \frac{3n+1}{2}, & \text{if } n \text{ odd;} \\ \frac{n}{2}, & \text{if } n \text{ even,} \end{cases}$$

and we take this as our definition throughout the rest of our work.

3 The commutation problem and compatible semiprimes

We let $(a_n)_{n=1}^{\infty}$ denote the sequence of numbers for which the arithmetic derivative and Collatz map commute. In our empirical investigation, we have found the first 30 numbers of this sequence and they are listed along with their prime factorization in Table 1. We note that all but one of these numbers are congruent to 9 mod 10 and can be written as a product of two distinct primes, one of which is congruent to 1 while the other is congruent to 9. All numbers with the exclusion of the exceptional case $n = 12$ are also congruent to 2 modulo 3. For $n = 12$, we see that a_n is divisible by 3. We note also that in the case of $n = 12$, the largest prime is at least an order of magnitude larger than all of the remaining primes in the table.

Our empirical results seem to indicate several constraints on the form of a_n, some of which can be proven by elementary methods. Take, for example, the fact that all numbers in the table are odd. We can easily show that this is always the case.

Proposition 2. *Let a_n be the sequence of positive integers for which $D(C(a_n)) = C(D(a_n))$. Then a_n is odd for all n.*

Proof. Suppose a_n were even for some n. Then there exists an $m \in \mathbb{N}$ such that $a_n = 2m$, and we must have $D(C(a_n)) = D(m)$ while $C(D(a_n)) = C(m + 2D(m))$. If $m + 2D(m)$ is even, it follows that $2D(m) = m + 2D(m)$, so that $m = 0$, a contradiction (since $a_n > 0$ by assumption). On the other hand, if $m + 2D(m)$ is odd, it follows that $D(m) = (3m + 6D(m) + 1)/2$, so that $4D(m) = -3m - 1$, which is again a contradiction. Thus, a_n is odd for all n. □

Our data suggests that a_n must be a product of distinct primes (an almost-prime), and we notice that no single prime appears in the table. This latter fact is indeed the case in general, as the next proposition shows.

n	a_n	Prime Factorization	$D(a_n)$	$C(a_n)$	Prime Factorization
1	114239	71×1609	1680	171359	349×491
2	144059	71×2029	2100	216089	281×769
3	933899	131×7129	7260	1400849	439×3191
4	1918199	79×24281	24360	2877299	241×11939
5	25054499	149×168151	168300	37581749	449×83701
6	30495419	1129×27011	28140	45743129	5099×8971
7	33065159	569×58111	58680	49597739	1801×27539
8	72602039	1511×48049	49560	108903059	5711×19069
9	255442559	809×315751	316560	383163839	2459×155821
10	353104079	1511×233689	235200	529656119	4691×112909
11	575473559	3631×158489	162120	863210339	12611×68449
12	606938385	$3 \times 5 \times 40462559$	323700487	910407578	$2 \times 47 \times 83 \times 116689$
13	808589879	2801×288679	291480	1212884819	8861×136879
14	846509819	2861×295879	298740	1269764729	9049×140321
15	1042804799	6871×151769	158640	1564207199	36709×42611
16	1055710979	2999×352021	355020	1583566469	9421×168089
17	1059728279	4079×259801	263880	1589592419	13411×118529
18	1184657879	2281×519359	521640	1776986819	7001×253819
19	1247085239	4751×262489	267240	1870627859	15889×117731
20	1791627599	8609×208111	216720	2687441399	38431×69929
21	2196997739	9059×242521	251580	3295496609	37199×88591
22	2323221179	7741×300119	307860	3484831769	27581×126349
23	2372469179	9091×260969	270060	3558703769	35899×99131
24	2591327159	10369×249911	260280	3886990739	46439×83701
25	3063507719	8191×374009	382200	4595261579	28211×162889
26	3276652079	5881×557159	563040	4914978119	18701×262819
27	4021840859	2909×1382551	1385460	6032761289	8821×683909
28	5489857619	2309×2377591	2379900	8234786429	6961×1182989
29	5716553879	3881×1472959	1476840	8574830819	11801×726619
30	6022735799	929×6483031	6483960	9034103699	2789×3239191

Table 1: First 30 elements of a_n.

Proposition 3. *Let a_n be the sequence of positive integers for which $D(C(a_n)) = C(D(a_n))$. Then a_n is a composite number for all n.*

Proof. Suppose $a_n = p$ were prime for some n and note that $p \neq 2$ by Proposition 2. Then $D(C(a_n)) = D\left(\frac{3p+1}{2}\right)$, while $C(D(a_n)) = C(1) = 2$. It follows that $D\left(\frac{3p+1}{2}\right) = 2$. If $\Omega(\frac{3p+1}{2}) = 1$, then $\frac{3p+1}{2}$ is prime and so $D\left(\frac{3p+1}{2}\right) = 1$, a contradiction. Then by Lemma 1, we have

$$2 = D\left(\frac{3p+1}{2}\right) \geq 2\sqrt{\frac{3p+1}{2}},$$

which produces a contradiction. Thus, a_n is composite for all n. □

It is worth noting that a weak connection between the arithmetic derivative and the Collatz conjecture can be constructed. Indeed, a tetration of a prime belongs to the sequence a_n if and only if its image under the Collatz map is a tetration of a prime.

