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Connections Between Combinations Without Specified Separations and Strongly Restricted Permutations, Compositions, and Bit Strings

Michael A. Allen
Physics Department
Faculty of Science
Mahidol University
Rama 6 Road
Bangkok 10400
Thailand
maa5652@gmail.com

Abstract

Let S_n and $S_{n,k}$ be, respectively, the number of subsets and k-subsets of $\mathbb{N}_n = \{1, \ldots, n\}$ such that no two subset elements differ by an element of the set $\mathcal{Q}$, the largest element of which is q. We prove a bijection between such k-subsets when $\mathcal{Q} = \{m, 2m, \ldots, jm\}$ with $j, m > 0$ and permutations π of $\mathbb{N}_{n+jm}$ with k excedances satisfying $\pi(i) - i \in \{-m, 0, jm\}$ for all $i \in \mathbb{N}_{n+jm}$. We also identify a bijection between another class of restricted permutation and the cases $\mathcal{Q} = \{1, q\}$ and derive the generating function for S_n when $q = 4, 5, 6$. We give some classes of $\mathcal{Q}$ for which S_n is also the number of compositions of $n + q$ into a given set of allowed parts. We also prove a bijection between k-subsets for a class of $\mathcal{Q}$ and the set representations of size k of equivalence classes for the occurrence of a given length-$(q + 1)$ subword within bit strings. We then formulate a straightforward procedure for obtaining the generating function for the number of such equivalence classes.

1 Introduction

Combinations without specified separations (or *restricted combinations*) refer to the subsets of $\mathbb{N}_n = \{1, \ldots, n\}$ with the property that no two elements of a subset differ by an element of a set $\mathcal{Q}$, the elements of which are independent of n. We denote the total number of such subsets and the number of k-subsets (i.e., subsets of size k) by $S_n^{\mathcal{Q}}$ and $S_{n,k}^{\mathcal{Q}}$, respectively. We drop the superscript when it is clear which set $\mathcal{Q}$ of disallowed differences we are dealing with. For any $\mathcal{Q}$, one always takes $S_{n<0}^{\mathcal{Q}} = S_{n,k<0}^{\mathcal{Q}} = S_{n<k,k}^{\mathcal{Q}} = 0$.

When there are no restrictions, we just have the number of ordinary combinations and so $S_n^{\{\}} = 2^n$ (A000079 in the OEIS [21]) and $S_{n,k}^{\{\}} = \binom{n}{k}$ (A007318). It appears to be well-known that $S_n^{\{1,\dots,q\}} = S_{n-1}^{\{1,\dots,q\}} + S_{n-q-1}^{\{1,\dots,q\}} + \delta_{n+q,0}$, where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise, and $S_{n,k}^{\{1,\dots,q\}} = \binom{n+q(1-k)}{k}$ (A329146). These reduce to the classic results $S_n^{\{1\}} = F_{n+2}$, where F_n is the n-th Fibonacci number (A000045, $F_n = F_{n-1} + F_{n-2} + \delta_{n,1}$, $F_{n<1} = 0$), and $S_{n,k}^{\{1\}} = \binom{n+1-k}{k}$ (A011973) [11]. Expressions for $S_n^{\mathcal{Q}}$ and $S_{n,k}^{\mathcal{Q}}$ have also been obtained when $\mathcal{Q} = \{q\}$ [14, 20, 16, 15, 3] and $\mathcal{Q} = \{m, 2m, \dots, jm\}$ with $j, m > 0$ [18, 1].

In recent work [2], which we henceforth refer to as All24, the author found a bijection between the k-subsets of $\mathbb{N}_n$ with disallowed differences set $\mathcal{Q}$, of which the largest element is q, and the restricted-overlap tilings of an $(n+q)$-board (an $(n+q)\times 1$ board) with squares and k combs. This enables generating functions for $S_n^{\mathcal{Q}}$ and recursion relations for $S_{n,k}^{\mathcal{Q}}$ to be readily obtained for a variety of families of $\mathcal{Q}$ [2]. A $(w_1, g_1, w_2, \dots, g_{t-1}, w_t)$-comb is a tile composed of $t > 0$ rectangular sub-tiles (called *teeth*) arranged in such a way that the i-th tooth, which has dimensions $w_i \times 1$, is separated from the $(i+1)$-th by a $g_i \times 1$ gap [2]. The *length* of a comb is the sum of the lengths of the teeth and gaps. Each $\mathcal{Q}$ corresponds to a length-$(q+1)$ comb C (which we refer to as a $\mathcal{Q}$-*comb*) defined as follows. We label the cells of C from 0 to q. Cell 0 is always part of the first tooth. Cell i (where $i = 1, \dots, q$) is part of a tooth iff $i \in \mathcal{Q}$. Note that we are free to include 0 in $\mathcal{Q}$ without affecting the results (and we do so in Theorems 16 and 17 for reasons of convenience and symmetry) since subset elements are distinct and so cannot differ by 0. This 0 element can then be regarded as corresponding to cell 0 of the comb. To ensure that there is only one way to specify the w_i and g_i of the $\mathcal{Q}$-comb, we insist that all teeth and gaps have positive integer length. A *restricted-overlap tiling* using squares and $\mathcal{Q}$-combs means that any comb can be placed so that its non-leftmost cells (i.e, cells 1 to q) overlap the non-leftmost cells of any other combs on the board and no other forms of overlap of tiles is allowed [2] (see Fig. 7 for examples). We let $B_n^{\mathcal{Q}}$ denote the number of restricted-overlap tilings of an n-board with $\mathcal{Q}$-combs and squares, and $B_{n,k}^{\mathcal{Q}}$ denote the number of such tilings that contain k combs. The following theorem is a consequence of the bijection.

Theorem 1 (Theorem 2 in All24)**.** *For all nonnegative n we have $S_n^{\mathcal{Q}} = B_{n+q}^{\mathcal{Q}}$ and $S_{n,k}^{\mathcal{Q}} = B_{n+q,k}^{\mathcal{Q}}$.*

Any tiling of an n-board can be expressed as a tiling using *metatiles*, which are gapless groupings of tiles that cannot be split into smaller gapless groupings by removing a gapless grouping of tiles [8]. The simplest metatiles when tiling a long enough board with squares and combs are a single square (denoted by S) and a $(w_1, g_1, \dots, w_t)$-comb with all the gaps filled by squares. We call the latter metatile a *filled comb* and it is denoted by CS^g, where $g = \sum_i g_i$. If the comb has one tooth it is just a w_1-omino (and also a metatile by itself), and S and C are then the only metatiles. Otherwise, as a consequence of the restricted-overlap nature of the tiling, a C can always be placed so that its leftmost cell occupies the next available empty cell of the yet-to-be-completed metatile. If more empty cells within the yet-to-be-completed metatile result (which, if a C is placed starting in the final cell of the final gap of the first comb on the board, is the case iff $2w_t < q$ [2, Lemma 2]), this can continue indefinitely and so there are an infinite number of possible metatiles.

A systematic way to generate all metatiles, and thus obtain generating functions for $B_n^{\mathcal{Q}}$ and $B_{n,k}^{\mathcal{Q}}$, is to first construct a directed pseudograph (henceforth simply referred to as a *digraph*) in which each node has a label giving a bit-string representation of the remaining gaps and filled cells (a 0 representing an empty cell, a 1 a filled one) starting at the next gap in the yet-to-be-completed metatile and ending at the final filled cell [9, 2] (see Figs. 3, 4, and 5 for examples). The *0 node* represents both the empty board and the completed metatile. Each arc represents the addition of a tile and has a label showing the type of tile and a subscript giving the increase in length of the incomplete metatile resulting from adding that tile, if that increase is not 0. The length increase from adding a C is given by $q + 1 - d$, where d is the number of binary digits in the node the arc starts from, but with $d = 0$ for the 0 node. A walk starting and ending at the 0 node, but not visiting it in between, represents a metatile and is denoted by the string of C and S corresponding to the arcs in the order visited. For example, in Fig. 4, the loop attached to the 0 node corresponds to the S metatile, and the cycle starting at the 0 node and then visiting the $0^3 1$ node (an abbreviation for 0001), the $0^2 1$ node, and then the 01 node before returning to the 0 node corresponds to the CS^2 (filled comb) metatile.

A cycle that does not include the 0 node is called an *inner cycle* (e.g., $SSC_{[4]}$ in Fig. 4, where including the length increment subscript quickly identifies which comb arc we are referring to). There are a finite number of possible metatiles iff the digraph lacks an inner cycle since a walk can traverse an inner cycle an arbitrary number of times before returning to the 0 node. If all the inner cycles have at least one node in common, that node is called a *common node* (e.g., the 01^{q-p} node is the common node in the Fig. 8 digraph; either the $0^2 1$ or the 01 node can be chosen as the common node in the digraph in Fig. 3; Fig. 4 and Fig. 5 show digraphs that have inner cycles but no common node). For digraphs with a common node, there are simple expressions for the recursion relations for the numbers of tilings $B_n^{\mathcal{Q}}$ and $B_{n,k}^{\mathcal{Q}}$ in terms of lengths of various cycles in the digraph [9, 2] (we rederive these in the form of generating functions in §2). If the digraph possesses inner cycles, but no common node, there is a class of digraph (which has 3 inner cycles) where an analogous simple procedure has been found for obtaining the recursion relations [1].

A systematic search of the OEIS for sequences $(S_n^{\mathcal{Q}})_{n\geq 0}$ for all $\mathcal{Q}$ with $q \leq 4$ shows that there are various connections between some types of restricted combinations and some types of restricted permutations. One also finds relations to compositions and bit strings. Here, in §3, we give bijections between between two classes of strongly restricted permutations and combinations with $\mathcal{Q} = \{1, q\}$ and $\mathcal{Q} = \{m, 2m, \ldots, jm\}$. As a consequence of the former bijection, we can prove a conjectured recursion relation for the permutations corresponding to the $\mathcal{Q} = \{1, 4\}$ case. In §2, we derive generating functions for $B_n^{\mathcal{Q}}$ and $B_{n,k}^{\mathcal{Q}}$ for tilings with metatiles generated by a particular class of digraph (which includes the 3 inner cycle class with no common node dealt with previously [1]) without a common node. This enables us to easily obtain a generating function for $S_n^{\{1,5\}}$. In §4, although it is not all that surprising given the tiling interpretation of restricted combinations, for completeness, we also show that $S_n^{\mathcal{Q}}$ is the number of compositions of $n+q$ into certain sets of parts for some classes of $\mathcal{Q}$. We point out an elementary connection between all instances of $\mathcal{Q}$ and bit strings in §5. This forms the basis for an algorithm for efficiently calculating $S_n^{\mathcal{Q}}$ and $S_{n,k}^{\mathcal{Q}}$ for any given $\mathcal{Q}$, as detailed via a heavily annotated C program listed in the Appendix. In §5, we also give a bijection between k-subsets for certain $\mathcal{Q}$ and equivalence classes for the occurrence of certain subwords within bit strings. The bijection enables us to show that a generating function for the number of equivalence classes is always straightforward to obtain. In the discussion (§6), we mention transfer matrices as an alternative way to obtain the $B_n^{\mathcal{Q}}$ and $B_{n,k}^{\mathcal{Q}}$ generating functions from digraphs.

