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Rational Dyck Paths

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Abstract

Given a positive rational q, we consider Dyck paths of height at most two, subject to constraints on the number of consecutive peaks and consecutive valleys depending on q. We introduce a general class of Dyck paths, named rational Dyck paths, and provide the associated generating function based on their semilength, along with a construction for this class. Moreover, we characterize certain subsets of rational Dyck paths that are enumerated by the $\mathbb{Q}$-bonacci numbers.

1 Introduction

Baril et al. [4] introduced a new class of binary strings, called Fibonacci q-decreasing words, and enumerated by the Fibonacci generalized numbers. Barcucci et al. [2, 3] defined a new class of Dyck paths (q-*Dyck paths*, for short) with the same enumeration. More recently,

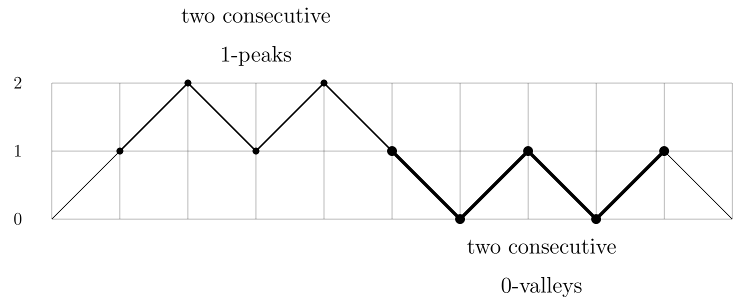

Figure 1: A path $P \in D_5$ with two consecutive 1-peaks and two consecutive 0-valleys.

Kirgizov introduced a class of words (the $\mathbb{Q}$-bonacci words [8]) larger than the one studied by Baril et al. [4] and enumerated by the $\mathbb{Q}$-bonacci numbers. Following a similar approach, the present authors defined [1] a class of Dyck paths (*Q-bonacci paths*, for short) larger than the one introduced by Barcucci et al. [2, 3] and enumerated by the $\mathbb{Q}$-bonacci numbers.

In this paper, we define a new class of Dyck paths, *rational Dyck paths*. We provide their construction and generating function (which turns out to be rational). We also present a very simple bijection between them and compositions of integers with parts in a set [7]. Moreover, since rational Dyck paths contain a subset of paths counted by the $\mathbb{Q}$-bonacci numbers, we give a bijection between this subset and a corresponding subset of paths in the set of Q-bonacci paths. Finally, we show that no bijection can exist between the remaining rational Dyck paths and the remaining Q-bonacci paths.

2 Definition

A Dyck path P is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with steps in $S = \{(1,1),(1,-1)\}$ that never passes below the x-axis. The step $(1,1)$ is abbreviated by the letter U and the step $(1,-1)$ by D, so that P can be represented by a string over the alphabet $\{U,D\}$.

The height of P is the maximum ordinate reached by its U steps. In this paper we consider the set $\mathcal{D}$ of Dyck paths with height at most 2.

The length of P is the number of its steps. We let D_n denote the subset of $\mathcal{D}$ consisting of Dyck paths with semilength n (i.e., the length divided by 2). The empty path, with semilength 0, is denoted by ε. Clearly, we have $\mathcal{D} = \bigcup_{n\geq 0} D_n$.

A *1-peak* of P is a substring UD of P where D attains ordinate 1, and a *0–valley* is a substring DU of P where D touches the x-axis. In Figure 1 an example is illustrated.

If L_1 and L_2 are two subsets of $\mathcal{D}$, then the concatenation $L_1 \cdot L_2$ is the set of all paths of the form PQ where $P \in L_1$ and $Q \in L_2$.

3 Construction and generating function

A path $P \in \mathcal{D}$ ($P \neq \varepsilon$) can be factorized by highlighting the blocks of maximal length of consecutive 1-peaks and consecutive 0-valleys as follows:

$$P = U(UD)^{p_1}(DU)^{v_1}(UD)^{p_2}(DU)^{v_2} \cdots (UD)^{p_k}(DU)^{v_k} D \ . \tag{1}$$

For clarity, the expression $(UD)^{p_j}$ (resp., $(DU)^{v_j}$) refers to the concatenation of p_j (resp., v_j) copies of (UD) (resp., (DU)). Moreover, the quantities p_j and v_j represent the number of 1-peaks and 0-valleys, respectively, occurring in the j-th block of maximal length that contains 1-peaks or 0-valleys. For example, the path $P = UUDUDDUDUDUUDDUDUD$ can be factorized as $P = U(UD)^2(DU)^3(UD)^1(DU)^2D$, where $k = 2, p_1 = 2, v_1 = 3, p_2 = 1, v_2 = 2$.

