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Realizability of Some Combinatorial Sequences

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Abstract

A sequence $a = (a_n)_{n=1}^{\infty}$ of non-negative integers is called realizable if there is a self-map $T : X \to X$ on a set X such that a_n is equal to the number of periodic points of T in X of (not necessarily exact) period n, for all $n \geq 1$. The sequence a is called almost realizable if there exists a positive integer m such that $(ma_n)_{n=1}^{\infty}$ is realizable. In this article, we show that certain wide classes of integer sequences are realizable, which contain many famous combinatorial sequences, such as the sequences of Apéry numbers of both kinds, central Delannoy numbers, Franel numbers, Domb numbers, Zagier numbers, and central trinomial coefficients. We also show that the sequences of Catalan numbers, Motzkin numbers, and large and small Schröder numbers are not almost realizable.

1 Introduction

Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers and $\mathbb{Z}^+$ the set of positive integers. In this paper, the serial numbers "Axxxxxx" associated with certain sequences in the paper all refer to the corresponding sequence numbers in the On-Line Encyclopedia of Integer Sequences (OEIS) [31]. We consider a property for sequences of non-negative integers which is inspired from dynamical systems.

Definition 1. A sequence $a = (a(n))_{n=1}^{\infty}$ of non-negative integers is called *realizable* if one of the two following equivalent conditions holds.

(1) there is a self-map $T : X \to X$ on a set X such that

$$a(n) = \#\{x \in X : T^n(x) = x\}$$

for all $n \geq 1$;

(2) $(\mu * a)(n)$ is a non-negative integer divisible by n, for all $n \geq 1$, where μ is the (classical) Möbius function and the operation $*$ is the Dirichlet convolution of arithmetic functions.

In this case, we also say that the sequence $a = (a(n))_{n=1}^{\infty}$ is realized via the map T and T realizes a.

Remark 2. For the equivalence, see [29, p. 398]. In the description (2), the condition

$$(\mu * a)(n) \geq 0$$

for all $n \geq 1$ is called the *sign condition*, and the condition

$$(\mu * a)(n) \equiv 0 \pmod{n}$$

for all $n \geq 1$ is called the *Dold condition*. In the description (1), by [37] one can equally require X to be an annulus and f a C^{∞} diffeomorphism of X.

Miska and Ward [23] defined the following generalization of realizability:

Definition 3. Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of non-negative integers. If there exists $m \in \mathbb{Z}^+$ such that the sequence $(ma_n)_{n=1}^{\infty}$ is realizable, then we say that a is *almost realizable*, and the minimal such $m \in \mathbb{Z}^+$ is denoted by $\mathrm{Fail}(a)$. When a is not almost realizable, we set $\mathrm{Fail}(a) = \infty$.

Example 4. We give some examples of sequences which are realizable, almost realizable, or not almost realizable.

(1) The sequence $(2^n - 1)_{n=1}^{\infty}$ A000225 is realized via the map $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, x \mapsto 2x \bmod 1$, the times-2 map on the circle $\mathbb{R}/\mathbb{Z}$.

(2) The sequence $(|(-2)^n - 1|)_{n=1}^{\infty}$ A062510 is realized via the map $z \mapsto z^{-2}$ on the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

(3) Let $d \in \mathbb{Z}^+$, $X = \{0, 1, \ldots, d-1\}^{\mathbb{Z}}$, and $T : X \ni (x_n)_n \mapsto (x_{n+1})_n \in X$ be the shift map on X. Then the sequence $(d^n)_{n=1}^{\infty}$ is realized via T.

(4) Consider a sequence $(U_n)_{n=1}^{\infty}$ given by $U_{n+2} = U_{n+1} + U_n, n \geq 1, U_1 = a, U_2 = b$, where $a, b \in \mathbb{Z}_{\geq 0}$. Note that $b = 3a = 3$ gives the *Lucas sequence* (L_n) (A000032) and $a = b = 1$ gives the *Fibonacci sequence* (F_n) (A000045). Puri and Ward [29] have showed that the sequence (U_n) is realizable if and only if $b = 3a$ if and only if (U_n) is a non-negative integer multiple of the Lucas sequence. In particular, the Fibonacci sequence (F_n) is not realizable. Indeed, Moss and Ward [25] proved that (F_n) is not almost realizable, but the sequence $(F_{n^2})_{n=1}^{\infty}$ (A054783) is almost realizable with $\mathrm{Fail}((F_{n^2})) = 5$.

(5) Write $S^{(1)}(n,k)$ for the *(signless) Stirling number of the first kind*, and $S^{(2)}(n,k)$ for the *Stirling number of the second kind*, where $n \geq k \in \mathbb{Z}_{\geq 0}$, see [33, pp. 32, 81]. For $k \geq 1$, set $S_k^{(1)} = (S^{(1)}(n+k-1,k))_{n=1}^{\infty}$ and $S_k^{(2)} = (S^{(2)}(n+k-1,k))_{n=1}^{\infty}$. Miska and Ward [23] have showed that $S_k^{(1)}$ is not almost realizable for every $k \geq 1$, and that $S_k^{(2)}$ is realizable if and only if $k \in \{1,2\}$, while $S_k^{(2)}$ is almost realizable with $\mathrm{Fail}(S_k^{(2)}) \mid (k-1)!$ for every $k \geq 1$.

(6) Write (E_n) for the sequence of *Euler numbers* A122045, given by the exponential generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Moss [24, Theorem 5.3.2] proved that the sequence $((-1)^n E_{2n})_{n=1}^{\infty}$ (A000364) is realizable.

(7) Write (B_n) for the sequence of *Bernoulli numbers*, given by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

cf. [32, Exercise 5.55]. For each $n \geq 1$, let $b_n \in \mathbb{Z}^+$ be the denominator of B_{2n} in the fraction in lowest terms (A002445) and write $|\frac{B_{2n}}{2n}| = \frac{\tau_n}{\eta_n}$ with $\tau_n, \eta_n \in \mathbb{Z}^+$, and $\gcd(\tau_n, \eta_n) = 1$. Then $(b_n)_{n=1}^{\infty}, (\tau_n)_{n=1}^{\infty}$ (A001067 taking absolute value), and $(\eta_n)_{n=1}^{\infty}$ (A006953) are realizable according to [13, Theorem 2.6] and [24, Theorem 5.5.3, Theorem 5.5.10]. Indeed, the sequence $(\eta_n)_{n=1}^{\infty}$ can be realized via an endomorphism of a group.

(8) Write (G_n) for the sequence of *Genocchi numbers* A226158, given by the exponential generating function

$$\frac{-2t}{e^{-t} + 1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}.$$

See [32, Exercise 5.8] for more on these numbers. It is easy to see that $G_{2n+1} = 0$ and $e_n := (-1)^n G_{2n} = (-1)^n 2(1-4^n) B_{2n}$ is a positive odd integer for $n \geq 1$. Since $\eta_1 = 12$ and $\tau_1 = 1$, by (7) and Fermat's little theorem, for every prime $p \geq 5$, we have

$$e_p = 2(4^p - 1) \times 2p \times \frac{\tau_p}{\eta_p} \equiv 4p(4-1)\frac{1}{12} \equiv 0 \not\equiv 1 = e_1 \pmod{p}.$$

If the sequence $(e_n)_{n=1}^{\infty}$ is almost realizable, then $p \mid \mathrm{Fail}((e_n))$ for every prime $p \geq 5$, contradicting the fact that $\mathbb{Z}^+ \ni \mathrm{Fail}((e_n)) < \infty$. Therefore, the sequence $(e_n)_{n=1}^{\infty}$ is not almost realizable.

(9) Consider a multiplicative arithmetic function f whose values are non-negative integers. Then $\mu * f$ is also multiplicative. Using multiplicativity, we see that $(f(n))_{n=1}^{\infty}$ is realizable

if and only if
$$0 \le f(p^m) - f(p^{m-1}) = (\mu * f)(p^m) \equiv 0 \pmod{p^m}$$
for every prime p and $m \in \mathbb{Z}^+$. For example, consider the *divisor function* $\sigma_k(n) = \sum_{d|n} d^k, n \in \mathbb{Z}^+$, for $k \in \mathbb{Z}^+$. As σ_k is multiplicative, from the above observation it is clear that $(\sigma_k(n))_{n=1}^\infty$ is realizable.

