Journal of Integer Sequences, Vol. 28 (2025),
Article 25.1.4

Continued Fractions with Predictable Patterns and Transcendental Numbers

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Abstract

The continued fractions that are the subject of this paper display a particular kind of "self-similar" structure: that is, their partial quotients are created by manipulating and moving the denominators of their convergents. An alternating series characterizes the real numbers that these continued fractions represent. Using an application of Roth's theorem, we prove the transcendence of these numbers.

1 Introduction

Let $x \in \mathbb{R}$. We write the continued fraction of x as

$$x = [a_0, a_1, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \tag{1}$$

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where the partial quotients a_n are integers, positive for $n > 0$. We have

$$x = \lim_{n\to\infty} [a_0, a_1, \ldots, a_n],$$

where

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Define p_n and q_n by

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, & \text{for } n \geq 1; \\ q_n = a_n q_{n-1} + q_{n-2}, & \text{for } n \geq 1, \end{cases} \tag{2}$$

with $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$. Then $[a_0, a_1, \ldots, a_n] = p_n/q_n$, which is called the nth convergent. We have the property

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}, \quad \text{for } n \geq 1. \tag{3}$$

Using this property and induction, we obtain

$$\frac{p_n}{q_n} = a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_{n-1} q_n}. \tag{4}$$

Continued fractions are useful in providing the best rational approximations.

We recall that the irrationality exponent $\mu(x)$ is defined for real numbers x to be the supremum of the set of μ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

is satisfied by an infinite number of coprime integer pairs (p, q) with $q > 0$. Rational numbers have irrationality exponent 1, while (as a consequence of Dirichlet's approximation theorem) every irrational number has irrationality exponent at least 2.

Referring back to [9], we recall a formula for the irrationality exponent of an irrational number in terms of its continued fraction expansion. For a real number $x = [a_0; a_1, a_2, \ldots]$ with convergents p_i/q_i, it holds:

$$\mu(x) = 1 + \limsup_{n\to\infty} \frac{\ln q_{n+1}}{\ln q_n} = 2 + \limsup_{n\to\infty} \frac{\ln a_{n+1}}{\ln q_n}. \tag{5}$$

A number $x \in \mathbb{R}$ with irrationality exponent $\mu(x) = \infty$ is called a Liouville number. In 1955, Roth [10] published his famous theorem about the rational approximation of algebraic real numbers.

Theorem 1. *Let x be a real algebraic number and let $\epsilon > 0$. Then the inequality*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}$$

has only a finite number of rational solutions p/q.

This theorem remains a very good tool to show the transcendence of real numbers. In fact, by this theorem, the irrationality exponent of any irrational algebraic number is exactly 2.

The results presented here are similar in spirit to those in a paper by Davison and Shallit [4], who found some continued fractions whose partial quotients are explicitly related to the denominators of their convergents, and used this to prove the transcendence of Cahen's constant:

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{x_n} = 0.64341\ldots,$$

where $\{S_n\}$ is the Sylvester sequence defined by the nonlinear recurrence:

$$S_0 = 2, S_{n+1} = S_n^2 - S_n + 1 \quad (n \geq 0),$$

and $x_n = S_n - 1$ satisfies the recurrence

$$x_{n+1} = x_n(x_n + 1) \quad (n \geq 0).$$

The partial quotients in continued fraction expansion of C appear as sequence A006280 in the On-Line Encyclopedia of Integer Sequences (OEIS) [8]. Recently, Duverney and Shiokawa [5] have proved that the exact value of its irrationality exponent is equal to 3.

The connections between infinite series representations for real numbers and continued fraction expansions are the subject of numerous studies. Several papers [2, 3, 4, 5, 6, 7] looked into this, and we'll also present some more fascinating examples. More precisely, in this paper, motivated by Davison and Shallit's concepts, we will reveal other analogous methods for displaying a real number as an infinite alternating series.

2 Main results

2.1 A first predictable pattern

There are many numbers in the literature with a repeating pattern with a period of 2. For example

$$\tan(1/n) = [0, n-1, 1, 3n-2, 1, 5n-2, 1, 7n-2, \ldots],$$

where n is a positive integer (OEIS A019426). We will see that there exist many other real numbers with a similar pattern which are transcendental numbers.