Clearly, in (1), it may hold that $p_1 = 0$ or $v_k = 0$ (or both). For example, in the path $P = UDUDUUDDUD = U(DU)^2(UD)^1(DU)^1D$ it is $k = 2, p_1 = 0, v_1 = 2, p_2 = 1, v_2 = 1$, while in $P = UUDUDD = U(UD)^2D$, we have $k = 1, p_1 = 2, v_1 = 0$. Finally, the path $P = UDUDUUDUDD = U(DU)^2(UD)^2D$ has $k = 2, p_1 = 0, v_1 = 2, p_2 = 2, v_2 = 0$. It may also happen that $k = 0$. In this case, the path P is UD.

Let $q = r/s$ be a positive rational number ($q \in \mathbb{Q}^+$), where the integers r and s are coprime.

Definition 1. Let $R_n^{r/s}$ be the set of Dyck paths $P \in \mathcal{D}$ of semilength n where

$$\frac{p_i}{v_i} \leq \frac{r}{s} \text{ or, equivalently, } v_i \geq \left\lceil p_i \cdot \frac{s}{r} \right\rceil$$

for $i = 1, 2, \ldots, k$.

Moreover, let $\mathcal{R}^{r/s}$ be the class defined as follows:

$$\mathcal{R}^{r/s} = \bigcup_{n \geq 0} R_n^{r/s}.$$

We refer to $\mathcal{R}^{r/s}$ as the class of *rational Dyck paths*.

In other words, if a rational Dyck paths P contains p consecutive 1-peaks, then they must be immediately followed by at least $\lceil p \cdot \frac{s}{r} \rceil$ consecutive 0-valleys. Moreover, in its factorization (1), we cannot have $v_k = 0$.

Example 2. If $q = 3/4$ and

$$P_1 = UUDDUDUUDUDDUDUD = U(UD)(DU)^2(UD)^2(DU)^2D,$$

then $k = 2, p_1 = 1, v_1 = 2, p_2 = 2, v_2 = 2$. According to Definition 1, we must have $v_2 \geq 3$. Therefore $P_1 \notin \mathcal{R}^{3/4}$.

The path

$$P_2 = UUDDUDUUDUDUDUDD = U(UD)(DU)^2(UD)^4D$$

is not allowed ($P_2 \notin \mathcal{R}^{3/4}$) since in this case $v_2 = 0$ (there are no valleys after the last block of 1-peaks).

The path

$$P_3 = UUDDUDUUDUDDUDUDUDUD = U(UD)(DU)^2(UD)^2(DU)^4D$$

is allowed.

A path P, different from the empty path ε $\big(P \in \{\mathcal{R}^{r/s} \backslash \varepsilon\}\big)$, can be recursively constructed as follows:

- $P = UD \cdot P'$, where $P' \in \mathcal{R}^{r/s}$ (possibly $P' = \varepsilon$), either
- $P = U(UD)^p(DU)^{\lceil ps/r \rceil - 1}D \cdot P'$, where $P' \in \{\mathcal{R}^{r/s} \setminus \varepsilon\}$, and $p \geq 1$.

Note that in the second case we must have $P' \neq \varepsilon$. For otherwise in the path P the p consecutive 1-peaks of the prefix $U(UD)^p(DU)^{\lceil ps/r \rceil - 1}D$ would not be followed by the correct number of consecutive 0-valleys. This number has to be $\lceil ps/r \rceil$, according to Definition 1. Observing that $\lceil ps/r \rceil - 1$ consecutive 0-valleys occur in the prefix itself, the missing one is given by the last D step of the prefix, followed by the first U step of P'.

Therefore, the construction of rational Dyck paths can be expressed by

$$\mathcal{R}^{r/s} = \varepsilon \ \cup \ UD \cdot \mathcal{R}^{r/s} \bigcup_{p \geq 1} U(UD)^p(DU)^{\lceil ps/r \rceil - 1}D \cdot \{\mathcal{R}^{r/s} \setminus \varepsilon\}. \tag{2}$$

We let $\delta^{r/s}(x) = \sum_{P \in \mathcal{R}^{r/s}} x^{|P|}$ denote the generating function of $\mathcal{R}^{r/s}$, according to the semilength of the paths, where $|P|$ denotes the semilength of P. From (2), it is possible to deduce the functional equation for the generating function $\delta^{r/s}(x)$, which is

$$\delta^{r/s}(x) = 1 + x\delta^{r/s}(x) + \sum_{p \geq 1} x^{p + \lceil ps/r \rceil} \big(\delta^{r/s}(x) - 1\big). \tag{3}$$