(10) Let $(\text{Bell}(n))_{n=0}^\infty$ be the sequence of *Bell numbers* A000110, which was introduced by Bell [6]. By the Touchard congruence [36], for every prime p, we have
$$\text{Bell}(p) - \text{Bell}(1) \equiv \text{Bell}(0) = 1 \pmod{p}.$$
As in (8), we see that the sequence $(\text{Bell}(n))_{n=1}^\infty$ is not almost realizable. For more information on Bell numbers, see [6, 5, 3] and the references therein.

(11) Let d_n be the *number of derangements* A000166 of the set $\{1, \dots, n\}$ for $n \ge 1$. It is well-known that $d_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$. Then, for every prime number p, we have
$$d_p - d_1 = p! \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} + (-1)^p - 0 \equiv (-1)^p \pmod{p}.$$
As in (8), we see that the sequence $(d_n)_{n=1}^\infty$ is not almost realizable.

In this article, we consider the realizability of some combinatorial sequences related to binomial coefficients, which are defined below.

Definition 5. For $r \in \mathbb{Z}^+$ and $n, s \in \mathbb{Z}_{\ge 0}$, define
$$A(n,r,s) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s.$$

Remark 6. Definition 5 includes many well-known sequences in combinatorics. For example, $(A(n))_{n=0}^\infty := (A(n,2,2))_{n=0}^\infty$ is the sequence of *Apéry numbers (of the first kind)* A005259, and $(\beta(n))_{n=0}^\infty := (A(n,2,1))_{n=0}^\infty$ is the sequence of *Apéry numbers of the second kind* A005258. The Apéry numbers of both kinds were introduced by Apéry [2] to prove the irrationality of $\zeta(3)$. For more information on Apéry numbers, see [7, 39, 30] and the references therein. The sequence of *central Delannoy numbers* A001850 is $(D(n))_{n=0}^\infty := (A(n,1,1))_{n=0}^\infty$, cf. [32, Example 6.3.8]. The number $D(n)$ equals the number of paths from the southwest corner $(0,0)$ of a square grid to the northeast corner (n,n), using only single steps north, northeast, or east. See [4, 35] to learn more about the central Delannoy numbers. Usually, the sequence $(f^{(3)}(n))$ A000172 is called the sequence of *Franel numbers*, which was first introduced by Franel [14]. Generally, for $r \in \mathbb{Z}^+$, we call $(f^{(r)}(n))_{n=0}^\infty := (A(n,r,0))_{n=0}^\infty$ the sequence of *Franel numbers of order* r. Note that $(f^{(1)}(n)) = (2^n)$ has been studied in (3) of Example 4, and that $(f^{(2)}(n))$ is the sequence of *central binomial coefficients* A000984.

Definition 7. For $r \in \mathbb{Z}^+$ and $n, s, t \in \mathbb{Z}_{\geq 0}$, define

$$D(n, r, s, t) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{2k}{k}^s \binom{2(n-k)}{n-k}^t.$$

Remark 8. The sequence of *Domb numbers* A002895 is $(\text{Domb}(n))_{n=0}^{\infty} := (D(n, 2, 1, 1))_{n=0}^{\infty}$, which was introduced by Domb [11]. For more information on Domb numbers, we refer the reader to [11, 9, 21] and the references therein. We call $(Z(n))_{n=0}^{\infty} := (D(n, 1, 1, 1))_{n=0}^{\infty}$ the sequence of *Zagier numbers* (A081085), which corresponds to Zagier [40, E. in Table 2].

Definition 9. The sequence $(P(n))_{n=0}^{\infty}$ of *Catalan-Larcombe-French numbers* A053175 is given by the formula

$$P(n) = 2^n \sum_{k=0}^{n} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k}, n \geq 0.$$

Remark 10. Catalan [8] showed that $(P(n))$ has the following recurrence relation:

$$(n + 1)^2 P(n + 1) = 8(3n^2 + 3n + 1)P(n) - 128n^2 P(n - 1), n \geq 1.$$

The number $P(n)$ is the "other" Catalan number in the sense of Larcombe-French [17]. Larcombe and French [17] showed that $P(n)$ can be given by elliptic integrals. We will show that $(P(n))_{n=1}^{\infty}$ is realizable (Remark 16), while the sequence $(C(n))_{n=1}^{\infty}$ of (true) Catalan numbers is not almost realizable ((1) of Theorem 21).

Definition 11. For $r \in \mathbb{Z}^+$ and $n, s, t, u \in \mathbb{Z}_{\geq 0}$, define

$$T(n, r, s, t, u) = \sum_{k=0}^{n} \binom{n}{2k}^r \binom{n+k}{k}^s \binom{2k}{k}^t \binom{2(n-k)}{n-k}^u.$$

Remark 12. For $n \geq 1$, the *central trinomial coefficient* $T(n)$ is defined to be the coefficient of x^n in $(x^2 + x + 1)^n$ (A002426). Clearly, $T(n) = T(n, 1, 0, 1, 0)$.

The sequences given in Definitions 5, 7, 9, and 11 are all realizable. In fact, the results can be generalized further, see Remarks 33 and 34.

Theorem 13. *For every $s \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}^+$, the sequence $(A(n, r, s))_{n=1}^{\infty}$ is realizable.*

Remark 14. In particular, the sequence of Apéry numbers (of the first kind), the sequence of Apéry numbers of the second kind, the sequence of central Delannoy numbers, and the sequence of Franel numbers of order $r \in \mathbb{Z}^+$ are all realizable.

Theorem 15. *For every $s, t \in \mathbb{Z}_{\geq 0}$, and $r \in \mathbb{Z}^+$, the sequence $(D(n, r, s, t))_{n=1}^{\infty}$ is realizable.*

Remark 16. In particular, the sequence of Domb numbers and the sequence of Zagier numbers are both realizable. Moreover, the sequence of Catalan-Larcombe-French numbers is also realizable. Larcombe and French [18, Theorem 3] proved that $P(n) = 2^n Z(n)$ for all $n \geq 0$. As in (3) of Example 4, the sequence $(2^n)_{n=1}^{\infty}$ is realized via the shift map $T : X = \{0,1\}^{\mathbb{Z}} \to X$. By Theorem 15, there exists a map $F : Y \to Y$ on a set Y such that

$$Z(n) = \#\{y \in Y : F^n(y) = y\}, n \geq 1.$$

Clearly, the sequence $(P(n))_{n=1}^{\infty} = (2^n Z(n))_{n=1}^{\infty}$ is realized via the map

$$T \times F : X \times Y \to X \times Y, (x, y) \mapsto (T(x), F(y)).$$

Remark 17. By Theorem 13, for all $m, r \in \mathbb{Z}^+$, $s \in \mathbb{Z}_{\geq 0}$, and every prime p, we have

$$A(p^m, r, s) \equiv A(p^{m-1}, r, s) \pmod{p^m}.$$

In fact, in the proof of Theorem 13, we will show that for every $m, n, r \in \mathbb{Z}^+$, $s \in \mathbb{Z}_{\geq 0}$, and every prime p, we have

$$A(np^m, r, s) \equiv A(np^{m-1}, r, s) \pmod{p^m}. \tag{1}$$

(The congruence (1) is a result of Theorem 13 and Lemma 30.) For certain values of r and s, results stronger than (1) have been proved. For example, Straub [34, Theorem 1.1] asserted that for $(V(n))$ in the 15 known sporadic Apéry-like sequences [34, pp. 1–2], arbitrary prime $p \geq 3$, and $m, n \in \mathbb{Z}^+$, we have

$$V(np^m) \equiv V(np^{m-1}) \pmod{p^{2m}}. \tag{2}$$

The 15 known sporadic Apéry-like sequences include $(f^{(3)}(n))_n$, $(f^{(4)}(n))_n$ (the sequence A005260), $(A(n))_n$, $(\beta(n))_n$, $(Z(n))_n$, $(\mathrm{Domb}(n))_n$, $(D(n, 2, 1, 0))_n$ (the sequence A002893), and some other sequences, see [20]. Similarly, in the proof of Theorem 15, we will show that for all $m, n, r \in \mathbb{Z}^+$, $s, t \in \mathbb{Z}_{\geq 0}$, and every prime p, we have

$$D(np^m, r, s, t) \equiv D(np^{m-1}, r, s, t) \pmod{p^m}. \tag{3}$$

(The congruence (3) is a result of Theorem 15 and Lemma 30.) Also, for restricted values of s, t, and p, results stronger than (3) have been proved. For example, Osburn and Sahu [27, Theorem 1.1] showed that for all $m, n, s, t \in \mathbb{Z}^+$, $r \in \{2, 3, 4, \ldots\}$, and every prime $p \geq 5$, we have

$$D(np^m, r, s, t) \equiv D(np^{m-1}, r, s, t) \pmod{p^{3m}}. \tag{4}$$

Theorem 18. *For every $r \in \mathbb{Z}^+$ and $s, t, u \in \mathbb{Z}_{\geq 0}$, the sequence $(T(n, r, s, t, u))_{n=1}^{\infty}$ is realizable.*

Next we consider some other famous sequences involving binomial coefficients as well, but they are not almost realizable.

