Journal of Integer Sequences, Vol. 27 (2024),
Article 24.6.8

# Elementary Functions Associated with Series Involving Reciprocals of Central Binomial Coefficients

Rimer Zurita
Carrera de Matematicas
Universidad Mayor de San Simon
Cochabamba
Bolivia
mauriciozurita.o@umss.edu

**Abstract**

In this paper we study some combinatorial series involving reciprocals of generalized central binomial coefficients multiplied by linear terms. We rewrite these series in terms of elementary functions, such as polynomials, inverse trigonometric functions, inverse hyperbolic functions, and radicals. These results generalize some particular cases studied by Sprugnoli.

## 1 Introduction

Binomial coefficients are present in many areas, such as graph theory, number theory, combinatorial analysis, probability and statistics, nuclear physics, etc. Central binomial coefficients (see sequence A000984) are defined for every integer $n \geq 0$ as

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}.$$

The current literature [2, 3] shows many results on the inverse of such coefficients. One of such results, due to Euler, is

$$\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} = \frac{\pi^2}{18}.$$

Sprugnoli [5] used techniques for solving recursive equations to obtain results for series of the forms

$$\sum_{k=0}^{\infty} \frac{4^k x^k}{(2k+1)\binom{2k}{k}}, \quad \sum_{k=0}^{\infty} \frac{4^k x^k}{(k+1)\binom{2k}{k}}, \quad \sum_{k=1}^{\infty} \frac{4^k x^k}{k^2\binom{2k}{k}},$$

where $|x| \le 1$, which he rewrote as expressions of elementary functions only.

Batir [1] derived an expression for the series $\sum_{k=1}^{\infty} \frac{x^k}{k^n\binom{2k}{k}}$ using representations in terms of appropriately expanded integrals. Similarly, Zhao and Wang [6] derived expressions for sums of the forms $\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2mk}{mk}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)\binom{2mk}{mk}}$. Another well known result, in terms of the generalized hypergeometric function, is

$$\sum_{k=1}^{\infty} \frac{1}{k^n\binom{2k}{k}} = \frac{1}{2}\ {}_{n+1}\mathrm{F}_{n-1}\left(\begin{matrix} \overbrace{1, 1, 1, \ldots, 1}^{n+1} \\ \frac{3}{2}, \underbrace{2, \ldots, 2}_{n-1} \end{matrix}; \frac{1}{4}\right).$$

This work aims to generalize the results by Sprugnoli, by studying series of the form

$$\sum_{k=0}^{\infty} \frac{x^k}{(2k+n)\binom{2mk}{mk}} \quad \text{or} \quad \sum_{k=0}^{\infty} \frac{x^k}{k^2\binom{2mk}{mk}},$$

for integers $m \ge 1, n \ge 0$, expressing them in terms of elementary functions, such as polynomials, inverse trigonometric functions, and radicals.

# 2 Definitions

Throughout this paper, let the following functions be defined for $x \in \mathbb{C}$ by

$$F(x) := \frac{4}{\sqrt{x(4-x)}} \arctan\left(\sqrt{\frac{x}{4-x}}\right),$$

$$H(x) := \frac{4\sqrt{x}\arcsin\left(\frac{\sqrt{x}}{2}\right)}{(4-x)^{3/2}} + \frac{4}{4-x},$$

$$L(x) := \frac{2\sqrt{x}\arcsin\left(\frac{\sqrt{x}}{2}\right)}{\sqrt{4-x}},$$

$$K(x) := 2\left(\arcsin\left(\frac{\sqrt{x}}{2}\right)\right)^2.$$

**Definition 1.** Let $l \geq 1$. We define the polynomial $Q_l(x) = \sum_{i=0}^{l} q_i x^i$ of degree $l$ such that its coefficients satisfy the recursive formulas

$$\begin{aligned} q_l &:= -\frac{4}{l}; \\ q_k &:= \left(\frac{4k+6}{k}\right) q_{k+1}, \quad \text{for } k = l-1, l-2, \ldots, 1; \\ q_0 &:= -12 q_1, \end{aligned}$$

and we define $C_l = -q_0/2$. Also, let $Q_0(x) = 8$ and $C_0 = -4$.