In Eq. (3), the sum $\sum_{p \geq 1} x^{p + \lceil ps/r \rceil}$ can be manipulated so that the functional equation simplifies to

$$\begin{aligned}
\delta^{r/s}(x) &= 1 + x\delta^{r/s}(x) + \big(\delta^{r/s}(x) - 1\big) \sum_{k=1}^{r} x^{k + \lceil ks/r \rceil} \sum_{j \geq 0} x^{(r+s)j} \\
&= 1 + x\delta^{r/s}(x) + \big(\delta^{r/s}(x) - 1\big) \cdot \frac{\sum_{k=1}^{r} x^{k + \lceil ks/r \rceil}}{1 - x^{r+s}} \ .
\end{aligned} \tag{4}$$

Thus, the generating function is

$$\delta^{r/s}(x) = \frac{1 - x^{r+s} - \sum_{k=1}^{r} x^{k+\lceil ks/r\rceil}}{(1-x)(1-x^{r+s}) - \sum_{k=1}^{r} x^{k+\lceil ks/r\rceil}}$$
$$= \frac{1 - \sum_{k=1}^{r-1} x^{k+\lceil ks/r\rceil} - 2x^{r+s}}{1 - x - \sum_{k=1}^{r-1} x^{k+\lceil ks/r\rceil} - 2x^{r+s} + x^{r+s+1}}. \tag{5}$$

From (2) and letting $\emptyset$ denote the empty set, it is not difficult to realize that the generation of the sets $R_n^{r/s}$ is given by

$$R_n^{r/s} = \begin{cases} \varepsilon, & \text{if } n = 0; \\ UD \cdot R_{n-1}^{r/s} \bigcup_{p\geq 1} \{Y_{n,p}\}, & \text{if } n \geq 1, \end{cases} \tag{6}$$

where

$$Y_{n,p} = \begin{cases} U(UD)^p(DU)^{\lceil ps/r\rceil - 1} D \cdot R_{n-p-\lceil ps/r\rceil}, & \text{if } n - p - \lceil ps/r\rceil \geq 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, denoting by $w_n^{r/s}$ the cardinality of $R_n^{r/s}$, the recurrence relation for $w_n^{r/s}$ is

$$w_n^{r/s} = \begin{cases} 1, & \text{if } n = 0; \\ w_{n-1}^{r/s} + \sum_{p\geq 1} u_{n,p}, & \text{if } n \geq 1, \end{cases} \tag{7}$$

where

$$u_{n,p} = \begin{cases} w_{n-p-\lceil ps/r\rceil}, & \text{if } n - p - \lceil ps/r\rceil \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

4 Links with other structures

If $r = s = 1$, then the generating function (5) becomes

$$\delta(x) = \frac{1 - 2x^2}{1 - x - 2x^2 + x^3}$$

which corresponds to the sequence $1, 1, 1, 2, 3, 6, 10, 19, 33, \ldots$. This is a shifted version of the sequence A028495 in The On-line Encyclopedia of Integer Sequences [10]. It enumerates the compositions of n with parts in $A = \{1, 2, 4, 6, \ldots\}$, according to their length. See Heubach et al. [7] and Baril et al. [5] for more information.

On closer inspection, our sets $R_{n+1}^{r/s}$ are in bijection with the sets of compositions $C_n^{A_{r/s}}$ of n with parts in the set

$$A_{r/s} = \{1\} \cup \{p + \lceil ps/r \rceil \mid p \geq 1\}.$$

The bijection is achieved by mapping the part 1 to UD, and the part $p + \lceil ps/r \rceil$ to $U(UD)^p(DU)^{\lceil ps/r \rceil - 1}D$. Finally, we add the factor UD at the end of the resulting Dyck path. For example, if $q = 3/4$, then $A_{3/4} = \{1, 3, 5, 7, 10, 12, 14, \ldots\}$ and the correspondence between $C_n^{A_{3/4}}$ and $R_{n+1}^{3/4}$ can be seen in Table 1.