Proposition 4. *Let p^p be a tetration of a prime. Then $a_n = p^p$ for some n if and only if $C(p^p) = q^q$ for some prime q.*

Proof. If $C(p^p) = q^q$ for some prime q, then

$$D(C(p^p)) = D(q^q) = q^q = C(p^p) = C(D(p^p)),$$

and we are done. Conversely, suppose p^p belongs to the sequence a_n. From Proposition 2, we know that $p \neq 2$. Then we have

$$D\left(\frac{3p^p+1}{2}\right) = D(C(p^p)) = C(D(p^p)) = \frac{3p^p+1}{2}. \tag{3}$$

We claim that the only positive fixed points of the arithmetic derivative are tetrations of primes. To see this, assume that $m = p_1^{\nu_1} \cdots p_k^{\nu_k}$ and let $D(m) = m$. It follows from (2) that

$$\sum_{j=1}^{k} \frac{\nu_j}{p_j} = 1,$$

and since all terms are non-negative, this tells us that $\nu_j \leq p_j$ for all j. Multiplying both sides by $p_1 \cdots p_k$ then produces

$$\sum_{j=1}^{k} \nu_j \prod_{i \neq j} p_i = p_1 \cdots p_k,$$

from which it follows that p_j divides ν_j. But since $\nu_j \leq p_j$, it follows that $\nu_j = 0$ or $\nu_j = p_j$. Of course, the latter cannot be true for more than one choice of j. Thus, the only positive fixed points are of the form q^q for some prime q, and it now follows from (3) that $C(p^p) = q^q$ for some prime q. □

Note that it is not known whether a prime tetration of the form in Proposition 4 exists, only that if it does exist, the connection outlined therein holds. Let us now dive deeper into our empirical results. By examining Table 1, we are led to believe that most values of a_n are squarefree semiprimes. It is therefore natural to wonder how the primes making up a_n are related to each other. In the next proposition, we give a bound for the difference between these two primes.

Proposition 5. *Let a_n be the sequence of positive integers for which $D(C(a_n)) = C(D(a_n))$. If $a_n = pq$ for some primes $q < p$, then*

$$p - q \geq 2\sqrt{30p^2 + 2} - 10p,$$

that is, the difference between the two primes grows at least linearly with the larger prime.

Proof. The proof is a simple calculation using Lemma 1. Note that neither p nor q is equal to 2, as n would then be even, contradicting Proposition 2. We can therefore assume that $m := p - q$ is even. Observe that

$$D(C(p(p-m))) = D\left(\frac{3p(p-m)+1}{2}\right)$$

and that

$$C(D(p(p-m))) = C(2p - m) = p - \frac{m}{2},$$

where we have used the fact that m is even in the last equality. If $\frac{3p(p-m)+1}{2}$ is prime, it follows that $m = 2(p-1)$. But this implies that $a_n = p(2-p)$, a contradiction. It therefore follows from Lemma 1 that

$$p - \frac{m}{2} \geq 2\sqrt{\frac{3p(p-m)+1}{2}}.$$

Squaring and rearranging the terms produces the inequality $20pm + m^2 - 8 \geq 20p^2$, and now factoring the left side gives

$$(m + 10p)^2 \geq 120p^2 + 8,$$

from which it follows that

$$m \geq 2\sqrt{30p^2 + 2} - 10p.$$

□

Proposition 5 eliminates several commonly paired primes. As every prime is obviously a distance zero from itself, squares of primes (A001248) cannot belong to $(a_n)_{n=1}^{\infty}$. The twin primes (A001359) differ from each other by two, and so their products are also eliminated, as are the products of the so-called cousin primes (A046132) which differ by four. Similarly, we are able to rule out products of Sophie Germain primes (A005384) with their corresponding safe prime; that is, products of the form $p(2p + 1)$ with both p and $2p + 1$ prime.

Corollary 6. *There does not exist an n such that a_n is a square of a prime, a product of twin primes, a product of cousin primes, or a product of a Sophie Germain prime and its corresponding safe prime.*

From Table 1, we see that when $D(a_n)$ is even, the known a_n are a product of two distinct primes, as are the corresponding $C(a_n)$. Moreover, when this is the case, we have $\frac{p+q}{2} = C(D(a_n)) = D(C(a_n)) = r + s$. The entries in the table are exhaustive in the sense that there are no additional elements of $(a_n)_{n=1}^{\infty}$ less than $a_{30} = 6022735799$. Using the observations just outlined, we can extend this list in a non-exhaustive manner with the following proposition.

Proposition 7. *Let p, q be primes such that both*

$$r_{\pm} = \frac{p + q \pm \sqrt{p^2 - 22pq + q^2 - 8}}{4}$$

are also prime. Then $C(pq) = r_+ r_-$ and pq is an element of $(a_n)_{n=1}^{\infty}$.