Definition 19. (1) The sequence of *Catalan numbers* A000108 $(C(n))_{n=0}^{\infty}$ is given by the formula

$$C(n) := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}, n \geq 0;$$

(2) The sequence of *Motzkin numbers* A001006 $(M(n))_{n=0}^{\infty}$ is given by the formula

$$M(n) := \sum_{k=0}^{n} \binom{n}{2k} C(k), n \geq 0;$$

(3) The sequence of *large Schröder numbers* A006318 is given by the formula

$$S(n) := \sum_{k=0}^{n} \binom{n+k}{2k} C(k), n \geq 0.$$

Remark 20. The Motzkin numbers are first appeared in Motzkin [26] in a circle chording setting. The *large Schröder number* $S(n)$ describes the number of paths from the southwest corner $(0,0)$ of a square grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the SW-NE diagonal.

Theorem 21. *The following sequences are not almost realizable.*

(1) $(C(n))_{n=1}^{\infty}$*;*

(2) $(M(n))_{n=1}^{\infty}$*;*

(3) $(S(n))_{n=1}^{\infty}$*.*

Remark 22. The sequence of *little Schröder numbers* A001003 $(s(n))_{n=1}^{\infty}$ is given by the formula

$$s(n) := \sum_{k=1}^{n} N(n,k)2^{k-1}, n \geq 1,$$

cf. [32, p. 178]. Here

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \in \mathbb{Z}^{+}$$

is the *Narayana number* A001263, cf. [32, Exercise 6.36]. It is well-known that $S(n) = 2s(n)$ for every $n \geq 1$, so from (3) of Theorem 21, we deduce that the sequence $(s(n))_{n=1}^{\infty}$ of little Schröder numbers is not almost realizable as well.

It is interesting to consider the realizability of combinatorial sequences of different types, which may not involve binomial coefficients. Motivated by computations, we make the following conjecture.

Conjecture 23. For $n \geq 1$, let $p(n)$ be the number of partitions of n, i.e., ways of writing n as an (unordered) sum of positive integers (A000041). Then the sequence

$$(p(n))_{n=1}^{\infty} = (1, 2, 3, 5, 7, 11, 15, \ldots)$$

of partition numbers is not almost realizable.

Remark 24. However, we show that the sequence $(p(n))_{n=1}^{\infty}$ satisfies the sign condition. Clearly, the sequence $(p(n))_{n=1}^{\infty}$ is increasing. By Remark 29, it suffices to check that

$$p(2n) \geq np(n), n \geq 1. \tag{5}$$

By [22, Corollary 3.1] and [19, Theorem 15.7], for all integers $n \geq 3$, we have

$$\frac{1}{14}e^{2\sqrt{n}} < p(n) < \frac{\pi}{\sqrt{6(n-1)}}e^{\pi\sqrt{\frac{2}{3}n}}.$$

Thus, for every integer $n \geq 523$, we have

$$\frac{p(2n)}{p(n)} > \frac{\sqrt{6(n-1)}}{14\pi}e^{(2\sqrt{2}-\pi\sqrt{\frac{2}{3}})\sqrt{n}} > n.$$

For integers $1 \leq n \leq 522$, the equation (5) has been checked by direct computation via a computer. Hence the sequence $(p(n))_{n=1}^{\infty}$ satisfies the sign condition.

In §2, we consider several useful lemmas. In §3, we complete the proofs of Theorems 13, 15, 18, and 21, and give some remarks.

2 Auxiliary lemmas

First, we need the following well-known theorem of Kummer [16].

Lemma 25 (Kummer's theorem)**.** *Given a prime number p and integers $n \geq m \geq 0$, the value of $\nu_p\left(\binom{n}{m}\right)$ is equal to the number of carries when m is added to $n-m$ in base p. Here ν_p denotes the standard p-adic valuation on $\mathbb{Q}$.*

The following lemma is [15, Corollary of p. 490] by Helou and Terjanian.

Lemma 26. *Let $n \geq m \geq 0$ be integers.*

(1) For every prime $p \geq 5$, we have

$$\binom{np}{mp} \equiv \binom{n}{m} \pmod{p^{3+\max(\nu_p(m),\nu_p(n-m))+\nu_p\left(\binom{n}{m}\right)}}.$$

(2) For $p = 3$, we have

$$\binom{3n}{3m} \equiv \binom{n}{m} \pmod{3^{2+\max(\nu_3(m),\nu_3(n-m))+\nu_3\left(\binom{n}{m}\right)}}.$$

(3) For $p = 2$, we have

$$\binom{2n}{2m} \equiv \binom{n}{m} \pmod{2^{1+\max(\nu_2(m),\nu_2(n-m))+\nu_2\left(\binom{n}{m}\right)}}.$$

Here, for two integers A, B, and a prime p, the expression $A \equiv B \pmod{p^\infty}$ means that $A \equiv B \pmod{p^N}$ for all $N \in \mathbb{Z}^+$, or equivalently, $A = B$.

Note that in Lemma 26, the conclusion trivially holds when $m = 0$.

Lemma 27. *Let $n \in \mathbb{Z}_{\geq 0}$ and p be a prime divisor of n. Set $m = \nu_p(n) \in \mathbb{Z}^+ \cup \{\infty\}$. Then*

$$\binom{n}{\lambda p} \equiv \binom{n/p}{\lambda} \pmod{p^{\max(m,\nu_p(n-\lambda p))}},$$

for every non-negative integer λ.

Proof. When $n = 0$, we have $\binom{n}{\lambda p} = 0 = \binom{n/p}{\lambda}$. Hence the conclusion is clear. We may assume that $n > 0$. Write $n = lp^m$. Observe that $\gcd(l, m) = 1$. If $\lambda \geq lp^{m-1}$ or $\lambda = 0$, then $\binom{n}{\lambda p} = \binom{n/p}{\lambda} \in \{0, 1\}$, so the conclusion holds. Now assume that $0 < \lambda < lp^{m-1}$. As $1 + \nu_p((n/p) - \lambda) = \nu_p(n - \lambda p)$, by Lemma 26, it suffices to show that

$$1 + \max\left(\nu_p(\lambda), \nu_p(lp^{m-1} - \lambda)\right) + \nu_p\left(\binom{lp^{m-1}}{\lambda}\right) \geq m.$$

Write $\lambda = p^t q$, where $t = \nu_p(\lambda) \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{Z}^+$ with $p \nmid q$. When $t \geq m - 1$, we have

$$1 + \max\left(\nu_p(\lambda), \nu_p(lp^{m-1} - \lambda)\right) + \nu_p\left(\binom{lp^{m-1}}{\lambda}\right) \geq 1 + \nu_p(\lambda) = 1 + t \geq m.$$

If $t \leq m - 2$, then $\nu_p(n/p) = m - 1 > m - 2 \geq t = \max(\nu_p(\lambda), \nu_p((n/p) - \lambda))$, so by Lemma 25 we see that

$$\nu_p\left(\binom{lp^{m-1}}{\lambda}\right) = \nu_p\left(\binom{lp^{m-1}}{qp^t}\right) \geq m - 1 - t.$$

Thus,

$$1 + \max\left(\nu_p(\lambda), \nu_p(lp^{m-1} - \lambda)\right) + \nu_p\left(\binom{lp^{m-1}}{\lambda}\right) \geq 1 + t + m - 1 - t = m.$$

□

The following lemma gives a sufficient condition for the sign condition.