	$C_n^{A_{3/4}}$	$R_{n+1}^{3/4}$
$n = 0$	ε	UD
$n = 1$	1	$UDUD$
$n = 2$	11	$UDUDUD$
$n = 3$	111, 3	$UDUDUDUD$, $UUDDUDUD$
$n = 4$	1111 13 31	$UDUDUDUDUD$ $UDUUDDUDUD$ $UUDDUDUDUD$
$n = 5$	11111 113 131 311 5	$UDUDUDUDUDUD$ $UDUDUUDDUDUD$ $UDUUDDUDUDUD$ $UUDDUDUDUDUD$ $UUDUDDUDUDUD$
$\cdots$	$\cdots$	$\cdots$

Table 1: $C_n^{A_{3/4}}$ and $R_{n+1}^{3/4}$

In a recent paper by Kirgizov [8] the concept of q-decreasing words, originally introduced for $q \in \mathbb{N}^+$ [4], has been generalized to the case where $q \in \mathbb{Q}^+$. If $q \in \mathbb{N}^+$, the q-decreasing words are enumerated [4] by the well-known q-generalized Fibonacci numbers [6, 9], whereas if $q \in \mathbb{Q}^+$, they are enumerated by the so-called $\mathbb{Q}$-bonacci numbers [8].

For certain values $q = r/s$, the n-length q-decreasing words are enumerated by sequences that, when properly filled with 1's at the beginning, coincide with sequences enumerating restricted compositions of n with a finite number of parts, depending on r/s. For example, the sequences A060961 and A117760 in The On-line Encyclopedia of Integer Sequences [10], corresponding to the values $q = 2/3$ and $q = 3/4$, enumerate the q-decreasing words and the compositions of n with parts in $\{1, 3, 5\}$ and $\{1, 3, 5, 7\}$, respectively.

More precisely, the involved compositions are the ones where the summands of n belong to the finite set

$$\widetilde{A}_{r/s} = \{1\} \cup \{p + \lceil ps/r \rceil \mid 1 \leq p \leq r\}.$$

This particular context, where restricted compositions, Dyck paths, and q-decreasing words appear to be connected in an intriguing way, prompted us to investigate the set $\mathcal{R}^{r/s}$ in order to

- find a subset of Dyck paths in bijection with $C_n^{\widetilde{A}_{r/s}}$;
- find a subset of Dyck paths with the same enumeration of the q-decreasing words according to the semilength, for $q \in \mathbb{Q}^+$.

4.1 Dyck paths and restricted compositions

Referring again to the factorization (1)

$$P = U(UD)^{p_1}(DU)^{v_1}(UD)^{p_2}(DU)^{v_2}\cdots(UD)^{p_k}(DU)^{v_k}D$$

for Dyck paths $P \in \mathcal{D}$, we impose a constraint on the number of consecutive 1-peaks within each block.

Definition 3. Let $\widetilde{R}_n^{r/s}$ be the set of Dyck paths $P \in \mathcal{D}$ of semilength n where

- $p_i \leq r$ for $i = 1, 2, \ldots, k$;
- for $i = 1, 2, \ldots, k$, the following condition must hold:

$$\frac{p_i}{v_i} \leq \frac{r}{s} \text{ or equivalently } v_i \geq \left\lceil p_i \cdot \frac{s}{r} \right\rceil .$$

Moreover, let $\widetilde{R}^{r/s}$ be the class defined as follows:

$$\widetilde{\mathcal{R}}^{r/s} = \bigcup_{n\geq 0} \widetilde{R}_n^{r/s}.$$

All the steps presented for $\mathcal{R}^{r/s}$ in Section 3 can be easily adapted to Definition 3. More precisely:

- the construction of $\widetilde{\mathcal{R}}^{r/s}$ can be described by

$$\widetilde{\mathcal{R}}^{r/s} = \varepsilon \ \cup \ UD \cdot \widetilde{\mathcal{R}}^{r/s} \ \bigcup_{p=1}^{r} U(UD)^p(DU)^{\lceil ps/r \rceil - 1}D \cdot \{\widetilde{\mathcal{R}}^{r/s} \setminus \varepsilon\};$$

- the corresponding functional equation for the generating function $\widetilde{\delta}^{r/s}(x)$ of $\widetilde{\mathcal{R}}^{r/s}$ is

$$\widetilde{\delta}^{r/s}(x) = 1 + x\widetilde{\delta}^{r/s}(x) + \sum_{p=1}^{r} x^{p+\lceil ps/r \rceil}\left(\widetilde{\delta}^{r/s}(x) - 1\right);$$

- the generating function is

$$\widetilde{\delta}^{r/s}(x) = \frac{1 - \sum_{p=1}^{r} x^{p+\lceil ps/r \rceil}}{1 - x - \sum_{p=1}^{r} x^{p+\lceil ps/r \rceil}}; \tag{8}$$

