Journal of Integer Sequences, Vol. 27 (2024),
Article 24.5.8

Generalized Impartial Two-player Pebbling Games on K_3 and C_4

Kayla Barker
Mathematics Program
Stockton University
101 Vera King Ferris Dr
Galloway, NJ 08205
USA
kaylaabarker5@gmail.com

Mia DeStefano
Dept. of Mathematics and Statistics
Vassar College
124 Raymond Ave
Poughkeepsie, NY 12604
USA
mdestefano@vassar.edu

Eugene Fiorini
DIMACS
Rutgers University
96 Frelinghuysen Road
Piscataway, NJ 08854
USA
efiorini@dimacs.rutgers.edu

Michael Gohn
Dept. of Mathematics and Computer Sci.
DeSales University
2755 Station Avenue
Center Valley, PA 18034
USA
mg7785@desales.edu

Joe Miller
Department of Mathematics
Iowa State University
411 Morrill Rd
Ames, IA 50011
USA
jmiller0@iastate.edu

Jacob Roeder
Dept. of Mathematics and Physics
Trine University
1 University Ave
Angola, IN 46703
USA
jwroeder19@my.trine.edu

Tony W. H. Wong
Department of Mathematics
Kutztown University of Pennsylvania
15200 Kutztown Road
Kutztown, PA 19530
USA
wong@kutztown.edu

Abstract

In a variation on the pebbling game played on a simple graph, a $(k+1:k)$-pebbling move comprises removing $k+1$ pebbles from a vertex and adding k pebbles to an adjacent vertex. We consider an impartial two-player game, where the winner of the game is the last player to make an allowable $(k+1:k)$-pebbling move. In this paper, we characterize the winning positions when the $(k+1:k)$-pebbling game is played on the complete graph K_3 and when the $(2:1)$-pebbling game is played on the cycle C_4.

1 Introduction

Given a simple graph G on vertices $v_1, v_2, \ldots, v_n$, every vertex v_i is assigned with a number of pebbles, specified by a nonnegative integer a_i for each $1 \leq i \leq n$. Fiorini et al. [2] describe an *impartial two-player* $(2:1)$*-pebbling game*, denoted by $\Gamma_G^{(2:1)}(a_1, a_2, \ldots, a_n)$, where players A and B take turns (player A taking the first turn) to make a $(2:1)$*-pebbling move*, which consists of removing two pebbles from a vertex and adding one pebble to an adjacent vertex. The first player having no available pebbling move loses the game. In this paper, we generalize this concept and consider a *impartial two-player* $(k+1:k)$*-pebbling game* $\Gamma_G^{(k+1:k)}(a_1, a_2, \ldots, a_n)$ for some positive integer k, where a $(k+1:k)$-pebbling move consists of removing $k+1$ pebbles from a vertex and adding k pebbles to an adjacent vertex.

In general, a two-player impartial game refers to a game with perfect information, no probabilistic moves, and finite number of moves before the game ends. Furthermore, at any point of the game, both players have exactly the same set of moves. Under the normal play condition where the first player without a legal move loses the game, the Sprague-Grundy theorem implies that in every impartial game, either the first player has a winning strategy, denoted as an N-game (N for the next player), or the second player has a winning strategy, denoted as a P-game (P for the previous player). It is clear that a game is an N-game if and only if there exists at least one available move that results in a P-game, whereas a game is a P-game if and only if there are no available moves, i.e., a *terminating game*, or every available move results in an N-game. Interested readers may refer to Berlekamp, Conway, and Guy [1] for more information on two-player impartial games.

Let $\mathcal{G}_{G,m}^{(k+1:k)}$ be the collection of all impartial two-player $(k+1:k)$-pebbling games with m initial pebbles in total on the underlying graph G. Due to the observation given in the previous paragraph, $\mathcal{G}_{G,m}^{(k+1:k)}$ can be partitioned into $\mathcal{P}_{G,m}^{(k+1:k)}$ and $\mathcal{N}_{G,m}^{(k+1:k)}$, the sets of all P-games and N-games in $\mathcal{G}_{G,m}^{(k+1:k)}$, respectively. Furthermore, if there exists a positive integer m such that $\mathcal{G}_{G,m}^{(k+1:k)} = \mathcal{N}_{G,m}^{(k+1:k)}$, then $\mathcal{G}_{G,m+2r}^{(k+1:k)} = \mathcal{N}_{G,m+2r}^{(k+1:k)}$ and $\mathcal{G}_{G,m+2r+1}^{(k+1:k)} = \mathcal{P}_{G,m+2r+1}^{(k+1:k)}$ for all nonnegative integers r. When such an m exists, this suggests there is a minimum m value where $\mathcal{G}_{G,m}^{(k+1:k)} = \mathcal{N}_{G,m}^{(k+1:k)}$, which we denote by $m^{(k+1:k)}(G)$. Fiorini et al. [2] established the following result when $k = 1$ and $G = K_n$.