Proof. Observe that

$$\frac{3pq+1}{2} = \left(\frac{p + q + \sqrt{p^2 - 22pq + q^2 - 8}}{4}\right)\left(\frac{p + q - \sqrt{p^2 - 22pq + q^2 - 8}}{4}\right)$$

so that $C(pq) = r_+ r_-$. Moreover, $r_+ + r_- = \frac{p+q}{2}$, and it follows that $D(C(pq)) = D(r_+ r_-) = r_+ + r_- = \frac{p+q}{2} = C(D(pq))$. □

We call such semiprimes pq and $r_+ r_-$ *compatible*. Equivalently, pq and rs are compatible if r and s are the roots of the quadratic equation $2x^2 - (p + q)x + 3pq + 1 = 0$. In a computer search for compatible semiprimes, we uncovered the elements of $(a_n)_{n=1}^{\infty}$ listed in Table 2, which is so long that we have banished it to the appendix. Note that the position in the sequence n is not labeled since this search is non-exhaustive. To facilitate a better understanding of the solutions to the arithmetic differential equation $D\left(\frac{3n+1}{2}\right) = \frac{D(n)}{2}$ for n odd, let us now derive several modular restrictions on compatible semiprimes.

Proposition 8. *Suppose pq and rs are compatible semiprimes. Then $pq \equiv 2 \equiv rs \pmod 3$. It follows that either $p \equiv 1$ and $q \equiv 2$ or $p \equiv 2$ and $q \equiv 1 \pmod 3$, and likewise for r, s.*

Proof. Observe that $rs = \frac{3pq+1}{2} \equiv 2 \pmod 3$, independent of p, q. Without loss of generality, take $r \equiv 1$ and $s \equiv 2$. Then $p + q = 2(r + s) \equiv 0$. Since p, q are prime, this congruence can only be satisfied by $p \equiv 1$ and $q \equiv 2$ or the reverse. □

We make use of the following lemma.

Lemma 9. *Suppose pq and rs are compatible semiprimes. Then p, q, r, s are all odd.*

Proof. That p and q are odd follows from Proposition 2. Without loss of generality, suppose $s = 2$. Then $\frac{p+q}{2} = 2 + r = 2 + \frac{3pq+1}{4}$, from which it follows that $2p + 2q = 3pq + 9 > 3pq$. Without loss of generality, we may assume $p \geq q > 2$. Then we have $2p + 2q > 3pq > 6p > 2p + 2p \geq 2p + 2q$, a contradiction. □

Proposition 10. *Suppose pq is compatible to another semiprime. Then $pq \equiv 3 \pmod 4$. It follows that either $p \equiv 1$ and $q \equiv 3$ or $p \equiv 3$ and $q \equiv 1 \pmod 4$.*

Proof. Since p, q are prime, they can only be congruent to 1 or 3 mod 4. Otherwise, they are divisible by 2, which is ruled out by Proposition 2. If $p \equiv 1 \equiv q$, then $rs \equiv 0$ or 2. But then rs is divisible by 2, which contradicts Lemma 9. The case of $p \equiv 3 \equiv q$ produces a contradiction similarly. □

Using Proposition 10 and quadratic reciprocity, we see that either p and q are both quadratic residues modulo each other or neither of them is a quadratic residue modulo the other. Before addressing the residues modulo 10 in an attempt to explain the consistent pattern in our data, recall the following method for solving Diophantine equations of the form $ax+by+cxy = d$. Observe that this equation is equivalent to $(cx+b)(cy+a) = ab+cd$. Thus, if a, b, c, d are known, we can look for solutions by factoring $ab + cd$ in all possible ways and setting the factors equal to $cx + b$ and $cy + a$. Using this method, we obtain the following lemma.

Lemma 11. *Let pq and rs be compatible semiprimes with $p > q$. Then $r, s \geq 241$ and we have $3q < r, s < 3p$.*

Proof. From the definition of compatible semiprimes, we have

$$\frac{p+q}{2} = r + s = \frac{3pq+1}{2s} + s,$$

so that we come to a Diophantine equation of the form

$$sp + sq - 3pq = 2s^2 + 1.$$

Now applying the method mentioned above with $a = b = s$, $c = -3$, and $d = 2s^2 + 1$, we have

$$(s - 3p)(s - 3q) = -5s^2 - 3.$$

By factoring the right hand side for each choice of s, it can now be verified computationally that the first prime s satisfying this equality so that p, q, r are also prime is $s = 241$. Moreover, since $-5s^2 - 3 < 0$, we have that $3q < s < 3p$. By symmetry, this argument holds also for r. □

Let us now determine the possible congruences modulo 5, so that the congruences modulo 10 can be determined, thereby explaining the pattern we observe in our data.

Proposition 12. *Let pq and rs be compatible semiprimes. Then $p \equiv 1$ and $q \equiv 4$ or $q \equiv 1$ and $p \equiv 4 \pmod 5$. Similarly, $r \equiv 1$ and $s \equiv 4$ or $s \equiv 1$ and $r \equiv 4 \pmod 5$.*

Proof. Let us begin by noting that if a product of two primes is congruent to 5, then one of the primes is necessarily equivalent to 5. Thus, eliminating all such cases can be done by showing that neither prime is 5. Suppose for the sake of a contradiction that $q = 5$. Then we have

$$2(r+s) = p + 5 = \frac{2rs-1}{15} + 5,$$

and so it follows that $15(r+s) = rs + 37 > rs$. Thus, $\frac{1}{r} + \frac{1}{s} > \frac{1}{15}$. But by Lemma 11, we have $\frac{1}{r} + \frac{1}{s} \leq \frac{2}{241}$, a contradiction. Thus, by symmetry, $p, q \neq 5$. We now proceed by eliminating the remaining cases one by one.