- the generation of the sets $\widetilde{R}_n^{r/s}$ is given by

$$\widetilde{R}_n^{r/s} = \begin{cases} \varepsilon, & \text{if } n = 0; \\ UD \cdot \widetilde{R}_{n-1}^{r/s} \bigcup_{p=1}^{r} \{\widetilde{Y}_{n,p}\}, & \text{if } n \geq 1, \end{cases}$$

where

$$\widetilde{Y}_{n,p} = \begin{cases} U(UD)^p(DU)^{\lceil ps/r \rceil - 1} D \cdot \widetilde{R}_{n-p-\lceil ps/r \rceil}, & \text{if } n - p - \lceil ps/r \rceil \geq 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, denoting by $\widetilde{w}_n^{r/s}$ the cardinality of $\widetilde{R}_n^{r/s}$, the recurrence relation for $\widetilde{w}_n^{r/s}$ is

$$\widetilde{w}_n^{r/s} = \begin{cases} 1, & \text{if } n = 0; \\ \widetilde{w}_{n-1}^{r/s} + \sum_{p=1}^{r} \widetilde{u}_{n,p}, & \text{if } n \geq 1, \end{cases} \tag{9}$$

where

$$\widetilde{u}_{n,p} = \begin{cases} \widetilde{w}_{n-p-\lceil ps/r \rceil}^{r/s}, & \text{if } n - p - \lceil ps/r \rceil \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

The bijection between the sets $C_n^{\widetilde{A}_{r/s}}$ and $\widetilde{R}_{n+1}^{r/s}$ is the same as the one between the sets $C_n^{A_{r/s}}$ and $R_{n+1}^{r/s}$, defined at the beginning of Section 4. Here, we note that the set $\widetilde{A}_{r/s}$, which contains the parts for the composition of n, is finite. This corresponds to the fact that the paths $P \in \widetilde{R}_n^{r/s}$ cannot have more than r consecutive 1-peaks. Moreover, due to the bijection, we observe that $\widetilde{w}_n^{r/s}$ represents the number of composition of $n-1$ for $n \geq 1$, with parts in $\widetilde{A}^{r/s}$. This is the reason why the sequence defined by (9) does not exactly match the sequence enumerating the compositions of n with parts in $\widetilde{A}^{r/s}$. Specifically, the sequence $(\widetilde{w}_n^{r/s})_{n\geq 0}$ is obtained by the one enumerating the compositions by inserting a 1 at the beginning. Indeed, the generating function

$$\frac{1}{1 - \sum_{\ell \in \widetilde{A}^{r/s}} x^{\ell}}$$

for the compositions of n with parts in $\widetilde{A}^{r/s}$ [7] is obtained as $\widetilde{\delta}^{r/s}(x) - 1$.

4.2 Dyck paths and q-decreasing words

Here, we consider the family of Dyck paths that have the same enumeration (according to their semilength) of q-decreasing words, as generalized by Kirgizov [8].

Recalling the factorization (1)

$$P = U(UD)^{p_1}(DU)^{v_1}(UD)^{p_2}(DU)^{v_2}\cdots(UD)^{p_k}(DU)^{v_k}D$$

for Dyck paths $P \in \mathcal{D}$, we maintain the restriction on the number of consecutive 1-peaks and relax the assumption on v_k.

Definition 4. Let $Q_n^{r/s}$ be the set of Dyck paths $P \in \mathcal{D}$ of semilength n where

- $p_i \le r$ for $i = 1, 2, \ldots, k$;
- for $i = 1, 2, \ldots, k-1$, the following condition must hold:

$$\frac{p_i}{v_i} \le \frac{r}{s} \qquad \text{or equivalently } v_i \ge \left\lceil p_i \cdot \frac{s}{r} \right\rceil;$$

- for $i = k$
  - if $p_k = r$, then $v_k \ge 0$;
  - if $p_k < r$, then either $v_k = 0$ or $v_k \ge \left\lceil p_k \cdot \frac{s}{r} \right\rceil$.

Moreover, let $\mathcal{Q}^{r/s}$ be the class defined as follows:

$$\mathcal{Q}^{r/s} = \bigcup_{n\ge 0} Q_n^{r/s}.$$

We refer to $Q_n^{r/s}$ as the class of *Q-bonacci paths.*

In other words, in a path $P \in Q_n^{r/s}$, a block B of p consecutive 1-peaks must be followed by at least $\lceil p \cdot \frac{s}{r} \rceil$ consecutive 0-valleys. The only case in which B can be followed by less than $\lceil p \cdot \frac{s}{r} \rceil$ consecutive 0-valleys is when B is the last block and $B = (UD)^r$. Moreover, the p consecutive 1-peaks in each B cannot exceed r.