Let $x = [a_0, a_1, \ldots]$ and $(p_n/q_n)_n$ be the sequence of its convergents. Let $\beta \in \mathbb{Z}_{>0}$. Let ω_0, ω_1, ω_2, ... be a sequence of positive integers. Assume that

$$\begin{cases} a_0 = 0, \\ a_1 = \omega_0, \\ a_{2n+2} = 1, & \text{for } n \geq 0; \\ a_{2n+3} = \omega_{n+1} q_{2n+1}^{\beta}, & \text{for } n \geq 0. \end{cases} \tag{6}$$

Then $q_0 = 1$, $q_1 = \omega_0$, and for $n \geq 0$

$$\begin{cases} q_{2n+3} = \omega_{n+1} q_{2n+1}^{\beta} q_{2n+2} + q_{2n+1}, \\ q_{2n+2} = q_{2n+1} + q_{2n}. \end{cases} \tag{7}$$

Using the equation (4), we see

$$\frac{p_{2n+1}}{q_{2n+1}} = \sum_{0 \leq i \leq 2n} \frac{(-1)^i}{q_i q_{i+1}} = [0, \omega_0, 1, \omega_1 q_1^{\beta}, 1, \omega_2 q_3^{\beta}, 1, \ldots, 1, \omega_n q_{2n-1}^{\beta}].$$

Hence, letting $n \to \infty$, we find the following way to express the continued fraction as an alternating series:

Proposition 2.

$$x = \sum_{i=0}^{\infty} \frac{(-1)^i}{q_i q_{i+1}} = [0, \omega_0, 1, \omega_1 q_1^{\beta}, 1, \omega_2 q_3^{\beta}, 1, \ldots, 1, \omega_n q_{2n-1}^{\beta}, \ldots]. \tag{8}$$

Theorem 3. *The irrationality exponent of the number x in Proposition 2 satisfies*

$$\mu(x) \geq 2 + \beta + \limsup_{n \to \infty} \frac{\ln \omega_n}{\ln q_{2n}} \geq 2 + \beta \geq 3. \tag{9}$$

Hence x is transcendental by Roth's theorem.

Proof. Let x be the number in equation (8), with continued fraction expansion $x = [a_0, a_1, \ldots]$. Let $\varepsilon > 0$ be arbitrary. From the equation (7) we have $q_{2n} \leq 2q_{2n-1}$ for large n. So

$$q_{2n} \leq q_{2n}^{\varepsilon} q_{2n-1},$$

which yields $(1-\varepsilon)\ln q_{2n} \le \ln q_{2n-1}$. Then by (5)

$$\begin{aligned}
\mu(x) &= 2 + \limsup_{n\to\infty} \frac{\ln a_{n+1}}{\ln q_n} \\
&= 2 + \limsup_{n\to\infty} \frac{\ln a_{2n+1}}{\ln q_{2n}} \\
&= 2 + \limsup_{n\to\infty} \frac{\ln(\omega_n q_{2n-1}^{\beta})}{\ln q_{2n}} \\
&= 2 + \limsup_{n\to\infty} \frac{\ln \omega_n}{\ln q_{2n}} + \beta \frac{\ln q_{2n-1}}{\ln q_{2n}} \\
&\ge 2 + (1-\varepsilon)\beta + \limsup_{n\to\infty} \frac{\ln \omega_n}{\ln q_{2n}},
\end{aligned}$$

which yields (9) since ε is arbitrarily small. □

We now prove the converse to Proposition 2.

Theorem 4. *Let (x_i) be a sequence of positive integers such that $x_0 = 1$, and $x = \sum_{i\ge 0}(-1)^i/x_i$ be an irrational number with continued fraction expansion*

$$x = [0, a_1, a_2, \ldots].$$

Let (p_n/q_n) be the sequence of convergents of x. Assume that

$$\sum_{i=0}^{n} \frac{(-1)^i}{x_i} = \frac{p_{n+1}}{q_{n+1}}$$

for all $n \ge 0$, and that $x_{2n+1} \mid x_{2n+2}$ and $x_{2n+3} = \left(\frac{x_{2n+2}}{x_{2n+1}} \frac{x_{2n}}{x_{2n-1}} \frac{x_{2n-2}}{x_{2n-3}} \cdots \frac{x_2}{x_1}\right)^2 + x_{2n+2}$ for all $n \ge 0$. Then for $n \ge 0$

(i) $x_n = q_n q_{n+1}$.