2 Generating functions for the numbers of walks on digraphs

We start by reviewing further terminology concerning digraphs. For a digraph possessing a common node, a path from the 0 node to the common node followed by a path from the common node back to the 0 node (and so the common node is only visited once) is called a *common circuit* (e.g., for either choice of common node, the digraph in Fig. 3 possesses a single common circuit, namely, $C_{[5]}S^2$). A cycle that includes the 0 node, but not the common node, is called an *outer cycle* (e.g., in Fig. 3, the S loop attached to the 0 node is the only outer cycle).

The proofs in this section require the following three lemmas [19]. The bivariate generating functions $G_i(x, y)$ we employ here are such that the coefficient of $x^l y^k$ of the generating function of a walk or set of walks between two nodes is the number of configurations of tiles that contribute l to the overall length of the tiling and contain k combs.

Lemma 2 (The Parallel Rule)**.** *If two nodes in a digraph are connected via m alternative*

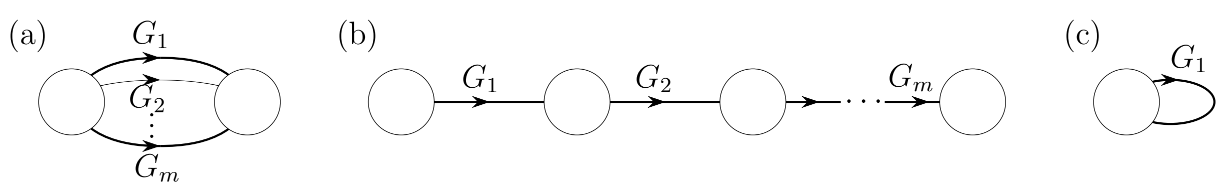

Figure 1: Schematic diagrams of connections between nodes corresponding to the construction of generating functions via (a) the parallel rule (Lemma 2), (b) the series rule (Lemma 3), and (c) the loop rule (Lemma 4).

walks (or sets of walks), as shown in Fig. 1a, that have associated generating functions $G_i(x,y)$, where $i = 1,\dots,m$, then the combined generating function for all of those walks connecting the two nodes is $\sum_{i=1}^{m} G_i(x,y)$.

Lemma 3 (The Series Rule)**.** *If two nodes in a digraph are connected via $m-1$ intermediate nodes, which are connected by walks (or sets of walks), as shown in Fig. 1b, that have associated generating functions $G_i(x,y)$, where $i = 1,\dots,m$, then the combined generating function for all of those walks connecting the two nodes is $\prod_{i=1}^{m} G_i(x,y)$.*

Lemma 4 (The Loop Rule)**.** *If a node is connected to itself by a walk (or set of walks), as shown in Fig. 1c, with a corresponding generating function $G_1(x,y)$, then the overall generating function corresponding to the traversal of the loop walk (or walks) zero or more times is*

$$1 + G_1(x,y) + G_1^2(x,y) + \cdots = \frac{1}{1 - G_1(x,y)}.$$

A form of the following theorem was originally stated by Edwards and Allen [9], and a more compact version re-derived later [2]. Here, using the above three lemmas, we obtain the result in terms of a bivariate generating function. We use this theorem in §3.1 to obtain the $S_n^{\{1,4\}}$ generating function.

Theorem 5. *For a digraph possessing a common node, let $l_{\mathrm{o}i}$ be the length of the i-th outer cycle ($i = 1,\dots,N_{\mathrm{o}}$) and let $k_{\mathrm{o}i}$ be the number of combs it contains, let L_r be the length of the r-th inner cycle ($r = 1,\dots,N$) and let K_r be the number of combs it contains, and let $l_{\mathrm{c}i}$ be the length of the i-th common circuit ($i = 1,\dots,N_{\mathrm{c}}$) and let $k_{\mathrm{c}i}$ be the number of combs it contains. Then the generating function such that the coefficient of $x^n y^k$ thereof is the number of tilings an n-board that contain k combs is given by*

$$G(x,y) = \frac{1 - \displaystyle\sum_{r=1}^{N} x^{L_r} y^{K_r}}{1 - \displaystyle\sum_{r=1}^{N} x^{L_r} y^{K_r} - \sum_{i=1}^{N_{\mathrm{o}}} \left(x^{l_{\mathrm{o}i}} y^{k_{\mathrm{o}i}} - \sum_{r=1}^{N} x^{l_{\mathrm{o}i}+L_r} y^{k_{\mathrm{o}i}+K_r} \right) - \sum_{i=1}^{N_{\mathrm{c}}} x^{l_{\mathrm{c}i}} y^{k_{\mathrm{c}i}}}. \tag{1}$$

Proof. The generating functions corresponding to a single traversal of the i-th outer cycle (represented by the loop attached to the 0 node in Fig. 2a), the i-th common circuit (represented by the path from the 0 node to the $\mathcal{C}$ node and back or the path $0 \to \mathcal{C} \to \mathcal{R} \to 0$), and the r-th inner cycle ($\mathcal{C} \to \mathcal{R} \to \mathcal{C}$) are, respectively, $x^{l_{\mathrm{o}i}} y^{k_{\mathrm{o}i}}$, $x^{l_{\mathrm{c}i}} y^{k_{\mathrm{c}i}}$, and $x^{L_r} y^{K_r}$. The N inner cycles are alternative paths from the common node back to itself and so, from the parallel and loop rules, the generating function corresponding to their traversal (starting from and returning to the common node) in any order zero or more times is $1/(1 - \sum_{i=1}^{N} x^{L_r} y^{K_r})$. Using the series rule, we multiply this by $x^{l_{\mathrm{c}i}} y^{k_{\mathrm{c}i}}$ to obtain the generating function for the walk that consists of the paths to and from $\mathcal{C}$ along the i-th common circuit and an arbitrary

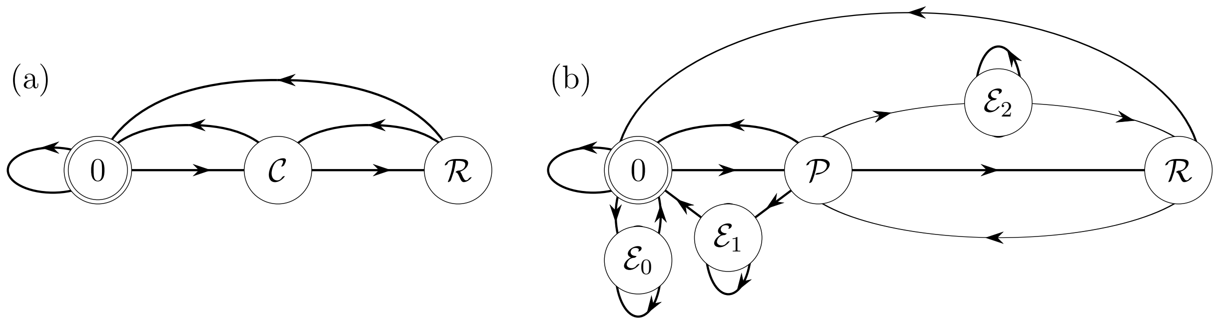

Figure 2: Schematic diagrams of digraphs with (a) a common node $\mathcal{C}$ and (b) a pseudo-common node $\mathcal{P}$ to which Theorem 5 and Theorem 6 apply, respectively. In these diagrams, each arc, with the exception of each errant loop (each arc connecting an $\mathcal{E}_i$ node to itself), potentially represents a number of alternative arcs in the specific instance of a digraph. Some nodes and arcs can be absent.

number of trips around the inner cycles. We now apply the parallel rule to all such cases and to all outer cycles, followed by the loop rule since these can be executed any number of times. This gives

$$G(x,y) = \frac{1}{1 - \sum_{i=1}^{N_\mathrm{o}} x^{l_{\mathrm{o}i}} y^{k_{\mathrm{o}i}} - \dfrac{\sum_{i=1}^{N_\mathrm{c}} x^{l_{\mathrm{c}i}} y^{k_{\mathrm{c}i}}}{1 - \sum_{r=1}^{N} x^{L_r} y^{K_r}}},$$

which yields (1) on rearranging. □

The following definitions are slightly modified versions of those given in earlier work on a class of digraph with inner cycles but no common node in order to incorporate the increased complexity of the type of digraphs we are about to consider [1]. For a digraph lacking a common node, we refer to any inner cycle that can be represented as a single arc linking a node to itself as an *errant loop* if the digraph would only have a common node $\mathcal{P}$ if all the errant loop arcs were removed. The node $\mathcal{P}$ is then referred to as a *pseudo-common node*. A node with an errant loop is called an *errant loop node*. Evidently, a particular errant loop node $\mathcal{E}$ cannot also be a pseudo-common node; if that were the case, removing all the other errant loop arcs except the errant loop at $\mathcal{E}$ would result in a digraph with a common node, implying that the loop at $\mathcal{E}$ was never an errant loop in the first place. For a digraph with at least one errant loop, a *common circuit* is now defined as two concatenated simple paths from the 0 node to $\mathcal{P}$ and from $\mathcal{P}$ to the 0 node. We also need to modify the definition of an outer cycle: it is now a cycle that includes the 0 node but not $\mathcal{P}$. Inner cycles (other than errant loops), outer cycles, and common circuits are said to be *plain* if they do not contain any errant loop nodes and *non-plain* otherwise.

Theorem 6. *If a digraph has plain outer cycles of lengths $l_{\mathrm{o}i}$ that contain $k_{\mathrm{o}i}$ combs ($i = 1 \ldots, N_{\mathrm{o}}$), non-plain outer cycles of lengths $\tilde{l}_{\mathrm{o}i}$ that contain $\tilde{k}_{\mathrm{o}i}$ combs with associated errant loops having lengths $l_{\mathrm{oe}i}$ and $k_{\mathrm{oe}i}$ combs ($i = 1 \ldots, \tilde{N}_{\mathrm{o}}$), plain inner cycles (excluding the errant loops) of lengths L_r that contain K_r combs ($r = 1, \ldots, N$), non-plain inner cycles of lengths $\tilde{L}_r$ that contain $\tilde{K}_r$ combs with associated errant loops having lengths $l_{\mathrm{e}r}$ and $k_{\mathrm{e}r}$ combs ($r = 1, \ldots, \tilde{N}$), plain common circuits of lengths $l_{\mathrm{c}i}$ that contain $k_{\mathrm{c}i}$ combs ($i = 1, \ldots, N_{\mathrm{c}}$), non-plain common circuits of lengths $\tilde{l}_{\mathrm{c}i}$ that contain $\tilde{k}_{\mathrm{c}i}$ combs with associated errant loops having lengths $l_{\mathrm{ce}i}$ and $k_{\mathrm{ce}i}$ combs ($i = 1, \ldots, \tilde{N}_{\mathrm{c}}$), and all non-plain cycles and circuits contain exactly one errant node, then the generating function whose coefficient of $x^n y^k$ gives the number of tilings of an n-board that use k combs is given by*

$$G(x,y) = \frac{1}{1 - \displaystyle\sum_{i=1}^{N_{\mathrm{o}}} x^{l_{\mathrm{o}i}} y^{k_{\mathrm{o}i}} - \sum_{i=1}^{\tilde{N}_{\mathrm{o}}} \frac{x^{\tilde{l}_{\mathrm{o}i}} y^{\tilde{k}_{\mathrm{o}i}}}{1 - x^{l_{\mathrm{oe}i}} y^{k_{\mathrm{oe}i}}} - \dfrac{\displaystyle\sum_{i=1}^{N_{\mathrm{c}}} x^{l_{\mathrm{c}i}} y^{k_{\mathrm{c}i}} + \sum_{i=1}^{\tilde{N}_{\mathrm{c}}} \frac{x^{\tilde{l}_{\mathrm{c}i}} y^{\tilde{k}_{\mathrm{c}i}}}{1 - x^{l}_{\mathrm{ce}i} y^{k_{\mathrm{ce}i}}}}{1 - \displaystyle\sum_{r=1}^{N} x^{L_r} y^{K_r} - \sum_{r=1}^{\tilde{N}} \frac{x^{\tilde{L}_r} y^{\tilde{K}_r}}{1 - x^{l_{\mathrm{e}r}} y^{k_{\mathrm{e}r}}}}}. \qquad (2)$$