If $pq \equiv 1$, then $rs \equiv 2$ and the possible congruences are therefore $(p,q) \equiv (1,1), (2,3), (4,4)$ and $(r,s) \equiv (1,2), (3,4)$, none of which satisfy $p+q = 2(r+s)$. If $pq \equiv 2$, then $rs \equiv 1$ and the possible congruences are therefore $(p,q) \equiv (1,2), (3,4)$ and $(r,s) \equiv (1,1), (2,3), (4,4)$, none of which satisfy $p+q = 2(r+s)$. If $pq \equiv 3$, then $rs \equiv 0$, which violates Lemma 11. Finally, if $pq \equiv 4$, then $rs \equiv 4$ and the possible congruences are therefore $(p,q) \equiv (1,4), (2,2), (3,3)$ and $(r,s) \equiv (1,4), (2,2), (3,3)$. The only ones satisfying $p+q = 2(r+s)$ are $(p,q) \equiv (1,4)$ and $(r,s) \equiv (1,4)$. □

We are now ready to give some justification for why all of the entries in Tables 1 and 2 are congruent to 9 modulo 10. The next proposition shows that this is always the case for compatible semiprimes, and so if all elements of $(a_n)_{n=1}^{\infty}$ are given by semiprimes compatible with another semiprime, then every entry is congruent to 9 modulo 10. Of course, not all elements of a_n are compatible semiprimes, given the exceptional case $n = 12$, but even if we restrict our attention to the case that $D(n)$ is even, it is not known that all remaining elements of $(a_n)_{n=1}^{\infty}$ are semiprimes compatible with another semiprime.

Proposition 13. *Let pq and rs be compatible semiprimes. Then $p \equiv 1$ and $q \equiv 9$ or $p \equiv 9$ and $q \equiv 1 \pmod{10}$. Similarly, $r \equiv 1$ and $s \equiv 9$ or $r \equiv 9$ and $s \equiv 1 \pmod{10}$.*

Proof. By Lemma 9, pq is odd, so there exists an n such that $pq = 1 + 2n$. On the other hand, $pq \equiv 4 \pmod 5$ by Proposition 12, so there exists an m such that $pq = 4 + 5m$. It follows that $2n = 3 + 5m$ and m is therefore odd. Thus, there is an integer k such that $m = 2k + 1$, and we have $pq = 4 + 5m = 9 + 10k \equiv 9 \pmod{10}$. Similarly, we have $rs \equiv 9 \pmod{10}$. The possible congruences are therefore $(p,q), (r,s) \equiv (1,9), (3,3), (7,7)$. The only choice satisfying $p + q = 2(r+s)$ is $(p,q) \equiv (1,9) \equiv (r,s)$. □

4 Sum of reciprocals of compatible semiprimes

It is not clear if there are infinitely many compatible semiprimes and therefore infinitely many solutions to our commutation problem. One method of proving that a given set has infinite cardinality is to show that a sum over that set diverges. For example, the sum of the reciprocals of the prime numbers diverges, showing that there are infinitely many prime numbers. Brun showed that the sum of the reciprocals of the twin primes actually converges, so that a similar argument cannot be made and the twin prime conjecture still remains unsolved. Let us prove a similar result related to our compatible semiprimes.

Theorem 14. *Let S denote the set of semiprimes compatible with another semiprime. Then the sum $\sum_{s\in S}\frac{1}{s}$ converges.*

Proof. We bound the number of semiprimes which are mapped to a semiprime by the Collatz map and use Abel's summation formula to show that the sum of the reciprocals of these numbers converges. It then follows that the sum over S converges, as S forms a subset of this set. Let x be large and fix a $y \le x$. For every semiprime $pq \le y$, define the set

$$S_{pq} := \left\{ n \le x : pq \mid n \text{ or } pq \mid \frac{3n+1}{2} \right\}.$$

Observe that if $n > y$ belongs to S_{pq} for some $pq \le y$, then at least one of n and $\frac{3n+1}{2}$ is not a semiprime. Indeed, since $pq \le y$ and $n > y$ by assumption, there must be a third prime factor in either case. It follows that the number of semiprimes $rs \le x$ such that $\frac{3rs+1}{2}$ is also semiprime is bounded above by the quantity

$$y + x - |\cup_{pq\le y} S_{pq}|.$$

Here the subtraction of the union of the sets S_{pq} removes those points $n > y$ that belong to S_{pq} since we have shown that they do not produce semiprimes under the Collatz mapping. However, it also removes points $n \le y$ which may satisfy this hypothesis, and so we make up for this with the addition of a factor of y. Now let ℓ be an even number. By the inclusion-exclusion principle, we have

$$|\cup_{pq\le y} S_{pq}| \ge \sum_{pq\le y} |S_{pq}| - \sum_{p_1q_1<p_2q_2\le y} |S_{p_1q_1} \cap S_{p_2q_2}| + \cdots - \sum_{p_1q_1<\cdots<p_\ell q_\ell \le y} |S_{p_1q_1} \cap \cdots \cap S_{p_\ell q_\ell}|. \quad (4)$$