Theorem 1.

(a) *If m is even, then* $\mathcal{N}^{(2:1)}_{K_2,m} = \{\Gamma^{(2:1)}_{K_2}(a_1, a_2) : a_1 + a_2 = m$ *and* $a_1 \not\equiv a_2 \pmod 3\}$*; if m is odd, then* $\mathcal{N}^{(2:1)}_{K_2,m} = \{\Gamma^{(2:1)}_{K_2}(a_1, a_2) : a_1 + a_2 = m$ *and* $a_1 \equiv a_2 \pmod 3\}$*. Hence, $m^{(2:1)}(K_2)$ does not exist.*

(b) $m^{(2:1)}(K_3) = 7$.

(c) $m^{(2:1)}(K_4) = 23$.

(d) *For all odd integers $n \geq 5$, we have* $m^{(2:1)}(K_n) = n + 2$.

(e) *For all even integers $n \geq 6$, we have* $m^{(2:1)}(K_n) = n + 7$.

In this paper, we expand our investigation on $m^{(k+1:k)}(K_n)$ when $k > 1$. Through computation using Mathematica (see Appendix A for the code), we obtain the values of $m^{(k+1:k)}(K_n)$ for some small k and n, excerpted in the table below.

n \ k	1	2	3	4	5	6
3	7	13	19	25	31	37
4	23	21		35		49
5	7	15	21	27	33	39
6	13	21	35	37	59	53
7	9	17	25	33	41	51
8	15	25	41	45	61	65
9	11	21	31	41	51	61
10	17	29	45	53	71	77

Table 1: Values of $m^{(k+1:k)}(K_n)$ for $1 \leq k \leq 6$ and $3 \leq n \leq 10$.

The first column of Table 1 corresponds to the result given in Theorem 1. This sequence appears in the On-Line Encyclopedia of Integer Sequences (OEIS) as A340631 [5]. On the other hand, the sequence given by the first row appears in the OEIS as A016921 [5]. This sequence is a result of the following theorem, which will be proved in Section 2.

Theorem 2. *For all positive integers k, we have* $m^{(k+1:k)}(K_3) = 6k + 1$.

The empty cells in the row corresponding to $n = 4$ are due to the following conjecture, suggested by our computational data.

Conjecture 3. For all odd integers $k \geq 3$, the value $m^{(k+1:k)}(K_4)$ does not exist.

Since $m^{(k+1:k)}(K_4)$ is conjectured to be nonexistent for odd $k \geq 3$, only the data corresponding to $n \geq 5$ in Table 1 appears in the OEIS (see A346197 and A347637 [5]).

We also study $m^{(k+1:k)}(G)$ when G is not a complete graph. In particular, we prove in Section 3 that $m^{(2:1)}(C_4)$ does not exist. In fact, we fully determine how $\mathcal{G}^{(2:1)}_{C_4,m}$ is partitioned into $\mathcal{P}^{(2:1)}_{C_4,m}$ and $\mathcal{N}^{(2:1)}_{C_4,m}$ for all positive integers m.

Before we proceed, we give several definitions that are useful for our discussions. After a player makes a $(k+1:k)$-pebbling move from vertex v_i to vertex v_j, if their opponent immediately makes a move from v_j to v_i, then we call this move an *echo pebbling move*. Next, for all $\gamma = \Gamma^{(k+1:k)}_G(a_1, a_2, \ldots, a_n)$, we let $|\gamma| = a_1 + a_2 + \cdots + a_n$ denote the total number of pebbles in this game. Furthermore, the game γ_{i+1} is called a *successor* of $\gamma_i = \Gamma^{(k+1:k)}_G(a_1, a_2, \ldots, a_n)$ if γ_{i+1} is a resultant game by performing one $(k+1:k)$-pebbling move on γ_i, and γ_i is called a *predecessor* of γ_{i+1}.