(ii) $q_{2n+1} \mid a_{2n+3}$.

(iii) $a_{2n+2} = 1$.

Proof.

(i) Clearly verified for $n = 0$. For $n \ge 1$ we have

$$\frac{p_{n+1}}{q_{n+1}} = \sum_{i=0}^{n} \frac{(-1)^i}{x_i} = \frac{p_n}{q_n} + \frac{(-1)^n}{x_n}.$$

Hence

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{x_n}.$$

By (3), we have

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n+1}q_n},$$

and hence $x_n = q_{n+1}q_n$.

(ii) Note that $x_{2n+1} \mid x_{2n+2}$ implies that $q_{2n+1}q_{2n+2} \mid q_{2n+2}q_{2n+3}$, and hence $q_{2n+1} \mid q_{2n+3}$. Since $q_{2n+3} = a_{2n+3}q_{2n+2} + q_{2n+1}$, so $q_{2n+1} \mid a_{2n+3}q_{2n+2}$. As $\gcd(q_{2n+1}, q_{2n+2}) = 1$, so $q_{2n+1} \mid a_{2n+3}$.

(iii) We have

$$\begin{aligned}\left(\frac{x_{2n+2}}{x_{2n+1}}\frac{x_{2n}}{x_{2n-1}}\frac{x_{2n-2}}{x_{2n-3}}\cdots\frac{x_2}{x_1}\right)^2 &= \left(\frac{q_{2n+2}q_{2n+3}}{q_{2n+1}q_{2n+2}}\frac{q_{2n}q_{2n+1}}{q_{2n-1}q_{2n}}\cdots\frac{q_2q_3}{q_1q_2}\right)^2 \\ &= q_{2n+3}^2 \\ &= x_{2n+3} - x_{2n+2} = q_{2n+3}q_{2n+4} - q_{2n+2}q_{2n+3}.\end{aligned}$$

This gives that $q_{2n+4} = q_{2n+3} + q_{2n+2}$.

□

Example 5. We put $\omega_n = 1$ for $n \geq 0$ and $\beta = 1$. Then $q_0 = 1$, $q_1 = 1$, and $q_{2n+3} = q_{2n+1}(q_{2n+2} + 1)$. Here are the first few values of the sequences q_n and x_n:

n	0	1	2	3	4	5	6	7	…
q_n	1	1	2	3	5	18	23	432	…
x_n	1	2	6	15	90	414	9936	196560	…

So we have

$$\begin{aligned}x = \sum_{i=0}^{\infty}\frac{(-1)^i}{q_iq_{i+1}} &= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{15} + \frac{1}{90} - \frac{1}{414} + \cdots \\ &= [0, 1, 1, q_1, 1, q_3, 1, \ldots, 1, q_{2n-1}, \ldots] \\ &= [0, 1, 1, 1, 1, 3, 1, 18, 1, 432, \ldots],\end{aligned}$$

is a transcendental number. The continued fraction expansion of this series is referenced as A380013 in the OEIS [8].

Example 6. We put $\beta = 1$, $\omega_0 = 1$, and $\omega_{n+1} = q_{2n+1}$ for $n \geq 0$. Then $q_0 = 1$, $q_1 = 1$, and $q_{2n+3} = q_{2n+1}^2 q_{2n+2} + q_{2n+1}$. So we have

$$\begin{aligned} x = \sum_{i=0}^{\infty} \frac{(-1)^i}{q_i q_{i+1}} &= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{15} + \frac{1}{240} + \cdots \\ &= [0, 1, 1, 1, 1, q_3^2, 1, q_5^2, 1, \ldots, 1, q_{2n-1}^2, \ldots] \\ &= [0, 1, 1, 1, 1, 9, 1, 2304, 1, 14923065600, \ldots], \end{aligned}$$

is a transcendental number.