Example 5. If $q = 4/5$ and

$$P_1 = UUDDUDUUDUDDUDUD = U(UD)(DU)^2(UD)^2(DU)^2D,$$

then $k = 2, p_1 = 1, v_1 = 2, p_2 = 2, v_2 = 2$. According to Definition 4, it should be $v_2 \ge 3$. Therefore $P_1 \notin \mathcal{Q}^{4/5}$.

The path

$$P_2 = UUDDUDUUDUDUDUDDUDUD = U(UD)(DU)^2(UD)^4(DU)^2D$$

is allowed ($P_2 \in \mathcal{Q}^{4/5}$) since in this case $p_2 = 4 = r$, so that the only requirement is $v_2 \geq 0$.

The path

$$P_3 = UUDDUDUUDUDUDD = U(UD)(DU)^2(UD)^3D$$

is allowed since $p_2 = 3 < r$ and $v_2 = 0$.

These Dyck paths were introduced by Barcucci et al. [1], who presented them with a different but equivalent approach. After defining them, and following a similar argument to that used in Section 3, we recall only the key points in order to arrive at the enumerating sequences.

- The construction of $\mathcal{Q}^{r/s}$ can be summarized by

$$\mathcal{Q}^{r/s} = \varepsilon \ \cup \ UD \cdot \mathcal{Q}^{r/s} \ \bigcup_{\ell=1}^{r+s-1} U\,\mathrm{pr}_\ell\big((UD)^r(DU)^{s-1}\big)D$$
$$\bigcup_{p=1}^{r} U(UD)^p(DU)^{\lceil ps/r\rceil-1}D \cdot \{\mathcal{Q}^{r/s} \setminus \varepsilon\},$$

  where, if A is a path, then $\mathrm{pr}_\ell(A)$ denotes the prefix of semilength ℓ of A.

  Specifically, a path $P \in \mathcal{Q}^{r/s}$, as in the case where $P \in \widetilde{\mathcal{R}}^{r/s}$ (Section 4.1), starts with the factor UD, or one of the factors $U(UD)^p(DU)^{\lceil ps/r\rceil-1}D$ with $p = 1, 2, \ldots, r$, concatenated with a suitable path in $\mathcal{Q}^{r/s}$. In this case ($P \in \mathcal{Q}^{r/s}$), the path P can also be a prefix of $(UD)^p(DU)^{\lceil ps/r\rceil-1}$, preceded by a step U and followed by a step D.

- The functional equation for the generating function $\chi^{r/s}(x)$ of $\mathcal{Q}^{r/s}$ is

$$\chi^{r/s}(x) = 1 + x\chi^{r/s}(x) + \sum_{j=2}^{r+s} x^j \ + \sum_{p=1}^{r} x^{p+\lceil ps/r\rceil}\big(\chi^{r/s}(x) - 1\big),$$

  where the last sum tracks the paths given by the prefixes of $(UD)^r(DU)^{s-1}$, preceded by U and followed by D.

- The generating function is

$$\chi^{r/s}(x) = \frac{1 + \displaystyle\sum_{j=2}^{r+s} x^j - \sum_{p=1}^{r} x^{p+\lceil ps/r\rceil}}{1 - x - \displaystyle\sum_{p=1}^{r} x^{p+\lceil ps/r\rceil}}.$$

- The generation of the sets $Q_n^{r/s}$ is given by

$$Q_n^{r/s} = \begin{cases} \varepsilon, & \text{if } n = 0; \\ UD, & \text{if } n = 1; \\ UD \cdot Q_{n-1}^{r/s} \cup U\,\mathrm{pr}_{n-1}\big((UD)^r (DU)^{s-1}\big) D \bigcup_{p=1}^{r} \{S_{n,p}\}, & \text{if } n \geq 2, \end{cases}$$

where

$$S_{n,p} = \begin{cases} U(UD)^p (DU)^{\lceil ps/r \rceil - 1} D \cdot Q_{n-p-\lceil ps/r \rceil}, & \text{if } n - p - \lceil ps/r \rceil \geq 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence, denoting by $v_n^{r/s}$ the cardinality of $Q_n^{r/s}$, the recurrence relation for $v_n^{r/s}$ is

$$v_n^{r/s} = \begin{cases} 1, & \text{if } n = 0; \\ v_{n-1}^{r/s} + \sum_{p=1}^{r} t_{n,p}, & \text{if } n \geq 1, \end{cases} \tag{10}$$

where

$$t_{n,p} = \begin{cases} v_{n-p-\lceil ps/r \rceil}^{r/s}, & \text{if } n - p - \lceil ps/r \rceil \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Denoting by $W_{r/s}(x)$ the generating function for the q-decreasing words as defined in [8], it is routine to check that

$$\chi^{r/s}(x) = 1 + xW_{r/s}(x) \ ,$$

so that the sequences $\left(v_n^{r/s}\right)_{n\geq 0}$ are obtained by inserting a 1 at the beginning of the sequences enumerating the q-decreasing words. Hence, all the sequences obtained in [8] have a different combinatorial interpretation in terms of Q-bonacci paths.