Proof. The proof is similar to that of Theorem 5; we simply have more possible types of walk. The generating functions corresponding to a single traversal of the i-th plain outer cycle, the i-th plain common circuit, and the r-th plain inner cycle are, respectively, $x^{l_{\mathrm{o}i}} y^{k_{\mathrm{o}i}}$, $x^{l_{\mathrm{c}i}} y^{k_{\mathrm{c}i}}$, and $x^{L_r} y^{K_r}$. On application of the series and loop rules, one sees that the generating functions corresponding to a single traversal of the i-th non-plain outer cycle, the i-th non-plain common circuit, and the r-th non-plain inner cycle in addition to an arbitrary number of traversals of their errant loops are, respectively, $x^{\tilde{l}_{\mathrm{o}i}} y^{\tilde{k}_{\mathrm{o}i}}/(1 - x^{l_{\mathrm{oe}i}} y^{k_{\mathrm{oe}i}})$, $x^{\tilde{l}_{\mathrm{c}i}} y^{\tilde{k}_{\mathrm{c}i}}/(1 - x^{l_{\mathrm{ce}i}} y^{k_{\mathrm{ce}i}})$, and $x^{L_r} y^{K_r}/(1 - x^{l_{\mathrm{e}r}} y^{k_{\mathrm{e}r}})$. In Fig. 2b these are represented, respectively, by the walks $0 \to \mathcal{E}_0 \to 0$, $0 \to \mathcal{P} \to \mathcal{E}_1 \to 0$ or $0 \to \mathcal{P} \to \mathcal{E}_2 \to \mathcal{R} \to 0$, and $\mathcal{P} \to \mathcal{E}_2 \to \mathcal{R} \to \mathcal{P}$, where, on reaching $\mathcal{E}_i$, its loop can be executed an arbitrary number of times. It should now be apparent where the terms in the denominator of (2) arise from. The first two sums are from the plain and non-plain outer cycles. The numerator of the final term is the generating function for all the common circuits (including arbitrary number of traversals of the errant loops in the case of the non-plain ones). On reaching $\mathcal{P}$ via the first part of a common circuit, the inner cycles (and their associated errant loops in the case of the non-plain ones) can be executed in any order any number of times, which is what the denominator of the final term corresponds to (by application of all three rules). □

To obtain the generating functions for $S_n^{\mathcal{Q}}$ and $S_{n,k}^{\mathcal{Q}}$, which we denote by $g^{\mathcal{Q}}(x)$ and $g^{\mathcal{Q}}(x,y)$, respectively, we require the following result.

Corollary 7. *If $G(x,y)$ is the bivariate generating function for $B_{n,k}^{\mathcal{Q}}$ then*

$$g^{\mathcal{Q}}(x,y) = \frac{1}{x^q}\left(G(x,y) - \frac{1-x^q}{1-x}\right),$$

where q is the largest element of $\mathcal{Q}$, and $g^{\mathcal{Q}}(x) = g^{\mathcal{Q}}(x, 1)$.

Proof. From Theorem 1 we have $S^{\mathcal{Q}}_{n,k} = B^{\mathcal{Q}}_{n+q,k}$, which implies that we can obtain $g^{\mathcal{Q}}(x, y)$ by removing the first q terms from $G(x, y)$, namely, $1 + x + x^2 + \cdots + x^{q-1}$ (since the only way to tile a board of length less than q is with no $\mathcal{Q}$-combs), and then by dividing the result by x^q to obtain the correct offset. □

3 Bijections between strongly restricted permutations and restricted combinations

A strongly restricted permutation π of the set $\mathbb{N}_n$ is a permutation for which the number of permissible values of $\pi(i) - i$ for each $i \in \mathbb{N}_n$ is less than a finite number independent of n [17]. Here we give two bijections between restricted combinations and types of strongly restricted permutations where the permutations satisfy $\pi(i) - i \in \mathcal{D}$.

3.1 Connection with the class $\mathcal{Q} = \{1, q\}$

Our first bijection concerns a strongly restricted permutation where $\mathcal{D} = \{-1, 0, 1\}$ and no two of q consecutive $\pi(i)$ may differ from one another by more than q.

Theorem 8. *There is a bijection between the k-subsets of $\mathbb{N}_n$ such that no two elements of the subset differ by an element of $\mathcal{Q} = \{1, q\}$, where $q = 2, 3, \ldots$, and the permutations π of $\mathbb{N}_{n+1}$ that have k pairs of consecutive items that have been exchanged such that for all $i = 1, \ldots, n + 2 - q$ the condition $\max_{j=0,\ldots,q-1} \pi(i + j) - \min_{j=0,\ldots,q-1} \pi(i + j) \leq q$ is met.*

Proof. The permutation π of $\mathbb{N}_{n+1}$ corresponding to an allowed subset $\mathcal{S}$ of $\mathbb{N}_n$ where $\mathcal{Q} = \{1, q\}$ is formed as follows. If $i \in \mathcal{S}$ (where $i = 1, \ldots, n$) then $\pi(i) = i + 1$ and $\pi(i + 1) = i$ (i.e., a pair of two consecutive items has been exchanged) whereas if neither i nor $i - 1$ are in $\mathcal{S}$ (for $i = 2, \ldots, n + 1$) then $\pi(i) = i$. As $\mathcal{Q}$ contains 1, once a pair of items has been exchanged, neither item will be moved again. Given that $\pi(i)$ can be no more than 1 away from i, the only way that the condition could be violated would be if $\pi(i+1) = i$ (and hence $\pi(i) = i + 1$) and $\pi(i + q) = i + q + 1$; this could only occur if both i and $i + q$ were in $\mathcal{S}$. This is impossible since $q \in \mathcal{Q}$. The process of forming a permutation from a subset is clearly reversible and so the bijection is established. □

For example, the allowed subsets of $\mathbb{N}_5$ when $\mathcal{Q} = \{1, 3\}$ followed by the corresponding permutation of $\mathbb{N}_6$ are {} 123456, {1} 213456, {2} 132456, {3} 124356, {4} 123546, {5} 123465, {1,3} 214356, {2,4} 132546, {3,5} 124365, {1,5} 213465, and {1,3,5} 214365.

Theorem 8 has connections to the following sequences in the OEIS [21]: A000930 ($S^{\{1,2\}}_n$) and A102547 ($S^{\{1,2\}}_{n,k}$), its associated triangle; A130137 ($S^{\{1,3\}}_n$), which we mention again in the section on bit strings; A263710 ($S^{\{1,4\}}_n$), which was the sequence that led to the discovery

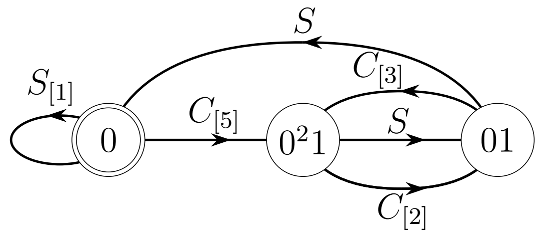

Figure 3: Digraph for restricted-overlap tiling with squares (S) and $(2,2,1)$-combs (C), which corresponds to the $\mathcal{Q} = \{1,4\}$ case. In this and subsequent figures showing digraphs, each number in square brackets gives the contribution to the length of the metatile resulting from the addition of the tile. The subscript is omitted if there is no contribution, as is the case for any square placed in a gap of a comb.

of the bijection and we now prove the conjectured generating function for that entry; along with A374737 ($S_n^{\{1,5\}}$) and A385870 ($S_n^{\{1,6\}}$), which the author recently added to the OEIS.

The case $\mathcal{Q} = \{1,4\}$ falls into the class covered by Theorem 8 in All24, which gives recursion relations for $B_n^{\mathcal{Q}}$ and $B_{n,k}^{\mathcal{Q}}$. However, in order to avoid introducing the notation used in that theorem at this stage, we instead obtain the generating function using Theorem 5, which is also straightforward and illustrative to do.

Proposition 9. *The generating function $g^{\{1,4\}}(x)$ for the number of restricted combinations of $\mathbb{N}_n$ with disallowed differences $\{1,4\}$ is given by*

$$g^{\{1,4\}}(x) = \frac{1 + x + x^2 + x^3 + 2x^4 + x^6}{1 - x - x^3 + x^4 - 2x^5 + x^6 - x^7}. \tag{3}$$

Proof. The $\mathcal{Q}$-comb is a $(2,2,1)$-comb. The metatile-generating digraph for tiling with squares and such combs (Fig. 3) has two inner cycles ($SC_{[3]}$ and $C_{[2]}C_{[3]}$). These have lengths $L_1 = 3$ and $L_2 = 5$. We choose $0^2 1$ as the common node. The common circuits ($C_{[5]}S^2$ and $C_{[5]}C_{[2]}S$) have lengths $l_{c1} = 5$ and $l_{c2} = 7$. There is a single outer cycle (S). This has length $l_{o1} = 1$. Then from Theorem 5 and Corollary 7 we get

$$g^{\{1,4\}}(x) = \frac{1}{x^4}\left(\frac{1 - x^3 - x^5}{1 - x^3 - x^5 - x + x^4 + x^6 - x^5 - x^7} - \frac{1 - x^4}{1 - x}\right), \tag{4}$$

which reduces to (3). $\square$

As our first application of Theorem 6, we also obtain the generating function for the next sequence in the series, i.e., that corresponding to $\mathcal{Q} = \{1,5\}$ (A374737).