Lemma 28. *Let $(b(n))_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Assume that there is a constant $C \geq 1.221$ such that $b(n+1) \geq Cb(n)$ for every $n \geq 1$. Then $(\mu * b)(n) \geq 0$ for every $n \geq 1$.*

Proof. Set $f = \mu * b$. By assumption we have $b(n+k) \geq C^k b(n)$ for every $n \geq 1$ and $k \geq 0$. In particular, $(b(n))_{n=1}^{\infty}$ is non-decreasing and non-negative. Trivially, $f(1) = b(1) \geq 0$. For every prime number p and $k \in \mathbb{Z}^+$, since $p^k > p^{k-1}$, we have

$$f(p^k) = b(p^k) - b(p^{k-1}) \geq 0.$$

For every pair of distinct prime numbers $p_1 \neq p_2$ and $k_1, k_2 \in \mathbb{Z}^+$, we have

$$\begin{aligned} f\left(p_1^{k_1} p_2^{k_2}\right) &= b\left(p_1^{k_1} p_2^{k_2}\right) - b\left(p_1^{k_1-1} p_2^{k_2}\right) - b\left(p_1^{k_1} p_2^{k_2-1}\right) + b\left(p_1^{k_1-1} p_2^{k_2-1}\right) \\ &\geq b\left(p_1^{k_1} p_2^{k_2}\right) - C^{p_1^{k_1-1} p_2^{k_2} - p_1^{k_1} p_2^{k_2}} b\left(p_1^{k_1} p_2^{k_2}\right) - C^{p_1^{k_1} p_2^{k_2-1} - p_1^{k_1} p_2^{k_2}} b\left(p_1^{k_1} p_2^{k_2}\right) + 0 \\ &\geq \left(1 - C^{3-3\times 2} - C^{2-3\times 2}\right) b\left(p_1^{k_1} p_2^{k_2}\right) \\ &\geq \left(1 - 1.221^{-3} - 1.221^{-4}\right) b\left(p_1^{k_1} p_2^{k_2}\right) \\ &\geq 0. \end{aligned}$$

The minimum positive integer with at least three distinct prime divisor is $30 = 2 \times 3 \times 5$, so it suffices to show that $f(n) \geq 0$ for $n \geq 30$. Assume that $n \geq 30$, and set $m = \lfloor \frac{n}{2} \rfloor \geq 15$. Then

$$\begin{aligned} f(n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) b(d) = b(n) + \sum_{d|n, d\neq n} \mu\left(\frac{n}{d}\right) b(d) \geq b(n) - \sum_{d|n, d\neq n} b(d) \\ &\geq b(n) - \sum_{d=1}^{m} b(d) \geq C^m b(m) - \sum_{d=1}^{m} C^{d-m} b(m) \geq (C^m - m) b(m) \\ &\geq (1.221^m - m) b(m) \geq 0, \end{aligned}$$

since $m \geq 15$. □

Remark 29. The converse of Lemma 28 is false, in general. For example, the trivial sequence $(1)_{n=1}^{\infty}$ provides a counterexample. Note that the number 1.221 in the statement can be replaced by the unique positive root $x_0 = 1.220744\cdots$ of the equation $x^4 = x + 1$. Puri [28] observed that if $(b(n))_{n=1}^{\infty}$ is a non-decreasing sequence of non-negative real numbers with

$$b(2n) \geq nb(n), \tag{6}$$

for all $n \geq 1$, then $(\mu * b)(n) \geq 0$ for all $n \geq 1$, cf. [23, proof of Lemma 8]. Note that Lemma 28 cannot be deduced directly from the observation of Puri since $1.221^n < n$ for $n \in \{2, 3, \ldots, 12\}$, and vice versa.

We consider the Dold condition now, which has an equivalent statement as follows.

Lemma 30. *Let $(V(n))_{n=1}^{\infty}$ be a sequence of integers. Then the following conditions are equivalent.*

(1) for every $n, m \in \mathbb{Z}^+$, and every prime number p, we have

$$V(np^m) \equiv V(np^{m-1}) \pmod{p^m};$$

(2) $(V(n))_{n=1}^{\infty}$ satisfies the Dold condition, i.e., $(\mu * V)(n) \equiv 0 \pmod{n}$, for every $n \geq 1$.

Proof. Set $g = \mu * V$. Given an arbitrary integer $n \geq 2$, write $n = p_1^{m_1} p_2^{m_2} \cdots p_l^{m_l}$, where $p_1, \ldots, p_l$ are pairwise distinct primes, and $l, m_1, \ldots, m_l \in \mathbb{Z}^+$. Set $n_1 = p_2^{m_2} \cdots p_l^{m_l}$. Here $n_1 = 1$ when $l = 1$. Then, we have

$$\begin{aligned} g(n) &= \sum_{d|n} \mu(d) V\left(\frac{n}{d}\right) = \sum_{d|n_1} \left(\mu(d) V\left(\frac{n}{d}\right) + \mu(dp_1) V\left(\frac{n}{dp_1}\right)\right) \\ &= \sum_{d|n_1} \mu(d) \left(V\left(\frac{n_1}{d} p_1^{m_1}\right) + \mu(p_1) V\left(\frac{n_1}{d} p_1^{m_1-1}\right)\right) \\ &= \sum_{d|n_1} \mu(d) \left(V\left(\frac{n_1}{d} p_1^{m_1}\right) - V\left(\frac{n_1}{d} p_1^{m_1-1}\right)\right). \end{aligned} \tag{7}$$

First assume that for every $n, m \in \mathbb{Z}^+$, and every prime number p, we have

$$V(np^m) \equiv V(np^{m-1}) \pmod{p^m}.$$

We show that the Dold condition holds. The congruence $g(1) \equiv 0 \pmod{1}$ trivially holds. For an arbitrary integer $n = p_1^{m_1} p_2^{m_2} \cdots p_l^{m_l} \geq 2$ as above, we have

$$g(n) = \sum_{d|n_1} \mu(d) \left(V\left(\frac{n_1}{d} p_1^{m_1}\right) - V\left(\frac{n_1}{d} p_1^{m_1-1}\right)\right) \equiv 0 \pmod{p_1^m}$$

by (7). Similarly, we have $g(n) \equiv 0 \pmod{p_j^{m_j}}$ for $1 < j \leq l$. Thus, we get $g(n) \equiv 0 \pmod{n}$, i.e., the Dold condition is verified.

Now assume that for every $n \in \mathbb{Z}^+$, we have $g(n) \equiv 0 \pmod{n}$. We will show that for every $n, m \in \mathbb{Z}^+$, and every prime number p, we have

$$V(np^m) \equiv V(np^{m-1}) \pmod{p^m}. \tag{8}$$

Write $n = n_1 p^s$, where $n_1 \in \mathbb{Z}^+$ and $s \in \mathbb{Z}_{\geq 0}$ with $\gcd(n_1, p) = 1$. Note that the congruence

$$V(n_1 p^{m+s}) \equiv V(n_1 p^{m+s-1}) \pmod{p^{m+s}}$$

implies the congruence (8), hence it suffices to show that (8) holds in the case $\gcd(n,p)=1$. We use induction on $n \in \mathbb{Z}^+$. When $n=1$, the congruence (8) follows from

$$V(p^m) - V(p^{m-1}) = g(p^m) \equiv 0 \pmod{p^m}.$$

Let n be an arbitrary integer at least 2 and assume that (8) hold for all smaller n (with $\gcd(n,p)=1$). Thus, by (7), the inductive hypothesis, and the Dold condition, we have

$$\begin{aligned} 0 \equiv g(np^m) &= \sum_{d \mid n} \mu(d) \left(V\left(\frac{n}{d}p^m\right) - V\left(\frac{n}{d}p^{m-1}\right)\right) \\ &\equiv \mu(1)\left(V\left(np^m\right) - V\left(np^{m-1}\right)\right) + \sum_{d \mid n, d>1} \mu(d) \cdot 0 \\ &= V\left(np^m\right) - V\left(np^{m-1}\right) \pmod{p^m}. \end{aligned}$$

(Here np^m, n, p, m correspond to n, n_1, p_1, m_1 in (7), respectively.) Therefore, we have showed that (8) holds for every $n, m \in \mathbb{Z}^+$, and every prime p. □

Remark 31. From the proof of Lemma 30, we can require that n and p are coprime in condition (1).