Example 7. We put $\beta = 1$, $\omega_0 = 1$, and $\omega_n = q_{2n}^n$ for $n \geq 1$. Then $q_0 = 1$, $q_1 = 1$, and $q_{2n+3} = q_{2n+2}^{n+2} q_{2n+1} + q_{2n+1} = q_{2n+1}(q_{2n+2}^{n+2} + 1)$. So we have

$$\begin{aligned} x = \sum_{i=0}^{\infty} \frac{(-1)^i}{q_i q_{i+1}} &= 1 - \frac{1}{2} + \frac{1}{10} - \frac{1}{35} + \frac{1}{12040} - \cdots \\ &= [0, 1, 1, q_2 q_1, 1, q_4^2 q_3, 1, q_6^3 q_5, \ldots, 1, q_{2n+2}^{n+1} q_{2n+1}, \ldots] \\ &= [0, 1, 1, 2, 1, 245, 1, 8859423442760, 1, \ldots]. \end{aligned}$$

Since $\mu(x) \geq 2 + \beta + \limsup_{n\to\infty} \dfrac{\ln \omega_n}{\ln q_{2n}} = +\infty$, then x is a Liouville number.

2.2 A second predictable pattern

In this part, we construct many real numbers with a repeated pattern with a period 3 analogous to those of $e^{\frac{1}{n}}$:

$$e^{\frac{1}{n}} = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, 1, 1, \ldots],$$

where n is a positive integer (OEIS A003417). We prove that these numbers are transcendental.

Let $x = [a_0, a_1, \ldots]$ and $(p_n/q_n)_n$ be the sequence of its convergents. As above, we let $\beta \in \mathbb{Z}_{>0}$ and ω_0, ω_1, ω_2, $\ldots$ be a sequence of positive integers. Assume that

$$\begin{cases} a_0 = 0, \\ a_1 = \omega_0, \\ a_{3n+2} = 1 & \text{for } n \geq 0; \\ a_{3n+3} = 1 & \text{for } n \geq 0; \\ a_{3n+4} = \omega_{n+1} q_{3n+2}^{\beta}, & \text{for } n \geq 0. \end{cases} \tag{10}$$

Then $q_0 = 1$, $q_1 = \omega_0$, and for $n \geq 0$

$$\begin{cases} q_{3n+2} = q_{3n+1} + q_{3n}, \\ q_{3n+3} = q_{3n+2} + q_{3n+1}, \\ q_{3n+4} = \omega_{n+1} q_{3n+3} q_{3n+2}^{\beta} + q_{3n+2}. \end{cases} \tag{11}$$

Using the equation (4), we see

$$\frac{p_{3n+1}}{q_{3n+1}} = \sum_{0\le i\le 3n} \frac{(-1)^i}{q_i q_{i+1}} = [0, \omega_0, 1, 1, \omega_1 q_2^\beta, 1, 1, \omega_2 q_5^\beta, 1, 1 \ldots, 1, \omega_n q_{3n-1}^\beta].$$

Hence, letting $n \to \infty$, we also find the following way to express the real x as an alternating series:

Proposition 8.

$$x = \sum_{i=0}^{\infty} \frac{(-1)^i}{q_i q_{i+1}} = [0, \omega_0, 1, 1, \omega_1 q_2^\beta, 1, 1, \omega_2 q_5^\beta, 1, 1 \ldots, 1, 1, \omega_n q_{3n-1}^\beta, \ldots]. \tag{12}$$

Theorem 9. *The irrationality exponent of the number x in Proposition 8 satisfies*

$$\mu(x) \ge 2 + \beta + \limsup_{n\to\infty} \frac{\ln \omega_n}{\ln q_{3n}} \ge 2 + \beta \ge 3. \tag{13}$$

Hence x is transcendental by Roth's theorem.

Proof. Let x be the number in equation (12), with continued fraction expansion $x = [a_0, a_1, \ldots]$. Let $\varepsilon > 0$ be arbitrary. From (11), we have $q_{3n} \le 2q_{3n-1}$ for large n. So

$$q_{3n} \le q_{3n}^{\varepsilon} q_{3n-1},$$

which yields $(1-\varepsilon)\ln q_{3n} \le \ln q_{3n-1}$. Then by (5)

$$\begin{aligned}
\mu(x) &= 2 + \limsup_{n\to\infty} \frac{\ln a_{n+1}}{\ln q_n} \\
&= 2 + \limsup_{n\to\infty} \frac{\ln a_{3n+1}}{\ln q_{3n}} \\
&= 2 + \limsup_{n\to\infty} \frac{\ln(\omega_n q_{3n-1}^\beta)}{\ln q_{3n}} \\
&= 2 + \limsup_{n\to\infty} \frac{\ln \omega_n}{\ln q_{3n}} + \beta \frac{\ln q_{3n-1}}{\ln q_{3n}} \\
&\ge 2 + (1-\varepsilon)\beta + \limsup_{n\to\infty} \frac{\ln \omega_n}{\ln q_{3n}},
\end{aligned}$$

which yields (13) since ε is arbitrarily small. □

The proof of the converse to Proposition 8 is similar to the proof of Theorem 4, so we omit it.