Proposition 10. *The generating function $g^{\{1,5\}}(x)$ for the number of restricted combinations of $\mathbb{N}_n$ with disallowed differences $\{1,5\}$ is given by*

$$g^{\{1,5\}}(x) = \frac{1 + 2x + 2x^2 + 2x^3 + 2x^4 + 4x^5 + 3x^6 + 2x^7 + x^8 + x^9}{1 - x^2 - x^3 - x^4 + x^5 - x^6 - x^7 - x^8 - x^{10}}. \tag{5}$$

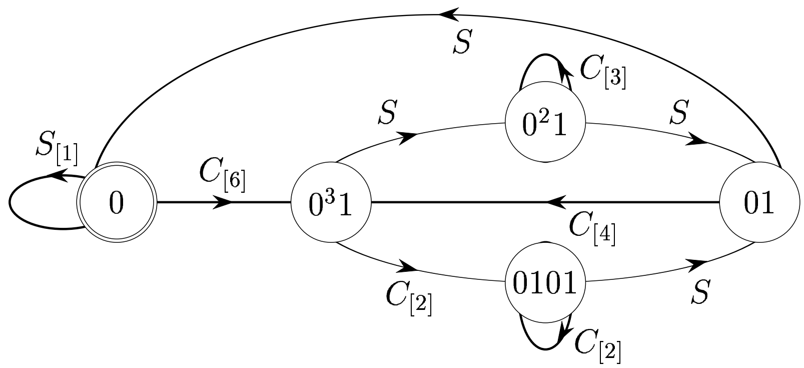

Figure 4: Digraph for restricted-overlap tiling with squares (S) and $(2,3,1)$-combs (C), which corresponds to the $\mathcal{Q} = \{1,5\}$ case.

Proof. In this case, the $\mathcal{Q}$-comb is a $(2,3,1)$-comb and the metatile-generating digraph (Fig. 4) falls into the category covered by Theorem 6. We choose the 0^31 node as the pseudo-common node $\mathcal{P}$. There is just one outer cycle (S). It is plain and its length $l_{\text{o}1} = 1$. The common circuits ($C_{[6]}S^3$ and $C_{[6]}C_{[2]}S^2$) are non-plain and have lengths $\tilde{l}_{\text{c}1} = 6$ and $\tilde{l}_{\text{c}2} = 8$; their respective errant loops ($C_{[3]}$ and $C_{[2]}$) have lengths $l_{\text{ce}1} = 3$ and $l_{\text{ce}2} = 2$. The inner cycles passing through $\mathcal{P}$ ($S^2C_{[4]}$ and $C_{[2]}SC_{[4]}$) are non-plain and have lengths $\tilde{L}_1 = 4$ and $\tilde{L}_2 = 6$; their respective errant loops ($C_{[3]}$ and $C_{[2]}$) have lengths $l_{\text{e}1} = 3$ and $l_{\text{e}2} = 2$. From (2) and Corollary 7 we have

$$g^{\{1,5\}}(x) = \frac{1}{x^5}\left(\frac{1}{1 - x - \dfrac{\dfrac{x^6}{1-x^3} + \dfrac{x^8}{1-x^2}}{1 - \dfrac{x^4}{1-x^3} - \dfrac{x^6}{1-x^2}}} - \frac{1-x^5}{1-x}\right),$$

which reduces to (5). □

The metatile-generating digraph corresponding to the next case in the $\{1,q\}$ series (Fig. 5) does not fall into any of the classes of digraph that we have derived a generating function for so far. However, as we now show, the systematic application of the digraph generating function rules (Lemmas 2, 3, and 4) can still be used to yield the generating function in this instance. To assist in breaking the derivation into more manageable chunks, we introduce the following notation. We let $G^{\{\text{Z}\}}_{\text{X}\to\text{Y}}$ denote the generating function corresponding to all possible walks from node X to node Y that do not include node X a second time and never

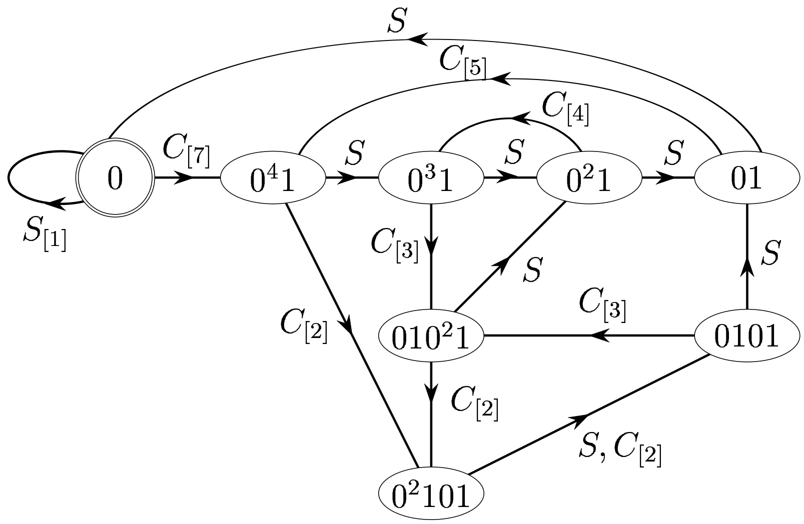

Figure 5: Digraph for restricted-overlap tiling with squares (S) and $(2,4,1)$-combs (C), which corresponds to the $\mathcal{Q} = \{1,6\}$ case. A comma-separated list of labels at an arc signifies a multiarc with those labels.

encounter node Z. We let $G_{\mathrm{X}}^{\{\mathrm{Z}\}}$ denote the generating function corresponding to all possible walks from node X to itself without visiting itself or node Z in between.

Proposition 11. *The generating function $g^{\{1,6\}}(x)$ for the number of restricted combinations of $\mathbb{N}_n$ with disallowed differences $\{1,6\}$ is given by*

$$g^{\{1,6\}}(x) = \frac{1+x+x^2+2x^3+2x^4+2x^5+5x^6+3x^8+2x^9-x^{10}+x^{11}-4x^{12}-x^{13}-2x^{14}-2x^{15}-x^{17}}{1-x-x^4-x^5+2x^6-4x^7+2x^8-2x^{10}+2x^{11}-4x^{12}+3x^{13}-x^{14}+2x^{16}-x^{17}+x^{18}}. \tag{6}$$

Proof. The generating function $G(x,y)$ corresponding to walks starting at the 0 node of the digraph in Figure 5 satisfies

$$\frac{1}{G(x,y)} = 1 - x - \frac{x^7 y\, G^{\{0\}}_{0^41\to 01}}{1 - G^{\{0\}}_{0^41}},$$

where, owing to the fact that the only way to return to the 0^41 node is via the $C_{[5]}$ arc from the 01 node, we have $G^{\{0\}}_{0^41} = x^5 y\, G^{\{0\}}_{0^41\to 01}$. The arcs leaving the 0^41 node are S and $C_{[2]}$ (the latter contributes a factor of x^2y to the corresponding term in the generating function) and the only way to reach the 01 node is from the 0^21 and 0101 nodes. Hence

$$G^{\{0\}}_{0^41\to 01} = \frac{G^{\{01\}}_{0^31\to 0^21} + G^{\{01\}}_{0^31\to 0101}}{1 - G^{\{01\}}_{0^31}} + \frac{x^2 y\big(G^{\{01\}}_{0^2101\to 0^21} + G^{\{01\}}_{0^2101\to 0101}\big)}{1 - G^{\{01\}}_{0^2101}}.$$

There is a direct route from 0^31 to 0^21 along with those that leave 0^31 via the $C_{[3]}$ arc (which contributes a factor of x^3y). On reaching 010^21 there are two possible loops it can execute ($C_{[2]}SC_{[3]}$ or $C_{[2]}^2C_{[3]}$) before reaching 0^21. Hence

$$G_{0^31\to0^21}^{\{01\}}=1+\frac{x^3y}{1-x^5y^2-x^7y^3}.$$

Since the only allowed route returning to the 0^31 node is the $C_{[4]}$ arc from the 0^21 node, $G_{0^31}^{\{01\}}=x^4y\,G_{0^31\to0^21}^{\{01\}}$. In a similar fashion we obtain

$$G_{0^31\to0101}^{\{01\}}=\frac{x^5y^2(1+x^2y)}{1-x^5y^2-x^7y^3},\qquad G_{0^2101\to0^21}^{\{01\}}=\frac{(1+x^2y)x^3y}{1-x^4y-x^7y^2},$$

and $G_{0^2101\to0^21}^{\{01\}}=1+x^2y$. The $1-x^7y^2/(1-x^4y)$ in

$$G_{0^2101}^{\{01\}}=\frac{(1+x^2y)x^5y^2}{1-\dfrac{x^7y^2}{1-x^4y}}$$

results from a loop within a loop: the $SC_{[4]}$ loop can be executed on arriving at the 0^31 node while traversing the $SC_{[4]}C_{[3]}$ loop. Combining the above results to give $G(x,y)$ and applying Corollary 7 yields (6) after simplifying. □

3.2 Connection with the class $\mathcal{Q}=\{m,2m,\dots,jm\}$

The proof we give of the second bijection (Theorem 13) requires the following bijection (established in previous work [9]) between strongly restricted permutations and tiling with $(\frac{1}{2},g)$-fences (which are just $(\frac{1}{2},g,\frac{1}{2})$-combs), where g is always a nonnegative integer. As the teeth (originally called posts in the context of fences) are not of integer width, it is convenient to regard each cell as being divided into two *slots* into which a tooth can fit [10].

Lemma 12. *There is a bijection between (i) the permutations of $\mathbb{N}_n$ satisfying $\pi(i)-i\in\mathcal{D}$ for each $i\in\mathbb{N}_n$ and (ii) the tilings of an n-board using $(\frac{1}{2},d_j)$-fences with the left tooth always placed in a left slot and d_j equal to the non-negative elements of $\mathcal{D}$ and $(\frac{1}{2},-d_k-1)$-fences with their gaps aligned with cell boundaries and d_k equal to the negative elements of $\mathcal{D}$. If the left slot of cell i of the n-board is occupied by the left (right) tooth of a $(\frac{1}{2},g)$-fence then $\pi(i)=i+g$ ($\pi(i)=i-g-1$).*

For example, the tiling in Fig. 6 corresponds to the permutation

$$\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13\\1&11&3&4&2&6&7&5&9&10&8&12&13\end{pmatrix}.$$

We refer to a fence corresponding to an excedance (i.e., a position i such that $\pi(i)>i$) as an *up fence* and a fence giving $\pi(i)-i<0$ as a *down fence*. The case of a fixed point ($\pi(i)=i$)

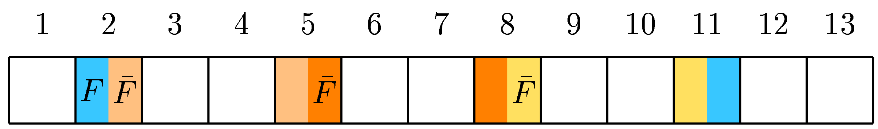

Figure 6: A 13-board tiled with an up $(\frac{1}{2},9)$-fence (F; blue), three down $(\frac{1}{2},2)$-fences ($\bar{F}$; yellow, orange, and ochre), and 9 squares.

corresponds to a gapless fence. This is just an ordinary square tile (that must be aligned with a cell).

The following bijection is an extension of that noted but not proved by Baltić [7]. The proof we give requires the notion of a comb with t teeth all of length w separated by gaps of length g, which we refer to as a $(w,g;t)$-comb [4, 5].