We show later in this work (see proof of Lemma 5) that $Q_l(4) = 8 \cdot 4^l$, which implies that $(4-x)$ divides the polynomial $(8x^l - Q_l(x))$.

**Definition 2.** Let $l \geq 1$. We define the polynomial $P_l(x) = \sum_{i=1}^{l} p_i x^i$ of degree $l$ and without constant term by

$$P_l(x) := \int \frac{8x^l - Q_l(x)}{2(4-x)}\, dx.$$

Also, let $P_0(x) = 0$.

Tables 1 and 2 show the first few polynomials $P_l(x)$ and $Q_l(x)$, respectively.

| $l$ | $C_l$ | $Q_l(x)$ |
|---|---|---|
| 0 | $-4$ | $8$ |
| 1 | $-24$ | $-4x + 48$ |
| 2 | $-120$ | $-2x^2 - 20x + 240$ |
| 3 | $-560$ | $-\frac{4}{3}x^3 - \frac{28}{3}x^2 - \frac{280}{3}x + 1120$ |
| 4 | $-2520$ | $-x^4 - 6x^3 - 42x^2 - 420x + 5040$ |
| 5 | $-11088$ | $-\frac{4}{5}x^5 - \frac{22}{5}x^4 - \frac{132}{5}x^3 - \frac{924}{5}x^2 - 1848x + 22176$ |

Table 1: The first five polynomials $Q_l(x)$.

**Definition 3.** Let $l \geq 0$. We define the polynomial $S_l(x) = \sum_{i=0}^{l} s_i x^i$ of degree $l$ by

$$\begin{aligned} s_l &:= \frac{8}{1-2l}; \\ s_k &:= \frac{8(k+1)}{(2k-1)} s_{k+1}, \quad \text{for } k = l-1, l-2, \ldots, 0. \end{aligned}$$

We show later in this work (see proof of Lemma 6) that $S_l(4) = 8 \cdot 4^l$, which implies that $(4-x)$ divides the polynomial $(8x^l - S_l(x))$.

| $l$ | $P_l(x)$ |
|---|---|
| 0 | $0$ |
| 1 | $-6x$ |
| 2 | $-\frac{5}{2}x(x+12)$ |
| 3 | $-\frac{7}{9}x(2x^2+15x+180)$ |
| 4 | $-\frac{1}{8}x(9x^3+56x^2+420x+5040)$ |
| 5 | $-\frac{11}{100}x(8x^4+45x^3+280x^2+2100x+25200)$ |

Table 2: The first five polynomials $P_l(x)$.

**Definition 4.** Let $l \geq 0$. We define the polynomial $T_{l-1}(x)$ of degree $l-1$ by

$$T_{-1}(x) := 0;$$
$$T_{l-1}(x) := \frac{x^{-1/2}}{2} \int \frac{8x^l - S_l(x)}{\sqrt{x}(4-x)}\,dx, \quad \text{if } l \geq 1.$$

Table 3 shows the first few polynomials $S_l(x)$ and $T_{l-1}(x)$.

| $l$ | $S_l(x)$ | $T_{l-1}(x)$ |
|---|---|---|
| 0 | $8$ | $0$ |
| 1 | $-8x+64$ | $-16$ |
| 2 | $-\frac{8}{3}x^2-\frac{128}{3}x+\frac{1024}{3}$ | $-\frac{32}{9}(x+24)$ |
| 3 | $-\frac{8}{5}x^3-\frac{64}{5}x^2-\frac{1024}{5}x+\frac{8192}{5}$ | $-\frac{16}{75}(9x^2+80x+1920)$ |
| 4 | $-\frac{8}{7}x^4-\frac{256}{35}x^3-\frac{2048}{35}x^2-\frac{32768}{35}x+\frac{262144}{35}$ | $-\frac{64}{3675}(75x^3+504x^2+4480x+107520)$ |

Table 3: The first five polynomials $S_l(x)$ and $T_{l-1}(x)$.