4.3 The classes $\mathcal{Q}^{r/s}$ and $\widetilde{\mathcal{R}}^{r/s}$

In this section, we prove that, for certain cases, some subsets of $\mathcal{Q}^{r/s}$ and $\widetilde{\mathcal{R}}^{r/s}$ are in bijection. More precisely, we state the following proposition.

Proposition 6. *If $s = tr + 1$ with $t \in \mathbb{N}$, then there exists a bijection $\phi : Q_n^{r/s} \to \widetilde{R}_{n+t+1}^{r/s}$, for $n \geq 0$.*

Proof. Let P be a path such that $P \in Q_n^{r/s}$. The general idea to obtain the corresponding path $\phi(P)$ in $\widetilde{R}_{n+t+1}^{r/s}$ is to add the suffix $(UD)^{t+1}$ (so that $\phi(P)$ ends with at least $t+1$ consecutive 0-valleys and its semilength is exactly $n+t+1$) and, eventually, replace a suitable number of the rightmost p_k consecutive 1-peaks of P with 0-valleys.

Referring to factorization (1), if $p_k = j$, let ν_j be the quantity $\nu_j = \lceil j \cdot \frac{s}{r} \rceil$, so that $v_k \geq \nu_j$, according to Definition 3. It is easy to check that

$$\nu_j = jt + 1, \tag{11}$$

being $1 \leq j \leq r$.

We consider the following cases:

- If $P \in \widetilde{R}_n^{r/s} (\subseteq Q_n^{r/s})$, then it is easily seen that $\phi(P) = P(UD)^{t+1} \in \widetilde{R}_{n+t+1}^{r/s}$;

- If $P = \alpha U(UD)^j D \in Q_n^{r/s}$, where $\alpha \in \widetilde{R}_{n-j-1}^{r/s}$, then the path $\phi(P)$ can be expressed by $\phi(P) = \alpha U(UD)^{j-h}(DU)^h D(UD)^{t+1}$ with h is the minimum integer such that $h+t+1 \geq \nu_{j-h} = t(j-h)+1$, taking into account formula (11). Hence $h = \lceil \frac{jt+1}{t+1} - 1 \rceil$.

  Suppose there exists $P' \in \widetilde{R}_n^{r/s}$ such that $\phi(P) = \phi(P')$. Then we can write $P' = \alpha U(UD)^{j-h}(DU)^h D$. According to formula (11), we have $h \geq (j-h)t + 1$, which implies $h \geq \lceil \frac{jt+1}{t+1} \rceil$. This leads to the inequality $h > \lceil \frac{jt+1}{t+1} - 1 \rceil$, against the preceding value of h.

- If $P = \alpha U(UD)^r (DU)^j D$, with $1 \leq j \leq s-1$, then we distinguish two cases:

  - if $j \geq s-t-1$, then $\phi(P) = P(UD)^{t+1} \in \widetilde{R}_{n+t+1}^{r/s}$;
  - if $1 \leq j < s-t-1$, then $\phi(P) = \alpha U(UD)^{r-h}(DU)^{j+h} D(UD)^{t+1}$, with h is the minimum integer such that $j+h+t+1 \geq \nu_{r-h} = (r-h)t+1$ thanks to (11), hence $h = \lceil \frac{rt-j+1}{t+1} - 1 \rceil$.

  By using similar arguments described in the previous bullet, one can show that there does not exists $P' \in \widetilde{R}_n^{r/s}$ such that $\phi(P) = \phi(P')$.

The inverse map ϕ^{-1} can be obtained by similar considerations. Given $P \in \widetilde{R}_{n+t+1}^{r/s}$ of the form $P = \alpha U(UD)^j (DU)^h D$, with $h \geq \nu_j \geq t+1$ and $\alpha \in \widetilde{R}_{n+t-j-h}^{r/s}$, we consider the following cases:

- if $h-(t+1) \geq \nu_j$, then $\phi^{-1}(P) = \alpha U(UD)^j (DU)^{h-t-1} D$, which belongs to $\widetilde{R}_n^{r/s} (\subseteq Q_n^{r/s})$;

- if $h-(t+1) < \nu_j$ and $j+h-t-1 \leq r$, then $\phi^{-1}(P) = \alpha U(UD)^{j+h-t-1} D$, which belongs to $Q_n^{r/s}$;

- if $h-(t+1) < \nu_j$ and $j+h-t-1 > r$, then $\phi^{-1}(P) = \alpha U(UD)^r (DU)^{j+h-t-1-r} D$, which belongs to $Q_n^{r/s}$.