2 The investigation on $m^{(k+1:k)}(K_3)$

In this section, since the focus is $G = K_3$ with $(k+1:k)$-pebbling moves, we will abbreviate the notation $\mathcal{G}^{(k+1:k)}_{K_3,m}$, $\mathcal{P}^{(k+1:k)}_{K_3,m}$, $\mathcal{N}^{(k+1:k)}_{K_3,m}$, and $\Gamma^{(k+1:k)}_{K_3}(a_1, a_2, a_3)$ as $\mathcal{G}_m$, $\mathcal{P}_m$, $\mathcal{N}_m$, and $\Gamma(a_1, a_2, a_3)$, respectively. Furthermore, due to the symmetry of K_3, the game $\Gamma(a_1, a_2, a_3)$ is equivalent to games given by every permutation of a_1, a_2, and a_3. We denote this equivalence class of games as $\Gamma\{a_1, a_2, a_3\}$. For convenience, we always assume $a_1 \geq a_2 \geq a_3$.

The technique we utilize to prove Theorem 2 is to show that $\mathcal{P}_{3,6k-1} = \Gamma\{4k+1, k-1, k-1\}$. We demonstrate this via a sequence of lemmas. The first three lemmas prove that every game in $\Gamma\{4k+1, k-1, k-1\}$ is a P-game.

Lemma 4. *Every game in $\Gamma\{2k+3, 0, 0\}$ is a P-game.*

Proof. Consider $\gamma_0 \in \Gamma\{2k+3, 0, 0\}$. Every successor γ_1 of γ_0 is in $\Gamma\{k+2, k, 0\}$, and player B can make one pebbling move on γ_1 to obtain $\gamma_2 \in \Gamma\{k, k, 1\}$. Clearly γ_2 is a P-game, so γ_0 is also a P-game. □

Lemma 5. *Let $0 \leq \ell \leq k-1$ and $\ell_1 \leq \ell_2$ be nonnegative integers such that $\ell_1 + \ell_2 = \ell$. Then every game in $\Gamma\{2k-1-\ell, k-\ell_1, k-\ell_2\}$ is a P-game.*

Proof. Consider $\gamma_0 \in \Gamma\{2k-1, k, k\}$. Every successor of γ_0 is in $\Gamma\{2k, k, k-2\}$, and the subsequent echo pebbling move by player B leads to $\gamma_2 \in \Gamma\{2k-2, k, k-1\}$. If player B repeats the strategy of echoing player A's every move, then for every $0 \leq \ell \leq k-1$, we have $\gamma_{2\ell} \in \Gamma\{2k-1-\ell, k-\ell_1, k-\ell_2\}$, where $\ell_1 \leq \ell_2$ and $\ell_1 + \ell_2 = \ell$. Note that all vertices in $\gamma_{2(k-1)}$ have at most k pebbles, which implies that $\gamma_{2(k-1)}$ is a P-game. Since player B has a strategy to bring every $\gamma_{2\ell} \in \Gamma\{2k-1-\ell, k-\ell_1, k-\ell_2\}$ to a P-game, the game $\gamma_{2\ell}$ is also a P-game. □

Lemma 6. *Every game in $\Gamma\{4k+1, k-1, k-1\}$ is a P-game.*

Proof. Consider $\gamma_0 \in \Gamma\{4k+1, k-1, k-1\}$. Regardless of the moves by player A, player B is going to perform the echo pebbling move every time when possible. The only scenario where player B does not have an echo pebbling move is when player B obtains a descendant $\gamma_{2k-2+2\ell} \in \Gamma\{3k+2-\ell, k-1-\ell, 0\}$ for some $0 \le \ell \le k-1$ and player A makes one pebbling move to obtain $\gamma_{2k-1+2\ell} \in \Gamma\{2k+1-\ell, k, k-1-\ell\}$. In this situation, player B can make one pebbling move to obtain $\gamma_{2k+2\ell} \in \Gamma\{2k-1-\ell, k, k-\ell\}$, which is a P-game by Lemma 5. $\square$

The next two lemmas together show that $\gamma \in \mathcal{G}_m$ is a P game if $m \le 6k$ is even and no vertex contains more than $m/2$ pebbles.