# 3 Auxiliary results

**Lemma 5.** *For every integer $l \geq 1$, we have*

$$\int x^l H(x)\,dx = P_l(x) + C_l\left(\arcsin\left(\frac{\sqrt{x}}{2}\right)\right)^2 + \frac{\arcsin\left(\frac{\sqrt{x}}{2}\right)\sqrt{x}}{\sqrt{4-x}}Q_l(x), \tag{1}$$

*where the constant $C_l$ and the polynomials $Q_l(x)$ are as in Definition 1, and $P_l(x)$ as in Definition 2.*

*Proof.* For $l = 0$, the proof is straightforward. For $l \geq 1$ fixed, let us prove the derivative of the right-hand side of Eq. (1) is equal to $x^l H(x)$. Indeed, we have that this equality is true if

$$P_l'(x) + C_l \arcsin\left(\frac{\sqrt{x}}{2}\right) \cdot \frac{1}{\sqrt{x(4-x)}} + \left(\frac{1}{2(4-x)} + \frac{2\arcsin(\sqrt{x}/2)}{\sqrt{x}(4-x)^{3/2}}\right) \cdot Q_l(x)$$
$$+ \frac{\arcsin\left(\frac{\sqrt{x}}{2}\right)\sqrt{x}}{\sqrt{4-x}} \cdot Q_l'(x) = x^l \left(\frac{4\sqrt{x}\arcsin\left(\sqrt{x}/2\right)}{(4-x)^{3/2}} + \frac{4}{4-x}\right),$$

which in turn is true if

$$P_l'(x) + \frac{Q_l(x)}{2(4-x)} = \frac{4x^l}{4-x} \tag{2}$$

and

$$\frac{C_l(4-x)}{x} + \frac{2Q_l(x)}{x} + (4-x)Q_l'(x) = 4x^l. \tag{3}$$

Let us first solve Eq. (3). Letting $Q_l(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_l x^l$, we have

$$C_l\left(\frac{4}{x} - 1\right) + \frac{2}{x}\left(q_0 + q_1 x + q_2 x^2 + \cdots + q_l x^l\right) + (4-x)\left(q_1 + 2q_2 x + \cdots + l q_l x^{l-1}\right) = 4x^l.$$

Comparing the coefficients, we can see that

$$\begin{gathered} C_l = -q_0/2; \\ q_l = -\frac{4}{l}; \\ 2q_{k+1} + 4(k+1)q_{k+1} - kq_k = 0 \Rightarrow q_k = \left(\frac{4k+6}{k}\right)q_{k+1}, \quad \text{for } k = l-1, l-2, \ldots, 1; \\ C_l = 6q_1. \end{gathered}$$

From Eq. (3) we conclude that $Q_l(4) = 8 \cdot 4^l$, which implies that $(4-x) \mid \left(8x^l - Q_l(x)\right)$, and from Eq. (2) we conclude

$$P_l(x) = \int \frac{8x^l - Q_l(x)}{2(4-x)}\, dx.$$

□

**Lemma 6.** *For every integer $l \geq 0$, we have that*

$$\int x^{l-1/2} H(x)\, dx = T_{l-1}(x)\sqrt{x} + \frac{\arcsin\left(\sqrt{x}/2\right)}{\sqrt{4-x}} S_l(x) \tag{4}$$

*where the polynomials $T_{l-1}(x)$ are as in Definition 4 and $S_l(x)$ as in Definition 3.*

*Proof.* Let us prove that the derivative of the right-hand side of Eq. (4) is equal to $x^{l-1/2}H(x)$. This is true if

$$T'_{l-1}(x)\sqrt{x}+\frac{T_{l-1}(x)}{2\sqrt{x}}+\left(\frac{\arcsin\left(\sqrt{x}/2\right)}{2(4-x)^{3/2}}+\frac{1}{2\sqrt{x}(4-x)}\right)S_l(x)$$
$$+\frac{\arcsin\left(\sqrt{x}/2\right)}{\sqrt{4-x}}S'_l(x)=x^{l-1/2}\left(\frac{4\sqrt{x}\arcsin\left(\sqrt{x}/2\right)}{(4-x)^{3/2}}+\frac{4}{4-x}\right).$$