Theorem 13. *There is a bijection between the k-subsets of $\mathbb{N}_n$ with disallowed differences $\mathcal{Q}=\{m,2m,\dots,jm\}$, where $j,m\geq 1$, and the permutations π of $\mathbb{N}_{n+jm}$ satisfying $\pi(i)-i\in\{-m,0,jm\}$ for all $i\in\mathbb{N}_{n+jm}$ that contain k excedances.*

Proof. From Lemma 12, the number of such permutations is the number of ways to tile an $(n+jm)$-board using squares aligned with the cell boundaries (S), up $(\frac{1}{2},jm)$-fences (F), and down $(\frac{1}{2},m-1)$-fences ($\bar{F}$). Suppose the left tooth of the first F on the board occupies the left slot of cell i. The only possible way to fill the right slot of cell i is with the left tooth of an $\bar{F}$. The right tooth of the $\bar{F}$ lies on the left slot of cell $i+m$. If $j=1$, the right slot of this cell is occupied by the right tooth of the first F. Otherwise it must again be occupied by the left tooth of another $\bar{F}$ and similarly until the right tooth of the first F is reached at cell $i+jm$. Hence this F has j $\bar{F}$ placed end-to-end in its interior. Combined, these $j+1$ tiles occupy exactly the same cells as a $(1,m-1;j+1)$-comb (Fig. 6 shows an example when $j=m=3$). All subsequent F on the board behave similarly. Note that owing to the fact that any $\bar{F}$ must have its left tooth in the right slot of a cell, such tiles can only appear on the board in conjunction with $j-1$ other $\bar{F}$ surrounded by an F. Hence there is a bijection between tilings of an $(n+jm)$-board using k F, jk $\bar{F}$ and $n+jm-(j+1)k$ S and the tilings of a board of the same length using k $(1,m-1;j+1)$-combs and $n+jm-(j+1)k$ squares. □

We let $P_n^{\mathcal{D}}$ denote the number of permutations π of $\mathbb{N}_n$ such that $\pi(i)-i\in\mathcal{D}$ for all $i\in\mathbb{N}_n$ and $P_{n,k}^{\mathcal{D}}$ denote the number of such permutations with k excedances. We now use the bijection and previously obtained results to relate $P_n^{\{-m,0,jm\}}$ and $P_{n,k}^{\{-m,0,jm\}}$ to some generalized Fibonacci numbers and the coefficients of their corresponding polynomials, respectively. We let $f_n^{(t)}$ denote the n-th $(1,t)$-bonacci number defined by $f_n^{(t)}=f_{n-1}^{(t)}+f_{n-t}^{(t)}+\delta_{n,0}$, $f_{n<0}^{(t)}=0$, and $f_n^{(t)}(x)$ denote the $(1,t)$-bonacci polynomial defined by $f_n^{(t)}(x)=f_{n-1}^{(t)}(x)+xf_{n-t}^{(t)}(x)+\delta_{n,0}$, $f_{n<0}^{(t)}(x)=0$.

Corollary 14. *For $m, j \geq 1$, $i, k \geq 0$, and $r = 0, \ldots, m-1$, we have*

$$P_{im+r}^{\{-m,0,jm\}} = \left(f_i^{(j+1)}\right)^{m-r}\left(f_{i+1}^{(j+1)}\right)^r, \tag{7}$$

$$P_{im+r,k}^{\{-m,0,jm\}} = [x^k]\Big(\left(f_i^{(j+1)}(x)\right)^{m-r}\left(f_{i+1}^{(j+1)}(x)\right)^r\Big), \tag{8}$$

where $[x^k]R(x)$ denotes the coefficient of x^k in $R(x)$.

Proof. Using a previously obtained result [1, Corollary 11], for $l \geq 0$, $m \geq 1$, $t \geq 2$, and $r = 0, \ldots, m-1$, we have

$$S_{lm+r}^{\{m,2m,\ldots,(t-1)m\}} = \left(f_{l+t-1}^{(t)}\right)^{m-r}\left(f_{l+t}^{(t)}\right)^r. \tag{9}$$

As a consequence of the bijection given in Theorem 13, $P_{n+jm}^{\{-m,0,jm\}} = S_n^{\{m,2m,\ldots,jm\}}$ and so $P_n^{\{-m,0,jm\}} = S_{n-jm}^{\{m,2m,\ldots,jm\}}$. Replacing n by $im + r$ in this and using (9) with t replaced by $j+1$ and l replaced by $i-t+1$ gives (7). Similarly, using another result from previous work [1, Corollary 10], we obtain

$$S_{lm+r,k}^{\{m,2m,\ldots,(t-1)m\}} = [x^k]\Big(\left(f_{l+t-1}^{(t)}(x)\right)^{m-r}\left(f_{l+t}^{(t)}(x)\right)^r\Big).$$

Combining this with the result from Theorem 13 that $P_{n+jm,k}^{\{-m,0,jm\}} = S_{n,k}^{\{m,2m,\ldots,jm\}}$ in the same way gives (8). □

Sequences in the OEIS that relate to Theorem 13 and Corollary 14 along with the corresponding $\mathcal{D}$ are as follows: A000045 $\{-1,0,1\}$, A006498 $\{-2,0,2\}$, A006500 $\{-3,0,3\}$, A031923 $\{-4,0,4\}$, A224809 $\{-2,0,4\}$, A224810 $\{-3,0,6\}$, A224811 $\{-2,0,8\}$, A224812 $\{-2,0,10\}$, A224813 $\{-2,0,12\}$, A224814 $\{-3,0,9\}$, A224815 $\{-4,0,8\}$.

4 Connections with compositions

A composition of a positive integer n is a way of expressing n as a sum of positive integers (referred to as *parts*) where the order of the parts is significant; thus, for example, 1+2 and 2+1 are different compositions of 3. For convenience, it is customary to take the number of compositions of 0 (any negative integer) as being 1 (0). It is well known that the number of compositions of n is the same as the number of tilings of an n-board using m-ominoes for arbitrary nonnegative integers m and hence that the number of compositions of n into distinct parts $m_1, m_2, \ldots, m_i, \ldots$ is given by $B_n = \delta_{n,0} + \sum_{i=1} B_{n-m_i}$, $B_{n<0} = 0$ for all integers n. It follows that any tiling in which all possible metatiles have distinct lengths corresponds to a composition into parts equal to the lengths of the metatiles.

4.1 Compositions with parts drawn from finite sets

We first examine two classes of $\mathcal{Q}$ where there are a finite number of possible metatiles (of distinct lengths) when restricted-overlap tiling with squares and $\mathcal{Q}$-combs. Recall that the number of possible metatiles is finite iff $2r \geq q$, where r is the length of the rightmost tooth of the $\mathcal{Q}$-comb [2, Lemma 2]. It is sometimes more convenient to express this condition as $r \geq s - 1$, where $s = q + 1 - r$ is the sum of the lengths of the teeth and gaps up to the rightmost tooth.

Lemma 15. *Suppose $q > 0$ is the largest element of $\mathcal{Q}$. Then $S_n^{\mathcal{Q}}$ is the number of compositions of $n+q$ into parts which are drawn from a finite set of distinct parts iff every metatile, when restricted-overlap tiling with squares and $\mathcal{Q}$-combs, contains no more than two combs. Then the parts are 1, $q + 1$, and $q + 1 + p_i$ (for $i = 1, \ldots, |\mathbb{N}_q \setminus \mathcal{Q}|$), where p_i is the cell number of the i-th empty cell in the $\mathcal{Q}$-comb.*

Proof. If there are a finite number of parts, there are a finite number of metatiles. This means that, in all cases, the start of the final comb in a metatile containing more than one comb must lie within the first comb (see the proof of Lemma 2 in All24). If there were a third comb present, the cells it occupies without overlapping (of which there must be at least one, i.e., its cell 0) could be replaced by squares and give rise to a metatile of the same length. The parts would then not be distinct. The square and filled comb metatiles are of length 1 and $q + 1$, respectively. The remaining metatiles contain two combs. If the start of the second comb is in cell p_i of the first comb, the resulting metatile (completed by filling any remaining empty cells with squares) is easily seen to have a length of $q + 1 + p_i$. □

In the proofs of the following two theorems we employ combs that, in general, have a periodic pattern of teeth and gaps before the rightmost tooth is reached. Extending the notation used for combs with teeth all the same length and gaps all the same length [4, 5], we call an $(l, g, l, g, \ldots, r)$-comb with t teeth an $(l, g, r; t)$-comb and we refer to an $(l, g, m, h, l, g, m, h, \ldots, r)$-comb with t teeth as an $(l, g, m, h, r; t)$-comb. Note that an r-omino (an (l, g, r)-comb) can be regarded as an $(l, g, r; t)$- or $(l, g, m, h, r; t)$-comb with $t = 1$ ($t = 2$). Recall that the inclusion of 0 in $\mathcal{Q}$ is merely for convenience in the statement of the theorems and is to be removed when giving particular instances of $\mathcal{Q}$ that the theorems apply to.

Theorem 16. *If*

$$\mathcal{Q} = \{0, 1, \ldots, l-1, l+g, l+g+1, \ldots, 2l+g-1, 2(l+g), 2(l+g)+1, \ldots, \\ (t-1)(l+g), \ldots, (t-1)(l+g)+r-1 \equiv q\}$$

for some $l \geq g > 0$, $t \in \mathbb{Z}^+$, $q > 0$, and, for $t > 1$, $r \geq (t-1)(l+g) - 1$, then $S_n^{\mathcal{Q}}$ is the number of compositions of $n + q$ into parts 1 and $q + 1$ when $t = 1$ and parts 1, $q + 1$, $q+l+1, \ldots, q+l+g$, $q+2l+g+1, \ldots, q+2(l+g), \ldots, q+(t-1)l+(t-2)g+1, \ldots, q+(t-1)(l+g)$ otherwise. Note that $\mathcal{Q}$ reduces to $\{0, 1, \ldots, r-1 = q\}$ when $t = 1$.

Figure 7: Construction of instances of metatiles containing two combs when the combs are (a) (2,2,7;3)-combs, (b) (5,2,1,2,9;3)-combs, and (c,d) (2,2,3,2,8;3)-combs. In each case, the second comb is displaced downwards a little from its final position so that parts of the cells of the first comb it overlaps are visible.

Proof. The $\mathcal{Q}$-comb in this case is an $(l, g, r; t)$-comb. When $t = 1$, there are no two-comb metatiles; the condition $r > 1$ ensures that the filled comb metatile (which in this case is just the comb) is not the same length as the square. For $t > 1$, the condition $r \geq (t-1)(l+g)-1$ is the same as $2r \geq q$ and thus establishes that there are a finite number of metatiles. Then the condition $l \geq g$ ensures that a metatile can have at most two combs (Fig. 7(a)). This is because if $g > l$, the leftmost tooth of a comb and the leftmost cell of another comb could both be placed within the first gap of the initial comb. The result then follows from Lemma 15. □

OEIS sequences corresponding to a $\mathcal{Q}$ that Theorem 16 applies to along with the set itself are as follows: A000045 {1}, A006498 {2}, A000930 {1,2}, A079972 {2,3}, A003269 {1,2,3}, A351874 {1,3,4}, A121832 {2,3,4}, A003520 {1,2,3,4}, A375985 {1,3,4,5}, A259278 {2,3,4,5}, A005708 {1,2,3,4,5}, A276106 {2,3,4,5,6}, A005709 {1,2,3,4,5,6}, A322405 {2,3,4,5,6,7}, A005710 {1,2,3,4,5,6,7}, A368244 {2,3,4,5,6,7,8}.