We must therefore endeavor to estimate $|S_{p_1q_1} \cap \cdots \cap S_{p_jq_j}|$ for arbitrary $j = 1, \ldots, \ell$. Observe that $n \in S_{p_1q_1} \cap \cdots \cap S_{p_jq_j}$ if and only if $n \equiv 0$ or $\frac{3n+1}{2} \equiv 0 \pmod{p_iq_i}$ for every $i = 1, \ldots, j$. By Proposition 2, we may assume that $p_i, q_i \ne 2$ so that the latter case is equivalent to $3n + 1 \equiv 0 \pmod{p_iq_i}$, and rearranging produces $3n \equiv -1 \pmod{p_iq_i}$. By Proposition 8, we may assume $p_i, q_i \ne 3$ so that $\gcd(3, p_iq_i) = 1$. It follows that this congruence has a unique solution which can be obtained from Euler's theorem. Thus, for each i there are exactly two possible residues modulo p_iq_i. Now, if n satisfies one of the two possible congruences for every i, then so does $n + mp_1q_1 \cdots p_jq_j$ for every $m \in \mathbb{Z}$. There are at most $\lfloor \frac{x}{p_1q_1\cdots p_jq_j} \rfloor + 1$ and at least $\lfloor \frac{x}{p_1q_1\cdots p_jq_j} \rfloor$ values of m such that $n + mp_1q_1 \cdots p_jq_j \le x$ ($\lfloor x \rfloor$ denotes the integer part of x). Moreover, since the two congruences are chosen independently for each i, there are 2^j possible choices. Putting these observations together, it follows that there is a $\lambda_j \in [-1, 1]$ such that

$$|S_{p_1q_1} \cap \cdots \cap S_{p_jq_j}| = \frac{2^j x}{p_1q_1 \cdots p_jq_j} + 2^j \lambda_j.$$

Thus, we have

$$x - |\cup_{pq\le y} S_{pq}| \le x - \sum_{pq\le y}\left(\frac{2x}{pq} + 2\lambda_1\right) + \cdots + \sum_{p_1q_1<\cdots<p_\ell q_\ell\le y}\left(\frac{2^\ell x}{p_1q_1\cdots p_\ell q_\ell} + 2^\ell\lambda_\ell\right).$$

Letting σ_k denote that k-th symmetric sum over the set $\{\frac{2}{pq} : pq \le y\}$, we have

$$x - |\cup_{pq\le y} S_{pq}| \le x(1 - \sigma_1 + \sigma_2 - \cdots + \sigma_\ell) + y^\ell,$$

where we have used the fact that there are at most $(y/2)^\ell$ terms in (4), as well as the estimate $|2^j\lambda_j| \le 2^\ell$. We need the following estimate on the alternating sum of the symmetric sums:

$$1 - \sigma_1 + \sigma_2 - \cdots - \sigma_k \le \prod_{j=1}^{n}(1 - a_j) \le 1 - \sigma_1 + \sigma_2 - \cdots + \sigma_\ell$$

for all k odd and ℓ even (the k-th symmetric sum is over the set $\{a_1, \ldots, a_n\}$ here). To see this, observe that the inequalities are trivially true for $n = 1$ and proceed by induction on n. Since ℓ is even, we have in our case that

$$x - |\cup_{pq\le y} S_{pq}| \le x\left(\prod_{pq\le y}\left(1 - \frac{2}{pq}\right) + \sigma_\ell\right) + y^\ell.$$

Now, the product is easily bounded above. We have

$$\begin{aligned}\prod_{pq\le y}\left(1 - \frac{2}{pq}\right) &= \prod_{pq\le y}\left(\left(1 - \frac{1}{pq}\right)^2 - \frac{1}{(pq)^2}\right)\\ &\le \left(\prod_{pq\le y}\left(1 - \frac{1}{pq}\right)\right)^2\\ &= \left(\exp\left(\log\left(\prod_{pq\le y}\left(1 - \frac{1}{pq}\right)\right)\right)\right)^2,\end{aligned}$$

and now we proceed by estimating the logarithm. Using the complete additivity property, we have

$$\log\left(\prod_{pq\le y}\left(1 - \frac{1}{pq}\right)\right) = \sum_{pq\le y}\log\left(1 - \frac{1}{pq}\right) \le -\sum_{pq\le y}\frac{1}{pq}.$$

Let $\pi_2(x)$ denote the number of semiprimes no greater than x. Using Abel's summation formula, we can write the partial sum of the reciprocal semiprimes as

$$-\sum_{pq\le y}\frac{1}{pq} = -\frac{\pi_2(x)}{x}\bigg|_2^y - \int_2^y \frac{\pi_2(x)}{x^2}dx.$$

Ishmukhametov and Sharifullina showed that $\pi_2(x) = \sum_{k=1}^{\sqrt{x}} \lfloor \pi\left(\frac{x}{p_k}\right) - k + 1 \rfloor$ [10], but all we really need is the result of Landau that $\pi_2(y) \sim \frac{x \log(\log(x))}{\log(x)}$ [14], from which it follows that

$$\begin{aligned}
-\sum_{pq \le y} \frac{1}{pq} &\sim -\frac{\log(\log(y))}{\log(y)} + \frac{\log(\log(2))}{\log(2)} - \int_2^y \frac{\log(\log(x))}{x\log(x)} dx \\
&= -\frac{\log(\log(y))}{\log(y)} + \frac{\log(\log(2))}{\log(2)} - \frac{1}{2}(\log(\log(y)))^2 + \frac{1}{2}(\log(\log(2)))^2 \qquad (5) \\
&= -\frac{1}{2}(\log(\log(y)))^2 + O(1).
\end{aligned}$$