Lemma 7. *Consider $\gamma_{2\ell} \in \Gamma\{a_1, a_2, a_3\}$, where $|\gamma_{2\ell}| = a_1 + a_2 + a_3 = 6k - 2\ell$ for some $0 \le \ell \le 2k-1$ and $k+1 \le a_1 \le 3k-\ell$. Then regardless of the move by player A, player B can obtain $\gamma_{2\ell+2} \in \Gamma\{a_1'', a_2'', a_3''\}$, where $a_1'' \le 3k-\ell-1$.*

Proof. If $a_3 = 0$, then $\gamma_{2\ell} \in \Gamma\{3k-\ell, 3k-\ell, 0\}$. In this case, the game $\gamma_{2\ell+1}$ is in $\Gamma\{4k-\ell, 2k-\ell-1, 0\}$ or

$$\begin{cases} \Gamma\{3k-\ell, 2k-\ell-1, k\}, & \text{if } 0 \le \ell \le k-1; \\ \Gamma\{3k-\ell, k, 2k-\ell-1\}, & \text{if } k \le \ell \le 2k-1, \end{cases}$$

and player B can make one pebbling move to obtain $\gamma_{2\ell+2}$ in

$$\begin{cases} \Gamma\{3k-\ell-1, 2k-\ell-1, k\}, & \text{if } 0 \le \ell \le k-1; \\ \Gamma\{3k-\ell-1, k, 2k-\ell-1\}, & \text{if } k \le \ell \le 2k-1, \end{cases}$$

thus establishing our assertion.

Now suppose $a_3 \ge 1$. If $a_1 \le 3k-\ell-1$, then regardless of the move by player A, player B is going to perform an echo pebbling move to obtain $\gamma_{2\ell+2} \in \Gamma\{a_1'', a_2'', a_3''\}$. Note that the resultant number of pebbles at every vertex does not increase after these two moves. Hence, we have $a_1'' \le a_1 \le 3k-\ell-1$.

It remains to consider the case where $a_1 = 3k-\ell$ and $a_2 \le 3k-\ell-1$. If player A makes one pebbling move that involves the vertex with $3k-\ell$ pebbles, then player B is going to perform an echo pebbling move to obtain $\gamma_{2\ell+2}$. Otherwise, player A obtains $\gamma_{2\ell+1} \in \Gamma\{3k-\ell, a_2', a_3'\}$ with $a_2' + a_3' = 3k-\ell-1$. Since one of the vertices with a_2 and a_3 pebbles receives k pebbles in this pebbling move by player A, we have $a_2' \ge k$. This implies that $a_3' \le 2k-\ell-1$. Hence, player B can make one pebbling move to obtain $\gamma_{2\ell+2} \in \Gamma\{a_1'', a_2'', a_3''\}$ such that $\{a_1'', a_2'', a_3''\} = \{2k-\ell-1, a_2', a_3'+k\}$. In both scenarios, the resultant $\gamma_{2\ell+2}$ satisfies the desired condition. $\square$

Lemma 8. *Consider $\gamma_{2\ell} \in \Gamma\{a_1, a_2, a_3\}$, where $|\gamma_{2\ell}| = 6k-2\ell$ for some $0 \le \ell \le 2k-1$ and $a_1 \le 3k-\ell$. Then $\gamma_{2\ell}$ is a P-game.*

Proof. Suppose player B inductively applies the strategy described in Lemma 7. Under this strategy, if player A obtains $\gamma_{2\ell+1}$ for some $0 \le \ell \le 2k-1$, then player B always has an available pebbling move to obtain $\gamma_{2\ell+2}$. Hence, player B can ensure that the game either terminates at $\gamma_{2\ell}$ for some $0 \le \ell \le 2k-1$ or does not terminate at γ_{4k-2}. If γ_{4k-2} is not a terminating game, then according to Lemma 7, player B can obtain $\gamma_{4k} \in \Gamma\{a_1^*, a_2^*, a_3^*\}$ with $a_1^* \le 3k-(2k-1)-1 = k$. This implies that γ_{4k} is a terminating game. Therefore, player B has a winning strategy, meaning that $\gamma_{2\ell}$ is a P-game. $\square$

The remaining preliminary results before the proof of Theorem 2 show that every $\gamma \in \mathcal{G}_{6k-1} \setminus \Gamma\{4k+1, k-1, k-1\}$ is an N-game.