Theorem 16 covers all cases where there are a finite number of metatiles for $q \leq 7$. The following generalization of the theorem (in which the $\mathcal{Q}$-combs are periodic in the first two teeth and gaps up to the final tooth) extends this to $q \leq 11$.

Theorem 17. *If*

$$\begin{aligned}\mathcal{Q} = \{&0, 1, \ldots, l-1, l+g, l+g+1, \ldots, l+g+m-1,\\ &l+g+m+h, l+g+m+h+1, \ldots, 2l+g+m+h-1,\\ &2(l+g)+m+h, 2(l+g)+m+h+1, \ldots, 2(l+g+m)+h-1, \ldots, *\},\end{aligned}$$

$$* = \begin{cases} \frac{t-1}{2}(l+g+m+h), \ldots, \frac{t-1}{2}(l+g+m+h)+r-1 = q > 0, & t \text{ odd};\\ \frac{t}{2}(l+g)+\left(\frac{t}{2}-1\right)(m+h), \ldots, \frac{t}{2}(l+g)+\left(\frac{t}{2}-1\right)(m+h)+r-1 = q, & t \text{ even};\end{cases}$$

where $g, m, h, t \in \mathbb{Z}^+$*, and, for* $t > 1$*,* $r \geq \frac{1}{2}(t-1)(l+g+m+h) - 1$ *if* t *is odd and* $r \geq \frac{1}{2}t(l+g) + (\frac{1}{2}t-1)(m+h) - 1$ *if* t *is even, and either (i)* $l \geq g+m+h$ *or (ii)* $l \geq g, h$ *and* $m \geq l-1+g$*, then* $S_n^{\mathcal{Q}}$ *is the number of compositions of* $n+q$ *into parts 1 and* $q+1$ *when* $t=1$ *and, when* $t>1$*, parts 1,* $q+1$*,* $q+l+1, \ldots, q+l+g$*,* $q+l+g+m+1, \ldots, q+l+g+m+h, \ldots, *$*, where*

$$* = \begin{cases} q + \dfrac{t-1}{2}(l+g+m) + \dfrac{t-3}{2}h + 1, \ldots, q + \dfrac{t-1}{2}(l+g+m+h), & t \text{ odd;} \\ q + \dfrac{t}{2}l + \left(\dfrac{t}{2}-1\right)(g+m+h) + 1, \ldots, q + \dfrac{t}{2}(l+g) + \left(\dfrac{t}{2}-1\right)(m+h), & t \text{ even.} \end{cases}$$

Proof. Here the $\mathcal{Q}$-comb is an $(l, g, m, h, r; t)$-comb. The $t = 1, 2$ cases are covered by the proof of Theorem 16. First note that if t is odd, the length of the comb $q+1 = \frac{1}{2}(t-1)(l+g+m+h)+r$ and it is $\frac{1}{2}t(l+g)+(\frac{1}{2}t-1)(m+h)+r$ if t is even from which the respective conditions on r (so that there are a finite number of metatiles) follow. There are two mutually exclusive sets of conditions that ensure that the metatiles contain no more than two combs. Condition (i) that $l \geq g+m+h$ (see Fig. 7(b)) arises as a result of the observation that if l were one less than $g+m+h$ and a second comb were placed starting at the first gap of the first comb, the start a third comb could be placed starting at the final cell of the second gap of the first comb. The first part of condition (ii), namely that $l \geq g, h$ is for the same reasons as the $l \geq g$ condition in Theorem 16. We must also have $l+g+m \geq g+m+h$ (see Fig. 7(c)). This also implies that $l \geq h$. Were $l+g+m$ one less than $g+m+h$, it can be seen that after placing the second comb at the start of the first gap, a third comb could be placed at the end of the second gap of the first comb. At the same time, we must ensure that m is not too small; otherwise the start of a third comb could be inserted at the start of the second gap of the first comb. Thus $l-1+g \leq m$ (see Fig. 7(d)). The result then follows from Lemma 15. □

4.2 Compositions with parts drawn from infinite sets

We now turn to compositions where the parts are drawn from infinite sets. In the context of tiling this means that there is an infinite number of possible metatiles and so the metatile-generating digraph has an inner cycle. We start by showing that the metatile lengths can only be all different if there is just one inner cycle.

Lemma 18. *If the metatile-generating digraph for a tiling contains a common node and more than one inner cycle then there exist distinct metatiles of the same length.*

Proof. Suppose the digraph contains inner cycles I_1 and I_2. If the sequence of nodes and arcs to create one metatile involves executing I_1 followed by I_2 then a distinct metatile of the same length can be generated by swapping the order in which the inner cycles are traversed. □

Four classes of $\mathcal{Q}$ whose associated digraphs possess common nodes have been investigated [2, Theorems 5–8]. One of these classes (Theorem 8 in All24) always has at least two inner

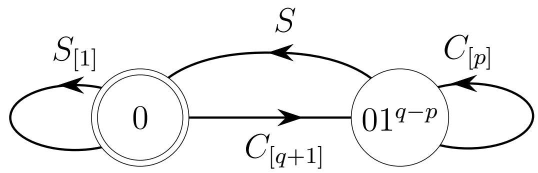

Figure 8: Digraph for restricted-overlap tiling with squares (S) and $(p, 1, q-p)$-combs (C), where $2p > q$. This corresponds to the $\mathcal{Q} = \{1, \ldots, p-1, p+1, \ldots, q\}$ case.

cycles; the simplest member of this class is $\mathcal{Q} = \{1, 4\}$ (see Fig. 3). Another class (Theorem 7 in All24) has just one inner cycle but always has the metatiles $C_{[q+1]}C_{[l]}^2$ and $C_{[q+1]}S^{a-1}C_{[2l]}S$, where $a = |\mathbb{N}_q \setminus \mathcal{Q}|$, which are of the same length (see Fig. 5 in All24). The metatile-generating digraph of one of the remaining classes has a inner cycles (Theorem 5 in All24). The $a = 1$ case leads to the following result.

Theorem 19. *If $\mathcal{Q} = \{1, \ldots, p-1, p+1, \ldots, q\}$, where $2p > q$, then $S_n^{\mathcal{Q}}$ is the number of compositions of $n+q$ into parts 1 and $q+1+jp$ for $j = 0, 1, 2, \ldots$.*

Proof. From the digraph (Fig. 8) it can be seen that the metatiles are S and $C_{[q+1]}C_{[p]}^jS$ for $j = 0, 1, 2, \ldots$. These are of lengths 1 and $q + 1 + jp$, respectively. □

OEIS sequences corresponding to a $\mathcal{Q}$ that Theorem 19 applies to along with the set itself are as follows: A130137 {1,3}, A317669 {1,2,4}, A375185 {1,2,3,5}, A375186 {1,2,4,5}.

The following theorem originates from Theorem 6 in All24 and as it is a rather general theorem, it employs some extra notation that we now describe. We let θ be the bit string representation of $\mathcal{Q}$ whereby the j-th bit from the right in θ is 1 if and only if $j \in \mathcal{Q}$ (we use the same bit string representation of $\mathcal{Q}$ in the program given in the Appendix). By $\lfloor \theta/2^b \rfloor$ we mean discarding the rightmost b bits in θ and shifting the remaining bits to the right b places (we employ the same notation in Theorem 21). We use $|$ to denote the bitwise OR operation.

Theorem 20. *Let $p_i \in \mathbb{N}_q \setminus \mathcal{Q}$ for $i = 1, \ldots, a$ and $p_i < p_{i+1}$. If $\theta \mid \lfloor \theta/2^{p_i-1} \rfloor$ for each $i = 1, \ldots, a-1$ is all ones after discarding the leading zeros, $a \geq 2$, and $p_a = q - r$ (which implies that $r \geq 1$), then if (a) $q = 2r + 1$ or (b) $q > 2r + 1$ and $1 \leq p_{a-1} \leq r$, then $S_n^{\mathcal{Q}}$ is the number of compositions of $n + q$ into parts 1, $q + 1$, $q + 1 + p_i$ for $i = 1, \ldots, a-1$, and $q + 1 + j(q - r)$ for $j \in \mathbb{Z}^+$.*

Proof. The digraph (Fig. 9) generates the metatiles S, $C_{[q+1]}S^a$, $C_{[q+1]}S^{i-1}C_{[p_i]}$ for $i = 1, \ldots, a-1$, and $C_{[q+1]}S^{a-1}C_{[q-r]}^jS$ for $j \in \mathbb{Z}^+$. Their respective lengths are 1, $q+1$, $q+1+p_i$, and $q + 1 + j(q - r)$. As $q - r = p_a > p_{a-1}$, the lengths are all distinct. □

The OEIS sequences to which Theorem 20 applies along with the corresponding $\mathcal{Q}$ are as follows: A224809 {2,4}, A375981 {1,4,5}, A375982 {2,3,5}, A375983 {2,4,5}.

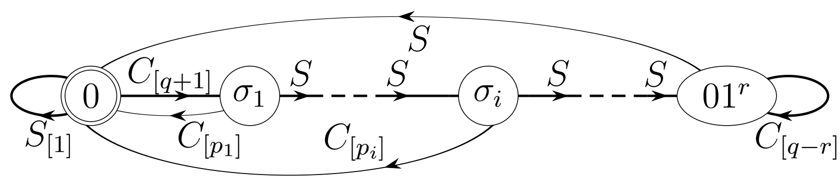

Figure 9: Digraph for tiling a board with squares and combs corresponding to $\mathcal{Q}$ specified in Theorem 20 [2]. In this figure and the next, node label σ_i denotes the bit string corresponding to filling the first $i-1$ empty cells of the $\mathcal{Q}$-comb with squares and discarding the leading 1s.

5 Connections with bit strings

Others have noted that the number of length-n bit strings (or *binary words*) having no two 1s that have positions in the string that differ by q (or, equivalently, are separated from one another by $q-1$ digits) is $S_n^{\{q\}}$ [15]. This can be generalized by considering the following bit string s representing subset $\mathcal{S}$: the j-th bit from the right in s is 1 if and only if $j \in \mathcal{S}$. (This representation is used in the program in the Appendix.) Then it is clear that $S_{n,k}^{\mathcal{Q}}$ is the number of length-n bit strings that contain k 1s placed in such a way that the difference in positions of any two 1s does not equal an element of $\mathcal{Q}$. We now mention an instance of this that corresponds to A130137 in the OEIS [21]. If one instead places the same restrictions on 0s rather than 1s, for the $\mathcal{Q}=\{1\}$ case, the bit strings (i.e., those containing no 00) are called *Fibonacci binary words*, so named as the number of such words of length n is F_{n+2}. A130137 is described as the number of length-n Fibonacci binary words that do not contain the substring 0110. This is $S_n^{\{1,3\}}$ since the three other possible disallowed substrings that only prevent the occurrence of two 0s whose positions differ by 3 (i.e., 0000, 0010, and 0100) do not need to be mentioned; they each contain 00 and so cannot be present in Fibonacci binary words.