Thus, we have

$$\exp\left(\log\left(\prod_{pq \le y}\left(1 - \frac{1}{pq}\right)\right)\right) \le A\exp\left(-\frac{1}{2}(\log(\log(y)))^2\right)$$

for some constant A and it follows that

$$\prod_{pq \le y}\left(1 - \frac{2}{pq}\right) \le Ce^{-(\log(\log(y)))^2} \le \frac{C}{\log^m(y)}$$

for some constant C and for every integer $m > 0$. All that remains in our estimate is to bound the symmetric sum σ_ℓ. Observe that

$$\left(\sum_{pq \le y} \frac{2}{pq}\right)^\ell = \sum_{p_1q_1, p_2q_2, \ldots, p_\ell q_\ell \le y} \frac{2^\ell}{p_1q_1 \cdots p_\ell q_\ell} = \ell! \sum_{p_1q_1 < p_2q_2 < \cdots < p_\ell q_\ell \le y} \frac{2^\ell}{p_1q_1 \cdots p_\ell q_\ell} + \sum_{pq \le y} \frac{2^\ell}{(pq)^\ell},$$

where the factor of $\ell!$ is the number of ways to order the ℓ semiprimes. It follows that

$$\sigma_\ell \le \frac{1}{\ell!}\left(\sum_{pq \le y} \frac{2}{pq}\right)^\ell,$$

and from (5), we have

$$\sigma_\ell \le \frac{1}{\ell!}\left(D(\log(\log(y))\right)^2 + B)^\ell$$

for some constants D and B. Thus, using $\ell^\ell \ge \ell!$ and $y \le x$, it follows that

$$\begin{aligned}
\sigma_\ell &\le \left(\frac{1}{\ell}\left(D(\log(\log(y)))^2 + B\right)\right)^\ell \\
&\le \left(\frac{1}{\ell}\left(D(\log(\log(x)))^2 + B\right)\right)^\ell
\end{aligned}$$

Let π_s denote the number of semiprimes which are mapped to a semiprime under the Collatz map. Putting these bounds together, we see that

$$\pi_s(x) \le y + x - |\cup_{pq \le y} S_{pq}| < y + x\left(\frac{C}{\log^m(y)} + \left(\frac{1}{\ell}\left(D(\log(\log(x)))^2 + B\right)\right)^\ell\right) + y^\ell.$$

But y, m, and ℓ are free to be chosen except for the restriction $y \leq x$. We tame the behavior of the term containing B and D by choosing $\ell \approx 2\big(D(\log(\log(x)))^2 + B\big)$. Now choosing $y = x^{\frac{1}{D(\log(\log(x)))^2+B}}$, we see that

$$\pi_s(x) \leq \frac{Mx\big(D(\log(\log(x))\big)^2 + B)^m}{(\log(x))^m}. \tag{6}$$

for a constant M. Let us now show that the sum $\sum_{s\in C_s} \frac{1}{s}$ over the set C_s of all semiprimes which are mapped to a semiprime under the Collatz map converges. By the Abel summation formula, we have

$$\sum_{s\in C_s} \frac{1}{s} = \frac{\pi_s(x)}{x}\bigg|_2^\infty + \int_2^\infty \frac{\pi_s(x)}{x^2}dx.$$

Using (6), the boundary term vanishes at infinity and the integral converges for large enough m. Thus, the sum converges. Since the compatible semiprimes form a subset, we finally have that $\sum_{s\in S} \frac{1}{s}$ converges, and this completes the proof. □

5 Conclusion

We have studied a curious class of pairs of semiprime numbers which we call compatible. These numbers form solutions to the arithmetic differential equation

$$D\left(\frac{3n+1}{2}\right) = \frac{D(n)}{2}, \tag{7}$$

which is a special case of the problem of searching for numbers for which the arithmetic derivative and Collatz map commute. In this latter problem, we have found an exhaustive list of the first 30 numbers providing a solution, and all but one of them is a solution to the special case (7). The compatible semiprimes form additional solutions to (7) which therefore solve the commutation problem as well, although the list we provide in Table 2 is likely not exhaustive.

To close out this work, we provide the following conjectures worth studying in future work. Note that Conjecture 15 implies Conjecture 16 which implies Conjecture 17. The latter logical implication holds because there are no even n for which (7) holds. Indeed, such an n can be written $n = 2m$ so that (7) reduces to $2D(6m+1) = 4D(m) + 8m + 1$. Since the left hand side is even while the right hand side is odd, such an n cannot exist. Note also that Conjecture 18 implies Conjecture 19.

Conjecture 15. There are infinitely many compatible semiprimes.

Conjecture 16. There are infinitely many solutions to the arithmetic differential equation (7).

Conjecture 17. There are infinitely many solutions to the commutation problem $D(C(n)) = C(D(n))$.