Lemma 9. *Consider $\gamma_{2\ell-1} \in \Gamma\{a_1, a_2, a_3\}$, where $|\gamma_{2\ell-1}| = 6k-2\ell+1$ for some $1 \le \ell \le 2k-1$, $a_1 \le 4k-\ell+1$, and $a_3 \le 2k-\ell$. Then $\gamma_{2\ell-1}$ is an N-game.*

Proof. Note that $a_2 \le \lfloor (6k-2\ell+1)/2 \rfloor = 3k-\ell$. If player A makes a pebbling move from the vertex with a_1 pebbles to the vertex with a_3 pebbles, then $\gamma_{2\ell}$ is in $\Gamma\{a_1', a_2', a_3'\}$ such that $\{a_1', a_2', a_3'\} = \{a_1-k-1, a_2, a_3+k\}$, where $a_1-k-1 \le 4k-\ell+1-(k+1) = 3k-\ell$ and $a_3+k \le 2k-\ell+k = 3k-\ell$. By Lemma 8, the game $\gamma_{2\ell}$ is a P-game, so $\gamma_{2\ell-1}$ is an N-game. $\square$

Corollary 10. *Consider $\gamma_1 \in \Gamma\{a_1, a_2, a_3\}$, where $|\gamma_1| = 6k-1$ and $a_1 \le 4k$. Then γ_1 is an N-game.*

Proof. Note that $a_3 \le \lfloor (6k-1)/3 \rfloor = 2k-1$. Hence, the conditions in Lemma 9 are satisfied with $\ell = 1$. $\square$

Lemma 11. *Consider $\gamma_{2\ell-1} \in \Gamma\{a_1, a_2, 0\}$, where $|\gamma_{2\ell-1}| = 6k-2\ell+1$ for some $1 \le \ell \le k-1$ and $a_2 \le k$. Then $\gamma_{2\ell-1}$ is an N-game.*

Proof. Player A can make a pebbling move to obtain $\gamma_{2\ell} \in \Gamma\{a_1-k-1, k, a_2\}$, and player B can only obtain $\gamma_{2\ell+1}$ in $\Gamma\{a_1-2k-2, 2k, a_2\}$ or $\Gamma\{a_1-2k-2, a_2+k, k\}$. Since

$$a_1 - 2k - 2 \le (6k-2\ell+1) - 2k - 2 = 4k - 2\ell - 1 \le 4k - (\ell+1) + 1$$

and $a_2 \le k \le 2k-(\ell+1)$, Lemma 9 applies and $\gamma_{2\ell+1}$ is an N-game, which implies our assertion. $\square$

Lemma 12. *Consider $\gamma_1 \in \Gamma\{a_1, a_2, a_3\}$, where $|\gamma_1| = 6k-1$ and $a_1 \ge 4k+1$. If $\gamma_1 \notin \Gamma\{4k+1, k-1, k-1\}$, then γ_1 is an N-game.*

Proof. Without loss of generality, let $\gamma_1 = \Gamma(a_1, a_2, a_3)$. Note that

$$a_3 \le ((6k-1)-(4k+1))/2 = k-1.$$

However, if $a_3 = k-1$, then $(a_1, a_2, a_3) = (4k+1, k-1, k-1)$, which is a contradiction. Hence, we have $a_3 \le k-2$.

Suppose that $a_2 \le k$. If $a_3 = 0$, then γ_1 is an N-game by Lemma 11, so we may assume that $a_3 \ge 1$. To start the game, player A can make a pebbling move to obtain