It is routine to show that $\phi(\phi^{-1}(P)) = P$. The map ϕ is the required bijection. Note that, the hypothesis $s = tr+1$ is not used in the definition of ϕ^{-1}. □

If $s \neq tr+1$, then $tr+1 < s < (t+1)r$. In this case, we have $t \leq \nu_{j+1} - \nu_j \leq t+1$ and of course, there exists j such that $\nu_{j+1} + \nu_j = t+1$. Let j_0 be the smallest integer such that $\nu_{r-j_0} - \nu_{r-(j_0+1)} = t+1$. We have $\nu_{r-j_0} = s - tj_0$ and $\nu_{r-(j_0+1)} = s - t(j_0+1) - 1$. We now show that there exist two paths, $P, P' \in \mathcal{Q}_n^{r/s}$, such that $\phi(P) = \phi(P')$. Let P, P' be the paths

$$P = \alpha U(UD)^r (DU)^{s-(t+1)(j_0+1)-1} D \in Q_n^{r/s} \setminus \widetilde{R}_n^{r/s}$$

and

$$P' = \alpha U(UD)^{r-(j_0+1)} (DU)^{s-t(j_0+1)-1} D \in \widetilde{R}_n^{r/s}.$$

By applying the map ϕ, we obtain $\phi(P) = \phi(P') = \alpha U(UD)^{r-(j_0+1)}(DU)^{s-t(j_0+1)+t} D$. Hence, the map ϕ is not injective, but it is surjective. Therefore, the sets $\widetilde{R}_{n+t+1}^{r/s}$ and $Q_n^{r/s}$ have different cardinalities, and the following proposition is proved.

Proposition 7. *The sets $\widetilde{R}_{n+t+1}^{r/s}$ and $Q_n^{r/s}$ are in bijection if and only if $s = tr+1$.*

We point out that in the class of Q-bonacci paths, there are rational Dyck paths. These are exactly the paths in the class $\widetilde{\mathcal{R}}^{r/s}$. In other words

$$\mathcal{R}^{r/s} \cap \mathcal{Q}^{r/s} = \widetilde{\mathcal{R}}^{r/s} \ .$$

5 Further work

In our opinion, there are several open questions that deserve to be addressed. One of the most intriguing is the fact that Q-bonacci paths in [1] and $\mathbb{Q}$-bonacci words in [8] are both enumerated by $\mathbb{Q}$-bonacci numbers. Nevertheless, an explicit bijection between the mentioned classes has not yet been defined.

A second one could deal with the definition of a Gray code for rational Dyck paths or/and Q-bonacci paths.

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References

[1] E. Barcucci, A. Bernini, S. Bilotta, and R. Pinzani, Dyck paths enumerated by the $\mathbb{Q}$-bonacci numbers, in S. Brlek and L. Ferrari, eds., *Proc. GASCom 2024*, Electron. Proc. Theor. Comput. Sci. (EPTCS), Vol. 403, 2024, pp. 49–53.

[2] E. Barcucci, A. Bernini, and R. Pinzani, From the Fibonacci to Pell numbers and beyond via Dyck paths, *Pure Math. Appl.* **30** (2022), 17–22.

[3] E. Barcucci, A. Bernini, and R. Pinzani, Sequences from Fibonacci to Catalan: A combinatorial interpretation via Dyck paths, *RAIRO Theor. Inform. Appl.* **58** (2024), #8.

[4] J.-L. Baril, S. Kirgizov, and V. Vajnovszki, Gray codes for Fibonacci q-decreasing words, *Theoret. Comput. Sci.* **297** (2022), 120–132.

[5] J.-L. Baril, J. L. Ramírez, and F. A. Velandia, Bijections between directed-column convex polyominoes and restricted compositions, *Theoret. Comput. Sci.* **1003** (2024), 114626.

[6] M. Feinberg, Fibonacci-Tribonacci, *Fibonacci Quart.* **1** (1963), 71–74.

[7] S. Heubach and T. Mansour, Compositions of n with parts in a set, *Congr. Numer.* **168** (2004), 127–143.

[8] S. Kirgizov, $\mathbb{Q}$-bonacci words and numbers, *Fibonacci Quart.* **60** (2022), 187–195.

[9] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices. *Amer. Math. Monthly* **67** (1960), 745–752.

[10] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`.

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