Conjecture 18. With the exception of 606938385, the only solutions to the commutation problem $D(C(n)) = C(D(n))$ are given by compatible semiprimes.

Conjecture 19. Every solution to the commutation problem $D(C(n)) = C(D(n))$ is squarefree.

It should also be noted that similar problems can be formulated for the generalized Collatz functions

$$C_{a,b}(n) := \begin{cases} \frac{an+b}{2}, & \text{if } n \text{ odd;} \\ \frac{n}{2}, & \text{if } n \text{ even,} \end{cases}$$

with $a \equiv b \pmod 2$. An exhaustive brute force search for numbers $n \leq 10^7$ solving the analogous commutation problem produced no solutions for $a = 5, b = 1$, while the slight variant $a = 5, b = 3$ has the eleven solutions $n = 12419$, 20171, 37727, 134579, 199259, 301799, 574319, 866891, 1580291, 1625411, 8014031, each of which is a squarefree semiprime. On the other hand, the case $a = 7, b = 1$ has only the solution $n = 429$ in this range, and 429 is a product of three distinct prime factors. The slight variant $a = 7, b = 3$ produced no solutions and the variant $a = 7, b = 5$ has the seven solutions $n = 29831, 38051, 76331, 568031, 888971, 1855871, 3095711$, each of which is a squarefree semiprime consisting of the product of a prime congruent to 3 and a prime congruent to 7 modulo 10. In each of these examples, the solutions are squarefree almost-primes, and for this reason we make the following conjecture.

Conjecture 20. The solutions to the commutation problem $D(C_{a,b}(n)) = C_{a,b}(D(n))$ are squarefree for every $a \equiv b \pmod 2$.

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Appendix

pq	Prime Factorization	$p+q$	rs	Prime Factorization
6371331359	16561×384719	401280	9556997039	77801×122839
8975589239	9241×971279	980520	13463383859	29201×461059
10657078799	20071×530969	551040	15985618199	83059×192461
12860887439	6089×2112151	2118240	19291331159	18539×1040581
13170630899	8681×1517179	1525860	19755946349	26839×736091
15512319959	4111×3773369	3777480	23268479939	12401×1876339
20713686659	27179×762121	789300	31070529989	108631×286019
23664471179	11369×2081491	2092860	35496706769	35099×1011331
24180093239	26959×896921	923880	36270139859	100291×361649
27169948919	29209×930191	959400	40754923379	110339×369361
30617640719	9721×3149639	3159360	45926461079	29629×1550051
41203174319	39929×1031911	1071840	61804761479	167971×367949
44444955899	37831×1174829	1212660	66667433849	144289×462041
45265630259	34949×1295191	1330140	67898445389	125941×539129
59663684459	50839×1173581	1224420	89495526689	241261×370949
60235146119	51329×1173511	1224840	90352719179	364621×247799
71047305239	38231×1858369	1896600	106570957859	130279×818021
72269351939	56681×1275019	1331700	108404027909	382271×283579
74550954899	43291×1722089	1765380	111826432349	153319×729371
74797926239	56239×1330001	1386240	112196889359	435481×257639
90923314259	59141×1537399	1596540	136384971389	247729×550541
91551073739	28909×3166871	3195780	137326610609	91141×1506749
92153208659	34949×2636791	2671740	138229812989	113041×1222829
100918727099	44119×2287421	2331540	151378090649	148861×1016909
105138878699	63079×1666781	1729860	157708318049	603689×261241
105567827399	22441×4704239	4726680	158351741099	69019×2294321
108216566279	49201×2199479	2248680	162324849419	170111×954229
108868846199	67511×1612609	1680120	163303269299	534581×305479
129264961559	60271×2144729	2205000	193897442339	219619×882881
131559702959	47881×2747639	2795520	197339554439	159349×1238411
138217769399	48991×2821289	2870280	207326654099	162971×1272169
138868264319	78079×1778561	1856640	208302396479	379649×548671
150260888399	75209×1997911	2073120	225391332599	310379×726181
152376066599	38561×3951559	3990120	228564099899	122029×1873031
178661152559	39929×4474471	4514400	267991728839	125731×2131469
182554595819	72739×2509721	2582460	273831893729	1023751×267479