3 Proofs of theorems

Proof of Theorem 13. Fix $s \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}^+$. Write $V(n) = A(n,r,s)$, $n \geq 1$, for simplicity of notation. Set $g = \mu * V$. For $n=1$, it is clear that $g(1) = V(1) = 1 + 2^s$ is divisible by 1. Given arbitrary $m, n \in \mathbb{Z}^+$, and an arbitrary prime number p, we consider the difference $V(np^m) - V(np^{m-1})$ modulo p^m. Let $0 \leq k \leq np^m$ be an integer. If $p \nmid k$, then we have

$$\binom{np^m}{k} \equiv 0 \pmod{p^m}$$

by Lemma 25. If $p \mid k$, then

$$\binom{np^m}{k} \equiv \binom{np^{m-1}}{k/p} \pmod{p^m}, \binom{np^m+k}{k} \equiv \binom{np^{m-1}+k/p}{k/p} \pmod{p^m}$$

by Lemma 27, since $\nu_p(np^m + k - k) \geq m$. Therefore,

$$\begin{aligned} V(np^m) - V(np^{m-1}) &= \sum_{k=0}^{np^m} \binom{np^m}{k}^r \binom{np^m+k}{k}^s - \sum_{k=0}^{np^{m-1}} \binom{np^{m-1}}{k}^r \binom{np^{m-1}+k}{k}^s \\ &\equiv 0 + \sum_{\lambda=0}^{np^{m-1}} \binom{np^m}{\lambda p}^r \binom{np^m+\lambda p}{\lambda p}^s - \sum_{k=0}^{np^{m-1}} \binom{np^{m-1}}{k}^r \binom{np^{m-1}+k}{k}^s \\ &\equiv \sum_{\lambda=0}^{np^{m-1}} \binom{np^{m-1}}{\lambda}^r \binom{np^{m-1}+\lambda}{\lambda}^s - \sum_{k=0}^{np^{m-1}} \binom{np^{m-1}}{k}^r \binom{np^{m-1}+k}{k}^s \\ &= 0 \pmod{p^m}. \end{aligned}$$

We get that

$$V(np^m) - V(np^{m-1}) \equiv 0 \pmod{p^m}.$$

By Lemma 30, we see that $(V(n))_{n=1}^{\infty}$ satisfies the Dold condition.

It suffices to show that $g(n) \geq 0$ for all $n \in \mathbb{Z}^+$. For every $n \geq 1$, we have

$$\begin{aligned}
V(n+1) &= \sum_{k=0}^{n+1} \binom{n+1}{k}^r \binom{n+1+k}{k}^s = 1 + \sum_{k=1}^{n+1} \binom{n+1}{k}^r \binom{n+1+k}{k}^s \\
&= 1 + \sum_{k=1}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right)^r \left(\binom{n+k}{k} + \binom{n+k}{k-1} \right)^s \\
&\geq 1 + \sum_{k=1}^{n+1} \left(\binom{n}{k}^r \binom{n+k}{k}^s + \binom{n}{k-1}^r \binom{n+k}{k-1}^s \right) \\
&\geq 1 + \sum_{k=1}^{n+1} \left(\binom{n}{k}^r \binom{n+k}{k}^s + \binom{n}{k-1}^r \binom{n+k-1}{k-1}^s \right) \\
&= 2 \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s = 2V(n).
\end{aligned}$$

As $2 > 1.221$, the Dold condition $g(n) \geq 0$ for all $n \in \mathbb{Z}^+$ follows from Lemma 28. Therefore, the sequence $(V(n))_{n=1}^{\infty}$ is realizable. □

Remark 32. In the proof of Theorem 13, we have showed that $A(n+1, r, s) \geq 2A(n, r, s)$, for all $n, s \geq 0$, and $r \geq 1$. For some specified values of r and s, some stronger results are known. Recall that $A(n) = A(n, 2, 2), \beta(n) = A(n, 2, 1), D(n) = A(n, 1, 1)$, and $f^{(r)}(n) = A(n, r, 0)$. Xia and Yao [38, Corollary 5] proved that the sequence $(A(n))_{n=0}^{\infty}$ is *strictly log-convex*, i.e.,

$$\frac{A(n+1)}{A(n)} > \frac{A(n)}{A(n-1)}, n \geq 1.$$

Hence, for all $n \geq 1$, we have

$$\frac{A(n+1)}{A(n)} \geq \frac{A(2)}{A(1)} = \frac{73}{5} > 2.$$

Xia and Yao also showed that the sequence $(D(n))_{n=0}^{\infty}$ is strictly log-convex [38, Corollary 6], hence we have

$$\frac{D(n+1)}{D(n)} \geq \frac{D(2)}{D(1)} = \frac{13}{3} > 2, n \geq 1.$$

The result [10, Theorem 5.2] of Chen and Xia implies that the sequence $(\beta(n))_{n=0}^{\infty}$ is strictly log-convex, hence we get

$$\frac{\beta(n+1)}{\beta(n)} \geq \frac{\beta(2)}{\beta(1)} = \frac{19}{3} > 2, n \geq 1.$$

The result [12, Corollary 4.3] of Došlić implies that for $r \in \{3,4\}$, the sequence $(f^{(r)}(n))_{n=0}^{\infty}$ is *log-convex*, i.e.,

$$f^{(r)}(n+1)f^{(r)}(n-1) \geq f^{(r)}(n)^2, n \geq 1.$$

Proof of Theorem 15. Fix $s, t \in \mathbb{Z}_{\geq 0}$, and $r \in \mathbb{Z}^+$. Write $W(n) = D(n,r,s,t)$, $n \geq 1$, for simplicity of notation. Set $g = \mu * W$. For $n = 1$, it is clear that $g(1) = W(1) = 2^t + 2^s$ is divisible by 1. Given $m, n \in \mathbb{Z}^+$, and an arbitrary prime number p, we consider the difference $A := W(np^m) - W(np^{m-1})$ modulo p^m. Let $0 \leq k \leq np^m$ be an integer. If $p \nmid k$, then we have

$$\binom{np^m}{k} \equiv 0 \pmod{p^m} \tag{9}$$

by Lemma 25. Now assume that $p \mid k$ and write $k = lp^u$, where $(l,u) = (0,m)$ when $k = 0$, and $l, u \in \mathbb{Z}^+$ with $\gcd(l,p) = 1$ when $k > 0$. We claim that

$$\binom{np^m}{k}^r \binom{2k}{k}^s \binom{2(np^m-k)}{np^m-k}^t \equiv \binom{np^{m-1}}{lp^{u-1}}^r \binom{2lp^{u-1}}{lp^{u-1}}^s \binom{2(np^{m-1}-lp^{u-1})}{np^{m-1}-lp^{u-1}}^t \pmod{p^m}. \tag{10}$$

Note that Lemma 27 implies that

$$\binom{np^m}{k}^r \equiv \binom{np^{m-1}}{lp^{u-1}}^r \pmod{p^m}.$$

By Lemma 25, we have $\nu_p\left(\binom{2M}{M}\right) \geq \delta_{2,p}$ for all $M \in \mathbb{Z}^+$, where $\delta_{2,p}$ is the Kronecker symbol. From Lemma 26, we see that

$$\binom{2k}{k} \equiv \binom{2lp^{u-1}}{lp^{u-1}} \pmod{p^{1+u}}, \tag{11}$$

$$\binom{2(np^m-k)}{np^m-k} \equiv \binom{2(np^{m-1}-lp^{u-1})}{np^{m-1}-lp^{u-1}} \pmod{p^{1+\min(u,m)}}. \tag{12}$$

Thus, the congruence (10) holds if $u \geq m-1$. Assume that $1 \leq u \leq m-2$. By the definition of u, we have $0 < k < np^m$. From Lemma 25, we get that

$$v_p\left(\binom{np^m}{k}\right) = v_p\left(\binom{np^{m-1}}{k/p}\right) \geq m-u. \tag{13}$$