In turn, this is true if

$$T'_{l-1}(x)\sqrt{x}+\frac{T_{l-1}(x)}{2\sqrt{x}}+\frac{S_l(x)}{2\sqrt{x}(4-x)}=\frac{4x^{l-1/2}}{4-x} \tag{5}$$

and

$$\frac{S_l(x)}{2(4-x)^{3/2}}+\frac{S'_l(x)}{\sqrt{4-x}}=\frac{4x^l}{(4-x)^{3/2}}. \tag{6}$$

Let us first solve Eq. (6), which is equivalent to solving

$$S_l(x)+(8-2x)S'_l(x)=8x^l. \tag{7}$$

Letting $S_l(x)=s_o+s_1x+\cdots+s_lx^l$ and comparing coefficients of Eq. (7), we have

$$s_l-2ls_l=8\Rightarrow s_l=8/(1-2l)$$
$$s_k+8(k+1)s_{k+1}-2ks_k=0\Rightarrow s_k=\frac{8(k+1)s_{k+1}}{(2k-1)},\quad \text{for } k=l-1,l-2,\ldots,0.$$

Also, from Eq. (7) we conclude that $S_l(4)=8\cdot 4^l$, which implies that $(4-x)|\big(8x^l-S_l(x)\big)$.

Let us now solve the first-order linear differential equation Eq. (5). We can rewrite it as

$$T'_{l-1}=-\frac{1}{2x}T_{l-1}+\left(\frac{4x^l}{x(4-x)}-\frac{S_l(x)}{2x(4-x)}\right),$$

which leads to

$$T_{l-1}(x)=\frac{x^{-1/2}}{2}\int\frac{8x^l-S_l(x)}{\sqrt{x}(4-x)}\,dx.$$

□

# 4 Main results

**Theorem 7.** *For $m\in\mathbb{N}$ and $|x|<4^m$, we have*

$$H_m(x):=\sum_{k=0}^{\infty}\frac{x^k}{\binom{2mk}{mk}}=\frac{1}{m}\sum_{j=0}^{m-1}H\left(w^jx^{1/m}\right), \tag{8}$$

*where $w=e^{2\pi i/m}$ is the $m$-th root of unity.*

*Proof.* We know that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, where $\Gamma$ is the Gamma function defined as $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$ for $\Re(s) > 0$. On the other hand, the Beta function, defined as $B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt$ for $\Re(z_1) > 0$ and $\Re(z_2) > 0$, satisfies $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$.

Let

$$
\begin{aligned}
F_m(x) :&= \sum_{k=0}^{\infty} \frac{x^k}{(2mk+1)\binom{2mk}{mk}} = \sum_{k=0}^{\infty} \frac{x^k \Gamma(mk+1)\Gamma(mk+1)}{\Gamma(2mk+2)} = \sum_{k=0}^{\infty} x^k B(mk+1, mk+1) \\
&= \sum_{k=0}^{\infty} x^k \int_0^1 t^{mk}(1-t)^{mk}\, dt = \int_0^1 \sum_{k=0}^{\infty} (x^{1/m} t(1-t))^{mk}\, dt.
\end{aligned}
$$

Using the geometric series and decomposing into partial fractions, we can see that, for every integer $m \geq 1$,

$$
\sum_{k=0}^{\infty} z^{mk} = \frac{1}{1-z^m} = \frac{1}{m} \sum_{l=0}^{m-1} \frac{w^l}{w^l - z}, \quad \text{for } |z| < 1,
$$

where $w = e^{2\pi i/m}$ is the $m$-th root of unity. Expanding $F_m(x)$, we have that, for $|x| < 4^m$,