Theorem 10. *Let (x_i) be a sequence of positive integers such that $x_0 = 1$, and $x = \sum_{i\geq 0}(-1)^i/x_i$ be an irrational number with continued fraction expansion*

$$x = [0, a_1, a_2, \ldots].$$

Let (p_n/q_n) be the sequence of convergents of x. Assume that

$$\sum_{i=0}^{n} \frac{(-1)^i}{x_i} = \frac{p_{n+1}}{q_{n+1}}$$

for all $n \geq 0$, and that $x_{3n+2} \mid x_{3n+3}$,

$$x_{3n+2} - x_{3n+1} = \left(\frac{x_{3n+1}}{x_{3n}} \frac{x_{3n-1}}{x_{3n-2}} \frac{x_{3n-3}}{x_{3n-4}} \cdots \frac{x_2}{x_1}\right)^2,$$

$$x_{3n+4} - x_{3n+3} = \left(\frac{x_{3n+3}}{x_{3n+2}}\right)^2 (x_{3n+2} - x_{3n+1})$$

for all $n \geq 0$. Then for $n \geq 0$ we have

(i) $x_n = q_n q_{n+1}$.

(ii) $q_{3n+2} \mid a_{3n+4}$.

(iii) $a_{3n+2} = a_{3n+3} = 1$.

Example 11. We put $\beta = 1$, $\omega_0 = 1$, and $\omega_{n+1} = q_{3n+2}$ for $n \geq 0$. Then

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{3n+2} = a_{3n+3} = 1, & \text{for } n \geq 0; \\ a_{3n+4} = q_{3n+2}^2, & \text{for } n \geq 0. \end{cases}$$

We have

$$\begin{aligned} x &= \sum_{i=0}^{\infty} \frac{(-1)^i}{q_i q_{i+1}} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{42} + \frac{1}{238} - \cdots \\ &= [0, 1, 1, 1, q_2^2, 1, 1, q_5^2, \ldots, 1, 1, q_{3n-1}^2, \ldots] \\ &= [0, 1, 1, 1, 4, 1, 1, 289, 1, 1, 81126049, \ldots], \end{aligned}$$

and this number is transcendental. The continued fraction expansion of this series is referenced as A380194 in the OEIS [8].

Example 12. We put $\beta = 1$, $\omega_0 = 1$, and $\omega_n = q_{3n}^n$ for $n \geq 1$. Then

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{3n+2} = a_{3n+3} = 1, & \text{for } n \geq 0; \\ a_{3n+4} = q_{3n+3}^{n+1} q_{3n+2}, & \text{for } n \geq 0. \end{cases}$$

We have

$$\begin{aligned} x = \sum_{i=0}^{\infty} \frac{(-1)^i}{q_i q_{i+1}} &= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{60} + \frac{1}{460} - \cdots \\ &= [0, 1, 1, 1, q_2 q_3, 1, 1, q_6^2 q_5, \ldots, 1, 1, q_{3n}^n q_{3n-1}, \ldots] \\ &= [0, 1, 1, 1, 6, 1, 1, 42527, 1, 1, 89468504340857877754313037, \ldots], \end{aligned}$$

is a Liouville number.

3 Acknowledgments

The authors wish to thank the referee for several helpful comments and suggestions that significantly improved the quality of the manuscript.

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2020 *Mathematics Subject Classification*: Primary 11A55; Secondary 11J70.
Keywords: continued fraction, transcendental number.

(Concerned with sequences A003417, A006280, A019426, A380013, and A380194.)

Received August 11 2024; revised versions received August 12 2024; October 8 2024; November 4 2024; January 31 2025; February 7 2025. Published in *Journal of Integer Sequences*, February 8 2025.

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