5.1 Binary word equivalence classes

The final bijection we present concerns equivalence classes for the occurrence of a subword within binary words. We say that two binary words of the same length n are *equivalent with respect to a subword* ω if ω occurs in the same position(s) in those words. One can represent the equivalence class by the set of subword positions, counting the leftmost position as 1. For example, all binary words of the form $xx10010010xx$, where each x can be 0 or 1, belong to the same equivalence class with respect to the subword 10010 and this class is represented by the set $\{3,6\}$.

Theorem 21. *Let ω be a length-l binary subword. We construct set $\mathcal{Q}$ as follows: $j \in \mathcal{Q}$ iff $\lfloor \omega/2^j \rfloor \neq \omega \bmod 2^{l-j}$ for $j=1,\ldots,l-1$. If q is the largest element of $\mathcal{Q}$ (and taking q as zero when $\mathcal{Q}=\{\}$) and $q=l-1$, then the number of equivalence classes of binary words* *of length n with respect to subword ω is $S^{\mathcal{Q}}_{n-q}$. Furthermore, $S^{\mathcal{Q}}_{n-q,k}$ is the number of such equivalence classes whose representation as a set is of size k.*

Proof. We show that the k-subsets of $\mathbb{N}_{n-q}$ satisfying the conditions for disallowed differences specified by $\mathcal{Q}$ are the same as the sets representing the equivalence classes of binary words of length n in which ω appears k times. In general, the possible positions of any subword of length l in a word of length n (and therefore the possible elements of the set representing an equivalence class) are $1, \ldots, n-l+1 = n-q$. From a restricted subset $\mathcal{S}$ of $\mathbb{N}_{n-q}$ we construct an equivalence class of words as follows. Starting with a length-n word of xs (each x representing 0 or 1), for each $s \in \mathcal{S}$ we place the start of subword ω at position s in the word. Note that $\lfloor \omega/2^j \rfloor$ gives the $l-j$ leftmost bits of ω and $\omega \bmod 2^{l-j}$ gives the $l-j$ rightmost bits. If the two are equal (and so $j \notin \mathcal{Q}$) then, on shifting the digits of subword ω by j places to the right and overlaying them on the original subword, none of the digits of the original subword are changed. This means that we have constructed a valid equivalence class once we have specified the restrictions on the remaining xs to avoid any further appearances of ω. Note that if there is an ω at position i and no other at $i+j$ for all $j = 1, \ldots, q$, we may place another ω at position $i+l$ or after this. This coincides with the fact that $l = q+1$ or any larger integers are not contained in the set $\mathcal{Q}$ of disallowed shifts. The reversible nature of the construction establishes the bijection. □

In the OEIS, the following sequences are connected with Theorem 21 (the corresponding $\mathcal{Q}$ and the subword starting with a 1 with the lowest numerical value it applies to are also given): A000079 ($\{\}$, 1), A000045 ($\{1\}$, 10), A000930 ($\{1,2\}$, 100), A003269 ($\{1,2,3\}$, 1000), A130137 ($\{1,3\}$, 1010), A003520 ($\{1,2,3,4\}$, 10000), A317669 ($\{1,2,4\}$, 10010; it was via this OEIS sequence that the connection to restricted combinations was made), A005708 ($\{1,2,3,4,5\}$, 100000), A375185 ($\{1,2,3,5\}$, 100010), A375186 ($\{1,2,4,5\}$, 100110), A177485 ($\{1,3,5\}$, 101010). Note that if the bits of subword ω are flipped (i.e., 0 changed to 1 and vice versa) and/or reversed, the corresponding $\mathcal{Q}$ remains unchanged.

In the rest of this section we show that the elements of $\mathcal{Q}$ concerned with the bijection given in Theorem 21 are always a well-based sequence and hence the generating function for the number of equivalence classes and the recursion relation for the number of equivalence classes whose set representation is of size k are straightforward to obtain. The sequence $q_1, q_2, \ldots, q_m$, where $q_i < q_{i+1}$ for all i, is said to be a *well-based sequence* if $q_1 = 1$ and for all 2-partitions of q_j, for $j = 2, \ldots, q_m$, at least one of the two parts is a q_i [12, 23]. Thus, for example, $1, 3, 4$ is not a well-based sequence since $4 = 2 + 2$ and 2 is not in the sequence. In the present context, we can regard q_i as being the elements of $\mathcal{Q}$ with $q_m = q$, its largest element. In the proof of Theorem 22, we use the following definition, which is equivalent aside from also classing the empty set as being well based: the sequence is well based if $a = |\mathbb{N}_q \setminus \mathcal{Q}|$ is zero or if, for all $i, j = 1, \ldots, a$ (where i and j can be equal), $p_i + p_j \notin \mathcal{Q}$, where the p_i are the elements of $\mathbb{N}_q \setminus \mathcal{Q}$. E.g., the only well-based sequences of length 3 are the elements of the sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, and $\{1,3,5\}$.

Theorem 22. *If $\mathcal{Q}$ is constructed from subword ω as described in Theorem 21, then the elements of $\mathcal{Q}$ are a well-based sequence with the additional condition that $\Lambda \notin \mathcal{Q}$ where Λ is any linear combination (with integer coefficients) of the p_i.*

Proof. Suppose ω is the length-l bit string $b_l b_{l-1} \cdots b_2 b_1$. Then $p_i, p_j \notin \mathcal{Q}$ iff $b_n = b_{n-p_i}$ and $b_n = b_{n-p_j}$ for $n \leq l$ with $n \geq p_i + 1$ and $n \geq p_j + 1$, respectively. Hence we also have $b_n = b_{n-p_i-p_j}$ for $1 + p_i + p_j \leq n \leq l$. This implies that $p_i + p_j \notin \mathcal{Q}$ and hence the elements of $\mathcal{Q}$ form a well-based sequence. In an analogous way, it is easily seen that we have $b_n = b_{n-\Lambda}$ where Λ is any integer-coefficient linear combination of the p_i such that $0 < \Lambda < n$ and hence $\Lambda \notin \mathcal{Q}$. □

The additional property in Theorem 22 means, for example, that the sets $\{1, 2, 5\}$ (for which $p_1 = 3$, $p_2 = 4$) and $\{1, 2, 3, 5, 7\}$ (for which $p_1 = 4$, $p_2 = 6$) do not correspond to any ω (since in both cases $p_2 - p_1 \in \mathcal{Q}$) although their elements form well-based sequences.

The following corollary gives a generating function for $E^{(\omega)}_{n,k}$, the number of equivalence classes of length-n binary words with respect to a length-l subword ω that have a set representation of size k.

Corollary 23. *The generating function $\tilde{g}(x, y)$, whose coefficient of $x^n y^k$ equals $E^{(\omega)}_{n,k}$, is given by*

$$\tilde{g}(x, y) = \frac{1 - \bar{c}}{(1 - x)c - xy}, \tag{10}$$

where $\bar{c} = \sum_{i=1}^{a} x^{p_i} y$, $a = |\mathbb{N}_q \setminus \mathcal{Q}|$, $c = 1 + \sum_{i=1}^{|\mathcal{Q}|} x^{q_i} y$, the set $\mathcal{Q}$ is obtained from ω using the procedure in Theorem 21, q_i are the elements of $\mathcal{Q}$ of which the largest is q (with $q = 0$ when $\mathcal{Q} = \{\}$), and p_i are the elements of $\mathbb{N}_q \setminus \mathcal{Q}$, provided that $q = l - 1$.

Proof. We first deal with the $q = 0$ case. Then $\mathcal{Q} = \{\}$ and so $\bar{c} = 0$ and $c = 1$. This leaves $\tilde{g}(x, y) = 1/(1 - x - xy)$. In this case, each binary word is in an equivalence class of its own and so $E^{(\omega)}_{n,k} = \binom{n}{k}$ since this is the number of length-n binary words that contain k 1s. The generating function for $\binom{n}{k}$ is $1/(1 - x - xy)$. For $q > 0$, since $E^{(\omega)}_{n} = E^{(\omega)}_{n,0} = 1$ for $0 \leq n \leq q - 1$, and, by Theorem 21, $E^{(\omega)}_{n,k} = S^{\mathcal{Q}}_{n-q,k}$ for $n \geq q$ and from Theorem 1

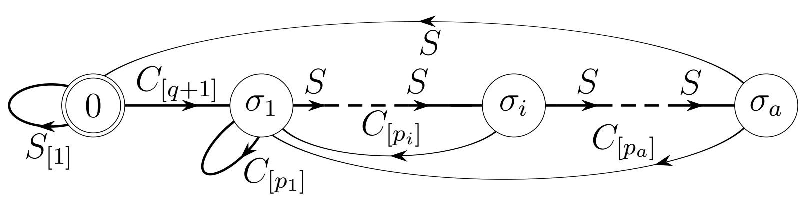

Figure 10: Digraph for tiling a board with squares and $\mathcal{Q}$-combs when the $q_i \in \mathcal{Q}$ are a well-based sequence [2].

$S^{\mathcal{Q}}_{n-q,k} = B^{\mathcal{Q}}_{n,k}$ for $n \geq q$, overall we have $E^{(\omega)}_{n,k} = B^{\mathcal{Q}}_{n,k}$. Hence $\tilde{g}(x, y) = G(x, y)$, the generating function such that the coefficient of $x^n y^k$ is the number of tilings of an n board that contain k combs. The digraph for tiling a board with squares and $\mathcal{Q}$-combs when the elements of $\mathcal{Q}$ are a well-based sequence is shown in Fig. 10 [2]. The digraph has a common node (σ_1) and the inner cycles ($S^{i-1}C_{[p_i]}$ for $i = 1, \ldots, a$) have lengths p_i. There is a single common circuit ($C_{[q+1]}S^a$), which is of length $q + 1$, and one outer cycle ($S_{[1]}$), which is of length 1. The common circuit and all the inner cycles contain a single comb. From Theorem 5 we then have

$$G(x, y) = \frac{1 - \bar{c}}{1 - \bar{c} - x + x\bar{c} - x^{q+1}y}. \tag{11}$$

Using the result that $c+\bar{c} = 1+\sum_{i=1}^{q} x^i y = 1+(x-x^{q+1})y/(1-x)$ to simplify the denominator gives (10). □

The following corollary, which is a generalization of a result obtained previously [12, Theorem 2] (and also Corollary 1 in All24), is then easily obtained from (11) by applying Corollary 7.

Corollary 24. *If the elements q_i of $\mathcal{Q}$ are a well-based sequence then the generating function such that the $x^n y^k$ coefficient thereof is $S^{\mathcal{Q}}_{n,k}$ is given by*

$$g(x, y) = \frac{c}{(1 - x)c - xy}, \tag{12}$$

where $c = 1 + \sum_{i=1}^{|\mathcal{Q}|} x^{q_i} y$.