$$
\begin{aligned}
F_m(x) &= \int_0^1 \frac{1}{m} \sum_{j=0}^{m-1} \left( \frac{w^j}{w^j - x^{1/m} t(1-t)} \right) dt = \frac{1}{m} \sum_{j=0}^{m-1} \int_0^1 \frac{1}{1 - \frac{x^{1/m}}{w^j} t(1-t)}\, dt \\
&= \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{\frac{x^{1/m}}{w^j}} \int_0^1 \frac{1}{(t-1/2)^2 + \left(\frac{w^j}{x^{1/m}} - \frac{1}{4}\right)}\, dt = \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{\frac{x^{1/m}}{w^j}} \int_{-1/2}^{1/2} \frac{1}{s^2 + \left(\frac{w^j}{x^{1/m}} - \frac{1}{4}\right)}\, ds \\
&= \frac{2}{m} \sum_{j=0}^{m-1} \frac{1}{\frac{x^{1/m}}{w^j}} \int_0^{1/2} \frac{1}{s^2 + \left(\frac{w^j}{x^{1/m}} - \frac{1}{4}\right)}\, ds \\
&= \frac{2}{m} \sum_{j=0}^{m-1} \frac{1}{\frac{x^{1/m}}{w^j}} \cdot \frac{1}{\sqrt{\frac{w^j}{x^{1/m}} - \frac{1}{4}}} \arctan\left( \frac{1}{2\sqrt{\frac{w^j}{x^{1/m}} - \frac{1}{4}}} \right) \\
&= \frac{1}{m} \sum_{j=0}^{m-1} \frac{4}{\sqrt{\frac{x^{1/m}}{w^j}\left(4 - \frac{x^{1/m}}{w^j}\right)}} \arctan\left( \sqrt{\frac{\frac{x^{1/m}}{w^j}}{4 - x^{1/m}/w^j}} \right) = \frac{1}{m} \sum_{j=0}^{m-1} F\left( \frac{x^{1/m}}{w^j} \right).
\end{aligned}
$$

Therefore,

$$
F_m(x) = \frac{1}{m} \sum_{j=0}^{m-1} F(w^j x^{1/m}), \quad \text{for } |x| < 4^m.
$$

By replacing $x$ with $x^{2m}$ and multiplying $F_m(x)$ by $x$, we have

$$
\sum_{k=0}^{\infty} \frac{x^{2mk+1}}{(2mk+1)\binom{2mk}{mk}} = \frac{x}{m} \sum_{j=0}^{m-1} F(w^j x^2), \quad \text{for } |x| < 2.
$$

Taking the derivative,

$$\sum_{k=0}^{\infty} \frac{x^{2mk}}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} \big(xF(w^j x^2)\big)' = \frac{1}{m} \sum_{j=0}^{m-1} \frac{\Big((w^{j/2}x)F\big((w^{j/2}x)^2\big)\Big)'}{w^{j/2}}.$$

Since $zF(z^2) = \frac{4}{\sqrt{4-z^2}} \arctan\left(\frac{z}{\sqrt{4-z^2}}\right) = \frac{4\arcsin(z/2)}{\sqrt{4-z^2}}$, by the chain rule we conclude that

$$\sum_{k=0}^{\infty} \frac{x^{2mk}}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} H(w^j x^2).$$

Finally, replacing $x$ with $x^{1/(2m)}$, we get

$$H_m(x) = \sum_{k=0}^{\infty} \frac{x^k}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} H(w^j x^{1/m}).$$

□

**Theorem 8.** *For $m \in \mathbb{N}$ and $|x| < 4^m$, we have*

$$L_m(x) := \sum_{k=1}^{\infty} \frac{x^k}{k\binom{2mk}{mk}} = \sum_{j=0}^{m-1} L(w^j x^{1/m}), \tag{9}$$

*where $w = e^{2\pi i/m}$ is the $m$-th root of unity.*

*Proof.* From Eq. (8), for $|x| < 2$ we have

$$\sum_{k=1}^{\infty} \frac{x^k}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} \big(H(w^j x^{1/m}) - 1\big),$$

which implies

$$\sum_{k=1}^{\infty} \frac{x^{k-1}}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} \left(\frac{H(w^j x^{1/m}) - 1}{x}\right), \quad \text{for } |x| < 2 \text{ and } x \neq 0.$$

Integrating with respect to $x$,

$$L_m(x) = \frac{1}{m} \sum_{j=0}^{m-1} \int \frac{H(w^j x^{1/m}) - 1}{x}\, dx + C_1.$$

Making the change of variable $u = w^j x^{1/m}$ we have

$$L_m(x) = \sum_{j=0}^{m-1} \int \frac{H(u) - 1}{u}\, du + C_1 = \sum_{j=0}^{m-1} L(w^j x^{1/m}) + C_1.$$

Since $L_m(0) = 0$, we must have $C_1 = 0$, and therefore

$$L_m(x) = \sum_{j=0}^{m-1} L(w^j x^{1/m}).$$

□

*Remark* 9. If we replace $x$ with $4x^2$ and let $m = 1$ in Eq. (9), we have

$$\sum_{k=1}^{\infty} \frac{(2x)^{2k}}{k\binom{2k}{k}} = \frac{2x \arcsin(x)}{\sqrt{1-x^2}}, \quad \text{for } |x| < 1,$$

which is a well-known result by Lehmer [4].