$\gamma_2 = \Gamma(a_1 - k - 1, a_2, a_3 + k)$. Then player B can only make a pebbling move from either v_1 or v_3. Let ℓ be the maximum nonnegative integer such that for each positive integer $\widetilde{\ell} \leq \ell$, player B makes a pebbling move from v_3 at the game $\gamma_{2\widetilde{\ell}}$ to obtain $\gamma_{2\widetilde{\ell}+1}$. Note that if $\ell = 0$, then player B must have made a pebbling move from v_1 at the game γ_2 to obtain γ_3. After each pebbling move by player B to obtain $\gamma_{2\widetilde{\ell}+1}$, player A can always perform an echo pebbling move to obtain $\gamma_{2\widetilde{\ell}+2}$ since $a_1 - k - 1 > a_2 \geq a_3$. With this strategy, player A can obtain $\gamma_{2\ell+2} = \Gamma(a_1 - k - 1 - \ell_1, a_2 - \ell_2, a_3 + k - \ell)$ for some nonnegative integers ℓ_1 and ℓ_2 such that $\ell_1 + \ell_2 = \ell$. Note that $a_3 + k - \ell \geq k$, so $\ell \leq a_3$. Now by the definition of ℓ, the next pebbling move by player B is from v_1, so $\gamma_{2\ell+3}$ is either $\Gamma(a_1 - 2k - 2 - \ell_1, a_2 + k - \ell_2, a_3 + k - \ell)$ or $\Gamma(a_1 - 2k - 2 - \ell_1, a_2 - \ell_2, a_3 + 2k - \ell)$. Since

- $a_1 - 2k - 2 - \ell_1 = (6k - 1 - a_2 - a_3) - 2k - 2 - \ell_1 \leq 4k - a_3 - 3 \leq 4k - \ell - 3 < 4k - (\ell + 2) + 1$,
- $a_3 + k - \ell \leq k - 2 + k - \ell = 2k - (\ell + 2)$, and
- $a_2 - \ell_2 = (6k - 1 - a_1 - a_3) - \ell_2 \leq (6k - 1 - (4k + 1) - \ell) - \ell_2 \leq 2k - (\ell + 2)$,

Lemma 9 implies that $\gamma_{2\ell+3} = \gamma_{2(\ell+2)-1}$ is an N-game. Therefore, player A has a winning strategy and γ_1 is an N-game.

Next, suppose that $a_2 \geq k + 1$. To start the game, player A can make a pebbling move to obtain $\gamma_2 = \Gamma(a_1 + k, a_2 - k - 1, a_3)$. Since $(a_2 - k - 1) + a_3 \leq (6k - 1) - (4k + 1) - k - 1 = k - 3$, player B can only make a pebbling move from v_1 to obtain γ_3. From here, we can inductively define $\gamma_{2\widetilde{t}+1}$ for each positive integer $\widetilde{t}$ as follows. Let $\gamma_i = \Gamma(a_1^{(i)}, a_2^{(i)}, a_3^{(i)})$ for positive integers i. If either $a_2^{(2\widetilde{t}+1)}$ or $a_3^{(2\widetilde{t}+1)}$ is at least $k + 1$, then player A can perform an echo pebbling move to obtain $\gamma_{2\widetilde{t}+2}$. After that, player B can only make a pebbling move from v_1 to obtain $\gamma_{2\widetilde{t}+3}$. Let t be the smallest positive integer such that both $a_2^{(2t+1)}$ and $a_3^{(2t+1)}$ are at most k. Let $i \in \{2, 3\}$ be such that $a_i^{(2t+1)} \leq a_{5-i}^{(2t+1)}$. Note that $a_{5-i}^{(2t+1)} = k$ and $a_i^{(2t+1)} \leq (6k - 1) - (4k + 1) - k - t = k - 2 - t$, thus $t \leq k - 2$. At this stage, player A can make a pebbling move from v_1 to v_i to obtain γ_{2t+2}, where $a_1^{(2t+2)} = a_1 - t - (k + 1)$, $a_i^{(2t+2)} \leq 2k - 2 - t$, and $a_{5-i}^{(2t+2)} = k$.