6 Discussion

Theorem 6, which we derived here in order to obtain the generating function for $S^{\{1,5\}}_n$, will be used elsewhere to obtain expressions for generating functions for $S^{\mathcal{Q}}_{n,k}$ in terms of the elements of $\mathcal{Q}$ for various other classes of $\mathcal{Q}$. As $|\mathbb{N}_q \setminus \mathcal{Q}|$ increases, the digraphs become too complicated for the application of general results analogous to Theorem 6. In such instances, the repeated use of Lemmas 2, 3, and 4 yields the generating function, as we demonstrate in §3.1 for the $\mathcal{Q} = \{1, 6\}$ case, although this becomes laborious for larger digraphs. Then a more direct approach is to employ a transfer matrix technique. For a digraph with v nodes, the basic method gives the following expression for the generating function [22]:

$$G(x, y) = \frac{\det(\mathbf{I}_{v-1} - \check{\mathbf{T}})}{\det(\mathbf{I}_v - \mathbf{T})}, \tag{13}$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix and the (i, j)-th entry of transfer matrix $\mathbf{T}$ is the generating function for the direct connection from the j-th node to the i-th node. For convenience, one insists that the 0 node is the first node. Then $\check{\mathbf{T}}$ is obtained from $\mathbf{T}$ by removing the first column and first row. For example, choosing the $0^2 1$ and 01 nodes to be, respectively, the second and third nodes, the transfer matrix for the digraph in Figure 3 is

$$\mathbf{T} = \begin{pmatrix} x & 0 & 1 \\ x^5 y & 0 & x^3 y \\ 0 & 1 + x^2 y & 0 \end{pmatrix},$$

and (13) gives $G(x,y) = (1 - x^3 y - x^5 y^2)/(1 - x - x^3 y + x^4 y - x^5(y + y^2) + x^6 y^2 - x^7 y^2)$. As expected, this reduces to the first term in parentheses in (4) on replacing y by 1. More sophisticated techniques involve first reducing the dimensions of the transfer matrices used in an analogous formula to (13) [13].

Given the simple form of (13), one might ask why we have gone to the trouble of introducing a lot of terminology concerning digraphs and deriving Theorems 5 and 6. There are two reasons. The first is that the determinant of the transfer matrix of a general digraph with an unspecified number of nodes (such as those shown in Figs. 9 and 10 in this article or Figs. 5 and 6 in All24) is not trivial to simplify to give the generating function; it is much easier to read off the lengths of the cycles and circuits in the digraph and plug them into the formula for $G(x,y)$. Once we have a general expression for $G(x,y)$ for a class of $\mathcal{Q}$, there is no longer any need to draw any digraphs (and deal with transfer matrices) for an instance of that class. Second, having concepts such as inner and outer cycles and common nodes can lead to quick proofs, as is the case for Lemma 18 in the present work and the main theorem in a forthcoming article [6].

7 Acknowledgments

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Appendix: C program for finding $S_n^{\mathcal{Q}}$ and $S_{n,k}^{\mathcal{Q}}$

There are two main features that contribute to the efficiency of the algorithm used in the program listed below. Rather than directly computing whether the difference of each pair x, y of elements of subset $\mathcal{S}$ is in $\mathcal{Q}$, we perform a bitwise AND operation on bit string representations of $\mathcal{S}$ shifted x places to the right and $\mathcal{Q}$. If the AND operation gives an answer of 0 then $y - x$ does not equal any element of $\mathcal{Q}$ for all $y > x$. We also exploit the result that if $\mathcal{S}$ is an allowed subset of $\mathbb{N}_n$ then it is an allowed subset of $\mathbb{N}_m$ for any $m > n$.

```
// Anything after // on the same line is a comment.
// Program name: rcl.c  (restricted combinations on a line).
// Purpose: counts subsets of {1,2,...,n} such that
// no two elements have a difference equal to an element of Q
// and finds S_n or S_{n,k} for n from 0 to MAXn, which is typically 32 or 64.
// To create an executable called rcl using the GNU C compiler enter:
```

```
// gcc rcl.c -O2 -o rcl

// include libraries
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
// sizeof( ) returns the number of bytes used by a variable type
// the variable type unsigned long int typically uses 4 bytes
// on 32-bit machines and 8 bytes on 64-bit machines
// hence MAXn is typically 32 or 64, respectively
#define MAXn 8*sizeof(unsigned long int)

// after including the libraries and defining MAXn, program starts here
// argc equals the number of command-line arguments plus 1
// argv is an array of character strings containing those arguments
int main(int argc,char **argv) {
// start by defining the variables used by the program
// j-th bit from the right in the bit string is 1 iff j is in the set it represents
  unsigned long int Q=0,s,ss; // bit strings giving Q, subset, and shifted subset
  char *p; // used for reading in Q (given as a comma-separated list) from command line
  unsigned short int tri, // true if S_{n,k} triangle rather than S_n is wanted
    i,j,k=0,test,lastn=0;
  unsigned long int S[MAXn+1][MAXn+2]; // S[n][k]=S_{n,k} if tri true; if not, S[n][0]=S_n
// end of definitions of variables
// give examples of usage if program run with no arguments
  if (argc==1) { // rcl called with no arguments
// puts( ) displays a string; \t is tab \n is new line
    puts("example usage:\t rcl 1,2,4\n\t\t rcl 1,2,4 t");
    return 1; // end the program
  } // end if
// p starts by pointing to the character before the start of the 1st argument
  for (p=argv[1]-1;p;p=strchr(p,',')) { // loop to read in Q from command line argument
// line(s) within these braces only executed if p is not zero
// ++p means add 1 to p first
// atoi( ) converts a string into an integer, stopping as soon as it reaches a non-digit
// x<<m means shift x in binary m places to the left
// | is the bitwise OR operation
    Q=Q|1<<(atoi(++p)-1); // add to existing Q
  } // end of loop: p is updated to point to the next comma in argument or zeroed if none
// && means a logical AND
  tri=(argc>2 && argv[2][0]=='t'); // true if argument 2 in command line is t
// initialize array that will contain S_{n,k}
// S_{n,0} is set to 1 in order to count the empty set
  for (i=0;i<=MAXn;i++) {
// cond?x:y returns x if cond is true and y otherwise
    for (j=0;j<=(tri?i+1:1);j++) {
      S[i][j]=j?0:1; // set k=0 totals to 1, zero k>0 totals
    }
  }
// ~0 means all ones in binary
```

```
  for (s=1;s<~0;s++) { // loop to test all possible nonempty subsets s
    if (tri) {
// if S_{n,k} wanted, get k = number of set bits of s
      for (k=0,ss=s;ss;ss>>=1) {
        if (ss&1) k=k+1; // & is the bitwise AND operation
      }
    }
    for (ss=s,i=1;i<=MAXn;i++) { // test subset s
      test=ss&1; // will only test shifted subset against Q if rightmost bit of ss is 1
      ss>>=1; // shift shifted subset ss to the right by 1
      if (ss==0) { // s is ok (no disallowed differences found)
        // i is now minimum n for which s is ok
        if (i>lastn) { // total(s) for previous n finished
          for (j=0;S[lastn][j];j++) {
            printf("%ld,",S[lastn][j]); // display S_n or S_{n,k} if nonzero
          }
          if (tri) puts(""); // new line
          else fflush(stdout); // display immediately
          lastn=i; // update lastn for next time it is used
        }
        for (j=i;j<=MAXn;j++) {
          S[j][k]++; // if s is ok for a given n, it is ok for all higher n
        }
        break; // done with current s
      }
      if (test && (Q&ss)) break; // disallowed difference found - done with current s
    }
  }
  for (j=0;S[lastn][j];j++) {
    printf("%ld,",S[lastn][j]); // lastn now equals MAXn
  }
  puts(""); // print a new line at the end
  return 0;
}
```

References

[1] M. A. Allen, On a two-parameter family of generalizations of Pascal's triangle, *J. Integer Sequences* **25** (2022), Article 22.9.8.

[2] M. A. Allen, Combinations without specified separations, *Commun. Comb. Optim.* (2024), https://doi.org/10.22049/cco.2024.29370.1959.

[3] M. A. Allen and K. Edwards, On two families of generalizations of Pascal's triangle, *J. Integer Sequences* **25** (2022), Article 22.7.1.

[4] M. A. Allen and K. Edwards, Identities involving the tribonacci numbers squared via tiling with combs, *Fibonacci Quart.* **61** (2023), 21–27.

[5] M. A. Allen and K. Edwards, Connections between two classes of generalized Fibonacci numbers squared and permanents of (0,1) Toeplitz matrices, *Linear Multilinear Algebra* **72** (2024), 2091–2103.

[6] M. A. Allen and K. Edwards, Identities relating permanents of some classes of (0,1) Toeplitz matrices to generalized Fibonacci numbers, *Fibonacci Quart.* **63** (2025), https://doi.org/10.1080/00150517.2025.24921984.

[7] V. Baltić, Applications of the finite state automata for counting restricted permutations and variations, *Yugoslav J. Oper. Res.* **22** (2012), 183–198.

[8] K. Edwards, A Pascal-like triangle related to the tribonacci numbers, *Fibonacci Quart.* **46/47** (2008/2009), 18–25.

[9] K. Edwards and M. A. Allen, Strongly restricted permutations and tiling with fences, *Discrete Appl. Math.* **187** (2015), 82–90.

[10] K. Edwards and M. A. Allen, A new combinatorial interpretation of the Fibonacci numbers cubed, *Fibonacci Quart.* **58** (5) (2020), 128–134.

[11] I. Kaplansky, Solution of the "problème des ménages", *Bull. Amer. Math. Soc.* **49** (1943), 784–785.

[12] S. Kitaev, Independent sets on path-schemes, *J. Integer Sequences* **9** (2006), Article 06.2.2.

[13] T. Kløve, Generating functions for the number of permutations with limited displacement, *Electron. J. Combin.* **16** (2009), R104.

[14] J. Konvalina, On the number of combinations without unit separation, *J. Combin. Theory Ser. A* **31** (1981), 101–107.

[15] J. Konvalina and Y.-H. Liu, Bit strings without q-separation, *BIT* **31** (1991), 32–35.

[16] J. Konvalina and Y.-H. Liu, Subsets without q-separation and binomial products of Fibonacci numbers, *J. Combin. Theory Ser. A* **57** (1991), 306–310.

[17] D. H. Lehmer, Permutations with strongly restricted displacements, in *Combinatorial Theory and its Applications II (Proceedings of the Colloquium, Balatonfured, 1969)*, North-Holland, Amsterdam, 1970, pp. 755–770.

[18] T. Mansour and Y. Sun, On the number of combinations without certain separations, *European J. Combin.* **29** (2008), 1200–1206.

[19] R. J. Mathar, Tilings of rectangular regions by rectangular tiles: Counts derived from transfer matrices, arxiv preprint arXiv:1406.7788 [math.CO], 2014. Available at https://arxiv.org/abs/1406.7788.

[20] H. Prodinger, On the number of combinations without a fixed distance, *J. Combin. Theor. Ser. A* **35** (1983), 362–365.

[21] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically, 2025. Available at https://oeis.org.

[22] R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, 2011.

[23] A. A. Valyuzhenich, Some properties of well-based sequences, *J. Appl. Ind. Math.* **5** (2011), 612–614.

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