**Theorem 10.** *For $m \in \mathbb{N}$ and $|x| < 4^m$, we have*

$$K_m(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2\binom{2mk}{mk}} = m \sum_{j=0}^{m-1} K(w^j x^{1/m}), \tag{10}$$

*where $w = e^{2\pi i/m}$ is the $m$-th root of unity.*

*Proof.* From Eq. (9) we have

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2\binom{2mk}{mk}} = \sum_{j=0}^{m-1} \int \frac{L(w^j x^{1/m})}{x}\, dx + C_2.$$

Making the change of variable $u = w^j x^{1/m}$, this becomes

$$K_m(x) = m \sum_{j=0}^{m-1} \int \frac{L(u)}{u}\, du + C_2 = m \sum_{j=0}^{m-1} K(w^j w^{1/m}) + C_2.$$

Since $K_m(0) = 0$, we must have $C_2 = 0$, which implies

$$K_m(x) = m \sum_{j=0}^{m-1} K(w^j x^{1/m}).$$

□

The following theorem is the main result of this work.

**Theorem 11.** *For every integer $m, n \geq 1$ and for $|x| < 4^m$, the polynomial*

$$P_{m,n}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(2k+n)\binom{2mk}{mk}} \tag{11}$$

*can be rewritten in one of the following ways.*

*(a) If $mn$ is even, then*

$$P_{m,n}(x) = \frac{1}{2x^{n/2}} \sum_{j=0}^{m-1} (-1)^{jn} \Bigg( P_{mn/2-1}(w^j x^{1/m}) + C_{mn/2-1} \arcsin^2\left(\frac{w^{j/2}x^{1/(2m)}}{2}\right)^2 + \frac{\arcsin\left(\frac{w^{j/2}x^{1/(2m)}}{2}\right) w^{j/2}x^{1/(2m)}}{\sqrt{4-w^j x^{1/m}}} \cdot Q_{mn/2-1}(w^j x^{1/m}) \Bigg),$$

*where the constant $C$ and the polynomials $Q$ are as in Definition 1, and $P$ as in Definition 2.*

*(b) If $mn$ is odd, then*

$$P_{m,n}(x) = \frac{1}{2x^{n/2}} \cdot \sum_{j=0}^{m-1} (-1)^{jn} \Bigg( T_{(mn-3)/2}(w^j x^{1/m}) w^{j/2} x^{1/(2m)} + \frac{\arcsin\left(\frac{w^{j/2}x^{1/(2m)}}{2}\right)}{\sqrt{4-w^j x^{1/m}}} \cdot S_{(mn-1)/2}(w^j x^{1/m}) \Bigg),$$

*where the polynomials $S$ are as in Definition 3 and $T$ as in Definition 4.*

*Proof.* From Eq. (8) we know that, for $|x| < 2$, we have

$$\sum_{k=1}^{\infty} \frac{x^k}{\binom{2mk}{mk}} = \left( \frac{1}{m} \sum_{j=0}^{m-1} H(w^j x^{1/m}) \right) - 1.$$

Then

$$\sum_{k=1}^{\infty} \frac{x^{2k+n-1}}{\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} H(w^j x^{2/m}) x^{n-1} - x^{n-1}.$$

Integrating with respect to $x$,

$$\sum_{k=1}^{\infty} \frac{x^{2k+n}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{m} \sum_{j=0}^{m-1} \int H(w^j x^{2/m}) x^{n-1}\, dx + C - \frac{x^n}{n},$$

which can be rewritten as

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{mx^n} \sum_{j=0}^{m-1} \int H(w^j x^{2/m}) x^{n-1}\, dx + \frac{C}{x^n}.$$

Let us apply the change of variable $u = w^j x^{2/m}$ in the integral:

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \int u^{mn/2-1} H(u)\, du + \frac{C}{x^n}. \tag{12}$$