pq	Prime Factorization	$p+q$	rs	Prime Factorization
185904249719	85889 × 2164471	2250360	278856374579	368551 × 756629
192709329959	71471 × 2696329	2767800	289063994939	256369 × 1127531
203788963199	84551 × 2410249	2494800	305683444799	912349 × 335051
218613696239	51151 × 4273889	4325040	327920544359	164089 × 1998431
233568190619	67741 × 3447959	3515700	350352285929	229189 × 1528661
240001727519	102911 × 2332129	2435040	360002591279	711649 × 505871
260131119179	103699 × 2508521	2612220	390196678769	462571 × 843539
262091228279	85889 × 3051511	3137400	393136842419	313109 × 1255591
262593906359	37649 × 6974791	7012440	393890859539	116191 × 3390029
288964258199	36871 × 7837169	7874040	433446387299	113359 × 3823661
296122234379	41341 × 7162919	7204260	444183351569	127849 × 3474281
296955821639	104119 × 2852081	2956200	445433732459	1056479 × 421621
306450940139	79411 × 3859049	3938460	459676410209	270619 × 1698611
322643081699	88169 × 3659371	3747540	483964622549	309359 × 1564411
341144024159	60919 × 5599961	5660880	511716036239	194101 × 2636339
343031424959	124991 × 2744449	2869440	514547137439	724991 × 709729
364522651739	125219 × 2911081	3036300	546783977609	587579 × 930571
368068728719	116689 × 3154271	3270960	552103093079	1159201 × 476279
395384049899	115891 × 3411689	3527580	593076074849	1311619 × 452171
400481245739	85619 × 4677481	4763100	600721868609	286771 × 2094779
405113920919	135119 × 2998201	3133320	607670881379	706291 × 860369
412288472879	136351 × 3023729	3160080	618432709319	714529 × 865511
436306401359	55529 × 7857271	7912800	654459602039	172981 × 3783419
473023514939	143981 × 3285319	3429300	709535272409	697729 × 1016921
481853046599	132529 × 3635831	3768360	722779569899	536191 × 1347989
484957912139	123239 × 3935101	4058340	727436868209	1564081 × 465089
523044914519	131111 × 3989329	4120440	784567371779	1555999 × 504221
526841686199	138511 × 3803609	3942120	790262529299	560081 × 1410979
560607218879	136519 × 4106441	4242960	840910828319	527581 × 1593899
561826681079	154351 × 3639929	3794280	842740021619	709729 × 1187411
579247503719	157831 × 3670049	3827880	868871255579	740359 × 1173581
651630487619	102059 × 6384841	6486900	977445731429	336211 × 2907239
656472916739	143159 × 4585621	4728780	984709375109	539641 × 1824749
736241831879	129119 × 5702041	5831160	1104362747819	2468131 × 447449
750468526739	97381 × 7706519	7803900	1125702790109	3588229 × 313721
844661031299	196139 × 4306441	4502580	1266991546949	1134871 × 1116419
852127113419	142231 × 5991149	6133380	1278190670129	497509 × 2569181
852545495159	141601 × 6020759	6162360	1278818242739	494359 × 2586821
883609516259	171529 × 5151371	5322900	1325414274389	663331 × 1998119

pq	Prime Factorization	$p+q$	rs	Prime Factorization
1067201903519	210359×5073241	5283600	1600802855279	941461×1700339
1069842593219	107509×9951191	10058700	1604763889829	4686961×342389
1086684073439	220009×4939271	5159280	1630026110159	1106491×1473149
1125176937359	175391×6415249	6590640	1687765406039	634241×2661079
1179415384139	226199×5214061	5440260	1769123076209	1644061×1076069
1205649411119	227569×5297951	5525520	1808474116679	1697191×1065569
1248830314859	221021×5650279	5871300	1873245472289	937481×1998169
1255569121199	135119×9292321	9427440	1883353681799	4272959×440761
1255865342579	233509×5378231	5611740	1883798013869	1693501×1112369
1267060830239	184351×6873089	7057440	1900591245359	663281×2865439
1353561815339	225821×5993959	6219780	2030342723009	2177449×932441
1409845008239	244561×5764799	6009360	2114767512359	1125169×1879511
1424445760319	248071×5742089	5990160	2136668640479	1823051×1172029
1428658219199	209249×6827551	7036800	2142987328799	783599×2734801
1459820224439	256489×5691551	5948040	2189730336659	1633531×1340489
1477666902419	161641×9141659	9303300	2216500353629	538939×4112711
1539898047899	166919×9225421	9392340	2309847071849	558209×4137961
1618340838659	179369×9022411	9201780	2427511257989	607951×3992939
1815603367859	204311×8886469	9090780	2723405051789	710089×3835301
2180605168559	275521×7914479	8190000	3270907752839	1087631×3007369
2271244344419	285161×7964779	8249940	3406866516629	2982799×1142171
2283132057839	319321×7149959	7469280	3424698086759	1617949×2116691
2312261656139	299539×7719401	8018940	3468392484209	1262731×2746739
2515975240439	253769×9914431	10168200	3773962860659	902521×4181579
2525623096319	339161×7446679	7785840	3788434644479	1929971×1962949
2759705814599	339289×8133791	8473080	4139558721899	1528771×2707769
2798487028499	354469×7894871	8249340	4197730542749	1826761×2297909
2914249202819	355339×8201321	8556660	4371373804229	2591551×1686779
3015166657439	355361×8484799	8840160	4522749986159	1608769×2811311
3029547727979	371299×8159321	8530620	4544321591969	2195071×2070239
3142431670799	376631×8343529	8720160	4713647506199	1982741×2377339
3207163248719	355609×9018791	9374400	4810744873079	1517939×3169261
3342365758439	338609×9870871	10209480	5013548637659	3777541×1327199
3388989990119	363119×9333001	9696120	5083484985179	3314219×1533841
3434499230039	376679×9117841	9494520	5151748845059	3068159×1679101
3727388517539	374729×9946891	10321620	5591082776309	3613559×1547251
3894649427759	421009×9250751	9671760	5841974141639	2484721×2351159
4196759425979	430739×9743161	10173900	6295139138969	2126191×2960759

Table 2: Additional solutions to the commutation problem $D(C(n)) = C(D(n))$ given by compatible semiprimes.

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