Set $N = \max(0, m - r(m-u))$. To show (10), it suffices to prove the congruences

$$\binom{2k}{k} \equiv \binom{2lp^{u-1}}{lp^{u-1}} \pmod{p^N} \text{ and } \binom{2(np^m-k)}{np^m-k} \equiv \binom{2(np^{m-1}-lp^{u-1})}{np^{m-1}-lp^{u-1}} \pmod{p^N}, \tag{14}$$

which follows from (11) and (12), since $r \geq 1$ and $1 \leq u \leq m-2$ imply

$$N = \max(0, ru - (r-1)m) < 1 + u = 1 + \min(u, m).$$

By (9) and (10), the difference $A = W(np^m) - W(np^{m-1})$ satisfies

$$\begin{aligned}
A &= \sum_{k=0}^{np^m} \binom{np^m}{k}^r \binom{2k}{k}^s \binom{2(np^m-k)}{np^m-k}^t - \sum_{k=0}^{np^{m-1}} \binom{np^{m-1}}{k}^r \binom{2k}{k}^s \binom{2(np^{m-1}-k)}{np^{m-1}-k}^t \\
&\equiv 0 + \sum_{\lambda=0}^{np^{m-1}} \binom{np^m}{\lambda p}^r \binom{2\lambda p}{\lambda p}^s \binom{2(np^m-\lambda p)}{np^m-\lambda p}^t - \sum_{k=0}^{np^{m-1}} \binom{np^{m-1}}{k}^r \binom{2k}{k}^s \binom{2(np^{m-1}-k)}{np^{m-1}-k}^t \\
&\equiv \sum_{\lambda=0}^{np^{m-1}} \binom{np^{m-1}}{\lambda}^r \binom{2\lambda}{\lambda}^s \binom{2(np^{m-1}-\lambda)}{np^{m-1}-\lambda}^t - \sum_{k=0}^{np^{m-1}} \binom{np^{m-1}}{k}^r \binom{2k}{k}^s \binom{2(np^{m-1}-k)}{np^{m-1}-k}^t \\
&= 0 \pmod{p^m}.
\end{aligned}$$

We get $W(np^m) \equiv W(np^{m-1}) \pmod{p^m}$. Hence, the Dold condition for $(W(n))_{n=1}^\infty$ follows from Lemma 30.

We consider the sign condition. It is clear that the sequence $\left(\binom{2n}{n}\right)_{n=0}^\infty$ is strictly increasing by looking the ratios of two adjacent terms. For all $n \geq 1$, we have

$$\begin{aligned}
W(n+1) &= \sum_{k=0}^{n+1} \binom{n+1}{k}^r \binom{2k}{k}^s \binom{2(n+1-k)}{n+1-k}^t \\
&= \binom{2(n+1)}{n+1}^t + \sum_{k=1}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1}\right)^r \binom{2k}{k}^s \binom{2(n+1-k)}{n+1-k}^t \\
&\geq \binom{2(n+1)}{n+1}^t + \sum_{k=1}^{n+1} \left(\binom{n}{k}^r + \binom{n}{k-1}^r\right) \binom{2k}{k}^s \binom{2(n+1-k)}{n+1-k}^t \\
&\geq \binom{2n}{n}^t + \sum_{k=1}^{n} \binom{n}{k}^r \binom{2k}{k}^s \binom{2(n-k)}{n-k}^t + \sum_{k=0}^{n} \binom{n}{k}^r \binom{2k}{k}^s \binom{2(n-k)}{n-k}^t \\
&= 2\sum_{k=0}^{n} \binom{n}{k}^r \binom{2k}{k}^s \binom{2(n-k)}{n-k}^t = 2W(n).
\end{aligned}$$

As $2 > 1.221$ and $W(n) > 0$ for all $n \geq 1$, the sign condition for $(W(n))_{n=1}^\infty$ holds by Lemma 28. Thus, the sequence $(W(n))_{n=1}^\infty$ is realizable. □

Remark 33. For every $r \in \mathbb{Z}^+$ and $s, t, u \in \mathbb{Z}_{\geq 0}$, set

$$C(n, r, s, t, u) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s \binom{2k}{k}^t \binom{2(n-k)}{n-k}^u, n \geq 1.$$

A very similar argument proves that the sequence $(C(n, r, s, t, u))_{n=1}^\infty$ is realizable, which generalizes both Theorem 13 and Theorem 15.

Proof of Theorem 18. Fix $r \in \mathbb{Z}^+$ and $s, t, u \in \mathbb{Z}_{\geq 0}$. As in Remark 33, applying Lemma 30, the combination of the proof of Theorem 15 and Lemma 30 with small modification implies the Dold condition for $(T(n, r, s, t, u) =: X(n))_{n=1}^{\infty}$. The only differences are that in many places k should be replaced by $2k$, so the formula (13) should have a correction term, i.e., (13) should be

$$v_p\left(\binom{np^m}{2k}\right) = v_p\left(\binom{np^{m-1}}{2k/p}\right) \geq m - u - \delta_{2,p},$$

and that N should be $\max(0, m - r(m - u - \delta_{2,p}))$. Since $m - r(m - u - 1) \leq 1 + u$ under the conditions $r \geq 1$ and $1 \leq u \leq m - 2$, we see that the proof is still valid.

For the sign condition, we first consider the easier case of central trinomial coefficients, i.e., the case when $(r, s, t, u) = (1, 0, 1, 0)$. We will use the observation of Puri stated as in Remark 29. Fix $n \in \mathbb{Z}^+$. For a polynomial $h(x)$ and $k \in \mathbb{Z}_{\geq 0}$, let $[x^k]h(x)$ denote the coefficient of x^k in $h(x)$. Then

$$T(2n) = [x^{2n}](x^2 + x + 1)^{2n} \geq ([x^n](x^2 + x + 1)^n)^2 = T(n)^2.$$

Clearly, the sequence of central trinomial coefficients is increasing. By Remark 29, it suffices to show that $T(n) \geq n$, for every $n \geq 1$. For $n \in \{1, 2, 3\}$, $T(n) \geq n$ trivially holds. When $n \geq 4$, we have

$$T(n) \geq 1 + \binom{n}{2\lfloor\frac{n}{2}\rfloor}\binom{2\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor} \geq 1 + 2\left\lfloor\frac{n}{2}\right\rfloor \geq n.$$

Therefore, the sequence $(T(n))_{n=1}^{\infty}$ satisfies the sign condition.

Now we consider the sign condition for the general case. We will prove that

$$X(n+1) \geq \frac{7}{5}X(n), n \geq 1, \tag{15}$$

from which the sign condition follows, applying Lemma 28.

Fix $n \in \mathbb{Z}^+$. When $n = 1$, we have $X(2) = 6^u + 3^s 2^{t+u} \geq 2 \times 2^u = 2X(1)$. When $n = 2$, we have

$$\begin{aligned} X(3) &= 20^u + 3^r 4^s 2^t 6^u \geq 2 \times 6^u - \delta_{u,0} + 3 \times 4^s 2^t 6^u \geq 2 \times 6^u - \delta_{u,0} + 1 + 2 \times 4^s 2^t 6^u \\ &\geq 2(6^u + 4^s 2^t 6^u) \geq 2(6^u + 3^s 2^{t+u}) = 2X(2). \end{aligned}$$

Similarly, it is easy to check that $X(4) \geq 2X(3)$ and $X(5) \geq 2X(4)$ hold. Now assume that $n \geq 5$. (Here $n \geq 5$ ensures that $n_1 - 1 \geq 1$ below.)

For integers $m, l \geq 0$, set

$$f_m(l) = \binom{m+l}{l}^s \binom{2l}{l}^t,$$

which is non-decreasing with respect to both m and l.