**Case 1:** Suppose $mn = 2l$ is even. Then

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \int u^{l-1} H(u)\, du + \frac{C}{x^n}.$$

By Eq. (1) we have

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \Bigg( P_{l-1}(u) + C_{l-1}\Big(\arcsin(\sqrt{u}/2)\Big)^2 + \frac{\arcsin(\sqrt{u}/2)\sqrt{u}}{\sqrt{4-u}} Q_{l-1}(u) \Bigg) + \frac{C}{x^n}.$$

Reverting the change of variable,

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \Bigg( P_{l-1}(w^j x^{2/m}) + C_{l-1}\Big(\arcsin(w^{j/2} x^{1/m}/2)\Big)^2 + \frac{\arcsin\big((w^{j/2}x^{1/m})/2\big) w^{j/2} x^{1/m}}{\sqrt{4-w^j x^{2/m}}} Q_{l-1}(w^j x^{2/m}) \Bigg) + \frac{C}{x^n}.$$

Multiplying by $x^n$ and evaluating on $x = 0$, we can see that $C = 0$. Finally, replacing $x$ with $x^{1/2}$, we get the desired result.

**Case 2:** Suppose $mn = 2l + 1$ is odd. From Eq. (12),

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \int u^{l-1/2} H(u)\, du + \frac{C}{x^n}.$$

By Eq. (4) we have

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \left( T_{l-1}(u)\sqrt{u} + \frac{\arcsin(\sqrt{u}/2)}{\sqrt{4-u}} S_l(u) \right) + \frac{C}{x^n}.$$

Reverting the change of variable,

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+n)\binom{2mk}{mk}} = \frac{1}{2x^n} \sum_{j=0}^{m-1} (-1)^{jn} \Bigg( T_{l-1}(w^j x^{2/m}) w^{j/2} x^{1/m} + \frac{\arcsin\big((w^{j/2}x^{1/m})/2\big)}{\sqrt{4-w^j x^{2/m}}} S_l(w^j x^{2/m}) \Bigg) + \frac{C}{x^n}.$$

As in the previous case, we can see that $C = 0$, and by replacing $x$ with $x^{1/2}$ we get the desired result. □

## 5 Applications

1. Let $m = 1$ or $m = 2$, and $x = \pm 1$ in Eq. (8). Then

$$\sum_{k=0}^{\infty} \frac{1}{\binom{2k}{k}} = \frac{4}{3} + \frac{2\pi\sqrt{3}}{27}, \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{\binom{2k}{k}} = \frac{4}{5} - \frac{4\sqrt{5}\,\operatorname{arcsinh}(1/2)}{25},$$

$$\sum_{k=0}^{\infty} \frac{1}{\binom{4k}{2k}} = \frac{16}{15} + \frac{\sqrt{3}}{27}\pi - \frac{2\sqrt{5}}{25}\operatorname{arcsinh}(1/2),$$

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{\binom{4k}{2k}} &\\ = \frac{-2\sqrt{17}}{289}\Bigg(&\left(3\sqrt{\sqrt{17}+4} - 5\sqrt{\sqrt{17}-4}\right)\arcsin\left(\frac{\sqrt{5-2\sqrt{2}} - \sqrt{2\sqrt{2}+5}}{4}\right) \\ &+ \left(5\sqrt{\sqrt{17}+4} + 3\sqrt{\sqrt{17}-4}\right)\log\left(\sqrt{2(\sqrt{17}-3)} + \sqrt{5-2\sqrt{2}} + \sqrt{2\sqrt{2}+5}\right) \\ &- 2\log(2)\left(5\sqrt{\sqrt{17}+4} + 3\sqrt{\sqrt{17}-4}\right)\Bigg) + \frac{16}{17} \\ \approx 0.&84660943050448617317.\end{aligned}$$

Sprugnoli [5] already showed these results in his paper.