We give a lower bound of $X(n+1) - X(n) =: B$ as follows. As in the proofs of Theorems 13 and 15, from the identity $\binom{m+1}{l+1} = \binom{m}{l} + \binom{m}{l+1}$, we get $\binom{m+1}{l+1}^v \geq \binom{m}{l}^v + \binom{m}{l+1}^v$, where

$m \ge l \ge 0$ and $v \ge 1$ are integers. Set $n_0 = \lfloor \frac{n+1}{2} \rfloor \in [3, n-2]$. Since the central binomial coefficient $\binom{2m}{m}$ is strictly increasing and $\binom{n+1+k}{k} \ge \binom{n+k}{k}$, we have

$$\begin{aligned} B &= \binom{2(n+1)}{n+1}^u + \sum_{k=1}^{n_0} \binom{n+1}{2k}^r f_{n+1}(k) \binom{2(n+1-k)}{n+1-k}^u - X(n) \\ &\ge \binom{2n}{n}^u + \sum_{k=1}^{n_0} \left(\binom{n}{2k}^r + \binom{n}{2k-1}^r \right) f_n(k) \binom{2(n+1-k)}{n+1-k}^u - X(n) \\ &\ge \sum_{k=1}^{n_0} \binom{n}{2k-1}^r f_n(k) \binom{2(n+1-k)}{n+1-k}^u. \end{aligned} \tag{16}$$

Using (16), we will give a lower bound of $X(n+1) - 2X(n)$. Set $n_1 = \left\lfloor \frac{n+3}{4} \right\rfloor = \lceil \frac{n}{4} \rceil \in [2, n-3]$. When $1 \le k \le n_1$, we have

$$2k - 1 \le 2 \left\lceil \frac{n}{4} \right\rceil - 1 \le \frac{n+1}{2}.$$

When $n_1 + 1 \le k \le n_0$, we have

$$2k - 1 \ge 2 \left\lceil \frac{n}{4} \right\rceil + 1 \ge \frac{n}{2} + 1.$$

Set

$$\Delta' = \left(\binom{n}{2n_1 - 1}^r f_n(n_1) - \binom{n}{2n_1 - 2}^r f_n(n_1 - 1) \right) \binom{2(n+1-n_1)}{n+1-n_1}^u$$

and

$$\Delta = \binom{n}{2n_1 - 2}^r (f_n(n_1) - f_n(n_1 - 1)) \binom{2(n+1-n_1)}{n+1-n_1}^u.$$

Then, we have

$$\begin{aligned} B &\ge \sum_{k=1}^{n_0} \binom{n}{2k-1}^r f_n(k) \binom{2(n+1-k)}{n+1-k}^u \\ &= \sum_{k=1}^{n_1} \binom{n}{2k-1}^r f_n(k) \binom{2(n+1-k)}{n+1-k}^u + \sum_{k=n_1+1}^{n_0} \binom{n}{2k-1}^r f_n(k) \binom{2(n+1-k)}{n+1-k}^u \\ &\ge \left(\sum_{k=1}^{n_1} \binom{n}{2k-2}^r f_n(k-1) \binom{2(n+1-k)}{n+1-k}^u + \Delta' \right) + \sum_{k=n_1+1}^{n_0} \binom{n}{2k}^r f_n(k) \binom{2(n-k)}{n-k}^u \\ &= \sum_{l=0}^{n_1-1} \binom{n}{2l}^r f_n(l) \binom{2(n-l)}{n-l}^u + \Delta' + \sum_{k=n_1+1}^{n_0} \binom{n}{2k}^r f_n(k) \binom{2(n-k)}{n-k}^u \\ &= \sum_{l=0, l \ne n_1}^{n_0} \binom{n}{2l}^r f_n(l) \binom{2(n-l)}{n-l}^u + \Delta', \end{aligned} \tag{17}$$

where we use (16) in the first inequality. Thus, we have

$$X(n+1) - 2X(n) \geq \Delta' - \binom{n}{2n_1}^r f_n(n_1) \binom{2(n-n_1)}{n-n_1}^u. \tag{18}$$

Note that $\Delta' \geq \Delta$, since

$$\left|2n_1 - 1 - \frac{n}{2}\right| \leq \left|2n_1 - 2 - \frac{n}{2}\right|$$

and $f_n(n_1) \geq f_n(n_1 - 1)$. Consequently, the inequality (18) implies

$$X(n+1) - 2X(n) \geq \Delta - \binom{n}{2n_1}^r f_n(n_1) \binom{2(n-n_1)}{n-n_1}^u. \tag{19}$$

Write $n = 4m + q$ with $m \in \mathbb{Z}^+$ and $q \in \{0, 1, 2, 3\}$. We see that $n_1 = m + 1 - \delta_{0,q}$. The proof of (15) then proceeds by cases, according to the value of q.

If $q \in \{1, 2\}$, then

$$\left|\frac{n}{2} - 2n_1\right| = \left|\frac{q}{2} + 2\delta_{0,q} - 2\right| = 2 - \frac{q}{2} \geq \frac{q}{2} = \left|\frac{q}{2} + 2\delta_{0,q}\right| = \left|\frac{n}{2} - (2n_1 - 2)\right|,$$

so we have $\binom{n}{2n_1-2} \geq \binom{n}{2n_1}$. Consequently, we have

$$\begin{aligned} \frac{1}{2}X(n) &\leq \sum_{\substack{0 \leq l \leq n_0, \\ l \notin \{n_1-1, n_1\}}} \binom{n}{2l}^r f_n(l) \binom{2(n-l)}{n-l}^u + \frac{1}{2} \sum_{l \in \{n_1-1, n_1\}}^{n_0} \binom{n}{2l}^r f_n(l) \binom{2(n-l)}{n-l}^u \\ &\leq \sum_{\substack{0 \leq l \leq n_0, \\ l \notin \{n_1-1, n_1\}}} \binom{n}{2l}^r f_n(l) \binom{2(n-l)}{n-l}^u + \binom{n}{2n_1-2}^r f_n(n_1) \binom{2(n+1-n_1)}{n+1-n_1}^u \\ &= X(n) + \Delta - \binom{n}{2n_1}^r f_n(n_1) \binom{2(n-n_1)}{n-n_1}^u. \end{aligned} \tag{20}$$

By (19) and (20), we get that

$$X(n+1) \geq \frac{3}{2}X(n).$$

Now assume that $q = 3$. We have

$$\left|\frac{n}{2} - 2n_1\right| = \frac{1}{2} = \left|\frac{n}{2} - (2n_1 - 1)\right| < \frac{3}{2} = \left|\frac{n}{2} - (2n_1 - 2)\right|,$$

so we get

$$\binom{n}{2n_1 - 1} = \binom{n}{2n_1} > \binom{n}{2n_1 - 2}.$$

Consequently, we have

$$\begin{aligned}\frac{1}{2}X(n) &\le \sum_{\substack{0\le l\le n_0,\\ l\notin\{n_1-1,n_1\}}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u + \frac{1}{2}\sum_{l\in\{n_1-1,n_1\}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u \\ &\le \sum_{\substack{0\le l\le n_0,\\ l\notin\{n_1-1,n_1\}}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u + \binom{n}{2n_1-1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u \qquad (21) \\ &= X(n)+\Delta' - \binom{n}{2n_1}^r f_n(n_1)\binom{2(n-n_1)}{n-n_1}^u.\end{aligned}$$

By (18) and (21), we see that

$$X(n+1) \ge \frac{3}{2}X(n).$$

At last assume that $q=0$. Then we have $n=4m, n_0=2m$, and $n_1=m\ge 2$. Note that

$$\binom{n+1}{2n_1}^r = \binom{4m+1}{2m}^r = \left(\binom{4m}{2m}\frac{4m+1}{2m+1}\right)^r \ge \frac{9}{5}\binom{4m}{2m}^r = \frac{9}{5}\binom{n}{2n_1}^r,$$

since $m\ge 2$ and $r\ge 1$. Thus, setting

$$E = X(n+1) - \frac{9}{5}\binom{n}{2n_1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u,$$

we have

$$\begin{aligned}E &\ge X(n+1) - \binom{n+1}{2n_1}^r f_{n+1}(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u \\ &= \binom{2(n+1)}{n+1}^u + \sum_{1\le k\le n_0, k\ne n_1} \binom{n+1}{2k}^r f_{n+1}(k)\binom{2(n+1-k)}{n+1-k}^u \\ &\ge \binom{2n}{n}^u + \sum_{1\le k\le n_0, k\ne n_1} \left(\binom{n}{2k}^r + \binom{n}{2k-1}^r\right) f_n(k)\binom{2(n+1-k)}{n+1-k}^u \\ &\ge X(n) - \binom{n}{2n_1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u + \sum_{k=1, k\ne n_1}^{n_0} \binom{n}{2k-1}^r f_n(k)\binom{2(n+1-k)}{n+1-k}^u. \end{aligned} \qquad (22)$$

Recall that $B = X(n+1) - X(n)$. From (22), we see that

$$B \ge \frac{4}{5}\binom{n}{2n_1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u + \sum_{k=1, k\ne n_1}^{n_0} \binom{n}{2k-1}^r f_n(k)\binom{2(n+1-k)}{n+1-k}^u. \qquad (23)$$

We claim that

$$\sum_{k=1,k\neq n_1}^{n_0} \binom{n}{2k-1}^r f_n(k)\binom{2(n+1-k)}{n+1-k}^u \geq \sum_{\substack{0\leq l\leq n_0,\\ l\notin\{n_1-1,n_1\}}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u. \tag{24}$$