2. Let $m = 1$ or $m = 2$, and $x = \pm 1$ in Eq. (10). Then

$$\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} = \frac{\pi^2}{18}, \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\binom{2k}{k}} = -2\operatorname{arcsinh}^2(1/2),$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2\binom{4k}{2k}} = \frac{\pi^2}{9} - 4\operatorname{arcsinh}^2(1/2),$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\binom{4k}{2k}} &= 8\arcsin^2\left(\frac{\sqrt{5-2\sqrt{2}} - \sqrt{2\sqrt{2}+5}}{4}\right) \\ &\quad - 8\left(2\log(2) - \log\left(\sqrt{2(\sqrt{17}-3)} + \sqrt{5-2\sqrt{2}} + \sqrt{2\sqrt{2}+5}\right)\right)^2 \\ &\approx -0.16321083867416013360.\end{aligned}$$

Sprugnoli [5] already showed the first two previous identities in his paper. The two other results are new.

3. Let $m = 3, n = 1$ in Eq. (11). Then

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)\binom{6k}{3k}}$$
$$= \frac{-2\sqrt{7}}{21}\Bigg(\bigg(3\sqrt{2\sqrt{21}+9} - 17\sqrt{3(2\sqrt{21}-9)}\bigg)\log\bigg(\sqrt{2(\sqrt{21}-3)} + \sqrt{7} + \sqrt{3}\bigg)$$
$$+ \arcsin\bigg(\frac{\sqrt{7}-\sqrt{3}}{4}\bigg)\bigg(17\sqrt{3(2\sqrt{21}+9)} + 3\sqrt{2\sqrt{21}-9}\bigg)$$
$$- 2\log(2)\bigg(3\sqrt{2\sqrt{21}+9} - 17\sqrt{3(2\sqrt{21}-9)}\bigg)\Bigg) + \frac{14\sqrt{3}}{9}\pi$$
$$\approx 1.0168860968050338303.$$

This is a new identity.

4. Let $m = 2, n = 3$ in Eq. (11). Then

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)\binom{4k}{2k}}$$
$$= -\frac{1}{17}\Bigg(\bigg(4080\arcsin\bigg(\frac{\sqrt{5-2\sqrt{2}} - \sqrt{2\sqrt{2}+5}}{4}\bigg) + \sqrt{17}\bigg(131\sqrt{\sqrt{17}+4} - 111\sqrt{\sqrt{17}-4}\bigg)\bigg)$$
$$\cdot \log\bigg(\sqrt{2(\sqrt{17}-3)} + \sqrt{5-2\sqrt{2}} + \sqrt{2\sqrt{2}+5}\bigg)$$
$$- \bigg(8160\log(2) + \sqrt{17}\bigg(111\sqrt{\sqrt{17}+4} + 131\sqrt{\sqrt{17}-4}\bigg)\bigg)\arcsin\bigg(\frac{\sqrt{5-2\sqrt{2}} - \sqrt{2\sqrt{2}+5}}{4}\bigg)$$
$$- 2\log(2)\sqrt{17}\bigg(131\sqrt{\sqrt{17}+4} - 111\sqrt{\sqrt{17}-4}\bigg)\Bigg) + 30$$
$$\approx 0.30192723676986869546.$$

This is a new identity.

# 6 Acknowledgments

The author would like to thank Diego Sejas-Viscarra for his valuable collaboration during the writing of this work.

# References

[1] N. Batir, Integral representations of some series involving $\binom{2k}{k}^{-1}k^{-n}$ and some related series, *Appl. Math. Comput.* **147** (2004), 645–667.

[2] H. W. Gould, *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, West Virginia University, 1972.

[3] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, 2008.

[4] D. H. Lehmer, Interesting series involving the central binomial coefficient, *Amer. Math. Monthly* **92** (1985), 449–457.

[5] R. Sprugnoli, Sums of reciprocals of the central binomial coefficients, *Integers* **6** (2006), #A27.

[6] F. Z. Zhao and T. Wang, Some results for sums of the inverses of binomial coefficients, *Integers* **5** (2005), #A22.

---

2020 *Mathematics Subject Classification*: Primary 11B65; Secondary 05A10.
*Keywords:* reciprocal of central binomial coefficient, combinatorial sum, elementary function.

---

(Concerned with sequence A000984.)

---

Received January 30 2024; revised versions received February 10 2024; May 22 2024; May 27 2024. Published in *Journal of Integer Sequences*, August 12 2024.

---

Return to Journal of Integer Sequences home page.