In fact, the inequality (24) follows from the inequalities

$$\sum_{k=1}^{n_1-1} \binom{n}{2k-1}^r f_n(k)\binom{2(n+1-k)}{n+1-k}^u \geq \sum_{k=1}^{n_1-1} \binom{n}{2k-2}^r f_n(k-1)\binom{2(n+1-k)}{n+1-k}^u$$

and

$$\sum_{k=n_1+1}^{n_0} \binom{n}{2k-1}^r f_n(k)\binom{2(n+1-k)}{n+1-k}^u \geq \sum_{k=n_1+1}^{n_0} \binom{n}{2k}^r f_n(k)\binom{2(n-k)}{n-k}^u.$$

Observe that the term

$$\binom{n}{2n_1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u = \binom{4m}{2m}^r f_{4m}(m)\binom{6m+2}{3m+1}^u$$

is not smaller than both

$$\binom{n}{2n_1}^r f_n(n_1)\binom{2(n-n_1)}{n-n_1}^u = \binom{4m}{2m}^r f_{4m}(m)\binom{6m}{3m}^u$$

and

$$\binom{n}{2n_1-2}^r f_n(n_1-1)\binom{2(n-(n_1-1))}{n-(n_1-1)}^u = \binom{4m}{2m-2}^r f_{4m}(m-1)\binom{6m+2}{3m+1}^u.$$

Hence, we have

$$\binom{n}{2n_1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u \geq \frac{1}{2}\sum_{l\in\{n_1-1,n_1\}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u. \tag{25}$$

Therefore, by (23), (24), and (25), we deduce that

$$\begin{aligned} B &\geq \frac{4}{5}\binom{n}{2n_1}^r f_n(n_1)\binom{2(n+1-n_1)}{n+1-n_1}^u + \sum_{\substack{0\leq l\leq n_0,\\ l\notin\{n_1-1,n_1\}}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u \\ &\geq \frac{2}{5}\sum_{l\in\{n_1-1,n_1\}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u + \frac{2}{5}\sum_{\substack{0\leq l\leq n_0,\\ l\notin\{n_1-1,n_1\}}} \binom{n}{2l}^r f_n(l)\binom{2(n-l)}{n-l}^u \qquad (26)\\ &= \frac{2}{5}X(n), \end{aligned}$$

which gives

$$X(n+1) \geq \frac{7}{5}X(n).$$

We have completed the proof of the claimed inequality (15) (and hence the theorem). □

Remark 34. For every $n, r_1, r_2, s, t, u \in \mathbb{Z}_{\geq 0}$ with $r_1 + r_2 \geq 1$, define

$$V(n, r_1, r_2, s, t, u) = \sum_{k=0}^{n} \binom{n}{k}^{r_1} \binom{n}{2k}^{r_2} \binom{n+k}{k}^{s} \binom{2k}{k}^{t} \binom{2(n-k)}{n-k}^{u}.$$

Here we follow the convention $0^0 = 1$. Note that $(V(n,1,1,0,0,0))_n$ is the sequence of *quadrinomial coefficients* A005725. When $r_1 r_2 = 0$, i.e., when $r_1 = 0$ or $r_2 = 0$, the sequence $(V(n, r_1, r_2, s, t, u))_{n=1}^{\infty}$ is realizable by Remark 33 and Theorem 18. Indeed, for all $r_1, r_2, s, t, u \in \mathbb{Z}_{\geq 0}$ with $r_1 + r_2 \geq 1$, an essentially same argument as the one given in the proof of Theorem 18 shows that the sequence $(V(n, r_1, r_2, s, t, u))_{n=1}^{\infty}$ is realizable. For shortening the paper, we omit the details.

Remark 35. Let r_1, r_2, s, t, u be non-negative integers with $r_1 + r_2 \geq 1$. Set $V(n) = V(n, r_1, r_2, s, t, u)$ for $n \geq 1$. Clearly, we have $V(1) > 0$. By checking the proof of Lemma 28, it is easy to see that $(\mu * V)(n) > 0$ for every $n \geq 1$. According to Remark 34, the sequence $(V(n))_n^{\infty}$ is realizable. Then $(\mu * V)(n) \geq n$ for all $n \geq 1$. Thus, for every map $T : X \to X$ realizing V and $n \in \mathbb{Z}^+$, there is at least one periodic orbit of T with exact period n in X.

Proof of Theorem 21.

(1) For an arbitrary prime number p, we have

$$C(p) \equiv (p+1)C(p) = \binom{2p}{p} = \frac{(p+1)\cdots(p+p-1)}{(p-1)!} \times 2 \equiv 2 \pmod{p}, \tag{27}$$

where the modulo is taken over the ring $\mathbb{Z}_p$. Thus, we have

$$(\mu * C)(p) = C(p) - C(1) = C(p) - 1 \equiv 1 \pmod{p}. \tag{28}$$

Assume that $(C(n))_{n=1}^{\infty}$ is almost realizable. By (28), we see that $p \mid \text{Fail}(C)$. However, the prime number p can be arbitrary large, contradicting the fact that $\mathbb{Z}^+ \ni \text{Fail}(C) < \infty$. Therefore, the sequence $(C(n))_{n=1}^{\infty}$ is not almost realizable.

(2) Let p be an arbitrary odd prime number. By using (27) and Lemma 25, we get

$$\begin{aligned}(\mu * M)(2p) &= M(2p) - M(p) - M(2) + M(1) \\ &= \sum_{k=0}^{p} \binom{2p}{2k} C(k) - \sum_{k=0}^{\frac{p-1}{2}} \binom{p}{2k} C(k) - 2 + 1 \\ &\equiv \binom{2p}{0} C(0) + \binom{2p}{2p} C(p) - \binom{p}{0} C(0) - 2 + 1 \\ &= C(p) - 1 \\ &\equiv 1 \pmod{p}.\end{aligned} \tag{29}$$

Assume that $(M(n))_{n=1}^{\infty}$ is almost realizable. By (29), we deduce that $p \mid \text{Fail}(M)$. Similar to (1), we see that $(M(n))_{n=1}^{\infty}$ is not almost realizable.

(3) Let p be an arbitrary odd prime number. For an integer $p/2 < k < p-1$, we have

$$C(k) = \frac{1}{k+1}\binom{2k}{k} \equiv 0 \pmod p$$

by Lemma 25. For an integer $0 < k < p/2$, we have $\binom{p+k}{2k} \equiv 0 \pmod p$ by Lemma 25. Note that

$$C(p-1) = \frac{1}{p}\binom{2p-2}{p-1} = \frac{(p+1)\cdots(2p-2)(2p-1)}{(p-1)!}\frac{1}{2p-1} \equiv -1 \pmod p.$$

Then with (27), we have

$$\begin{aligned}(\mu * S)(p) = S(p) - S(1) &= \sum_{k=0}^{p}\binom{p+k}{2k}C(k) - 2 \\ &\equiv \binom{p}{0}C(0) + \binom{2p-1}{2p-2}C(p-1) + \binom{2p}{2p}C(p) - 2 \\ &\equiv C(p) - C(p-1) - 1 \\ &\equiv 2 - (-1) - 1 = 2 \pmod p.\end{aligned}$$

Similar to (1), we see that $(S(n))_{n=1}^{\infty}$ is not almost realizable.

□

Remark 36. Although the sequences in Theorem 21 are not almost realizable, they all satisfy the sign condition. Clearly, we have $C(n+1) = \frac{2(2n+1)}{n+2}C(n) \geq 2C(n)$, for every $n \geq 1$. Hence, by Lemma 28, the sequence $(C(n))_{n=1}^{\infty}$ satisfies the sign condition. Aigner [1, Proposition 3] proved that the sequence $(M(n))_{n=0}^{\infty}$ is log-concave, so the sign condition follows from $M(2)/M(1) = 2$ and Lemma 28. Xia and Yao [38, Corollary 7] showed that the sequence $(s(n))_{n=0}^{\infty}$ of little Schröder numbers is strictly log-concave, so the sign condition for $(s(n))_{n=1}^{\infty}$ follows from $s(2)/s(1) = 3$ and Lemma 28. Clearly, the sequence $(S(n))_{n=1}^{\infty}$ of large Schröder numbers also satisfies the sign condition.

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