Now we can inductively define $\gamma_{2t+2\widetilde{\tau}}$ for each positive integer $\widetilde{\tau}$ as follows. At the game $\gamma_{2t+2\widetilde{\tau}}$, if player B makes a pebbling move from v_i to obtain $\gamma_{2t+2\widetilde{\tau}+1}$, then player A can always perform an echo pebbling move to obtain $\gamma_{2t+2\widetilde{\tau}+2}$ since $a_1 - t - (k + 1) > k > k - 2 - t$. Let τ be the smallest positive integer such that at $\gamma_{2t+2\tau}$, player B makes a pebbling move from v_1 to obtain $\gamma_{2t+2\tau+1}$. Note that $a_i^{(2t+2\tau-1)} = a_i^{(2t+1)} - (\tau - 1) \leq k - 2 - t - (\tau - 1)$, and thus $t + \tau + 1 \leq k$. Then

$$
\begin{aligned}
a_1^{(2t+2\tau+1)} &\leq a_1^{(2t+2)} - (k + 1) \\
&= (6k - 1) - a_2 - a_3 - t - 2(k + 1) \\
&\leq (6k - 1) - (k + 1) - t - 2(k + 1) \\
&= 3k - 4 - t < 4k - (t + \tau + 1) - 1,
\end{aligned}
$$

and

$$\min\left(a_2^{(2t+2\tau+1)}, a_3^{(2t+2\tau+1)}\right) \leq \max(2k-1-t-\tau, k) = 2k-(t+\tau+1).$$

As a result, $\gamma_{2t+2\tau+1}$ is an N-game by Lemma 9. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Note that $\mathcal{P}_{6k-1} = \Gamma\{4k+1, k-1, k-1\}$ by Lemma 6, Corollary 10, and Lemma 12. Hence, $\mathcal{N}_{6k} = \Gamma\{3k+1, 2k, k-1\}$ since they are the only predecessors of the games in $\Gamma\{4k+1, k-1, k-1\}$. The predecessors of the games in $\Gamma\{3k+1, 2k, k-1\}$ are given by

$$\Gamma\{4k+2, k, k-1\} \cup \Gamma\{3k+1, 2k+1, k-1\} \cup \Gamma\{2k+1, 2k, 2k\} \cup \Gamma\{3k+1, 2k, k\}. \quad (1)$$

The set of games in (1) contains the only candidates for inclusion in $\mathcal{P}_{6k+1}$. However,

- the games in $\Gamma\{4k+2, k, k-1\} \cup \Gamma\{3k+1, 2k+1, k-1\}$ are predecessors of the games in $\Gamma\{3k+1, 2k-1, k\}$, which are P-games since they are not in $\mathcal{N}_{6k}$;
- the games in $\Gamma\{2k+1, 2k, 2k\} \cup \Gamma\{3k+1, 2k, k\}$ are predecessors of the games in $\Gamma\{3k, 2k, k\}$, which are also P-games.

Therefore, all the games in (1) are not in $\mathcal{P}_{6k+1}$. This implies that $\mathcal{P}_{6k+1} = \emptyset$, and $m = 6k+1$ is the minimum integer such that $\mathcal{G}_m = \mathcal{N}_m$. $\square$

3 Classifying the $(2:1)$-pebbling games on C_4

The focus of this section is $G = C_4$ with $(2:1)$-pebbling moves, so we will abbreviate the notation $\mathcal{G}^{(2:1)}_{C_4,m}$, $\mathcal{P}^{(2:1)}_{C_4,m}$, $\mathcal{N}^{(2:1)}_{C_4,m}$, and $\Gamma^{(2:1)}_{C_4}(a_0, a_1, a_2, a_3)$ as $\mathcal{G}_m$, $\mathcal{P}_m$, $\mathcal{N}_m$, and $\Gamma(a_0, a_1, a_2, a_3)$, respectively. We will fully determine the sets $\mathcal{P}_m$ and $\mathcal{N}_m$, thus proving that $m^{(2:1)}(C_4)$ does not exist.

Let the vertices of C_4 be v_0, v_1, v_2, v_3, where v_i is adjacent to v_{i+1} for all $i \in \{0,1,2,3\}$. Note that in this section, the addition in the indices is always performed modulo 4. Additionally, we define some specialty moves to facilitate the discussions. Suppose that one player makes a pebbling move from v_j to $v_{j+\epsilon}$ for some $j \in \{0,1,2,3\}$ and $\epsilon \in \{-1,1\}$. Then several response moves by their opponent are as follows. A *reverse rotational move* is a pebbling move by the opponent from v_j to $v_{j-\epsilon}$ (Figure 1), a *rotational symmetric move* is from v_{j+2} to $v_{j-\epsilon}$ (Figure 2), and a *reflectional move* is from $v_{j-\epsilon}$ to v_{j+2} (Figure 3).

Next, we define five subsets of $\bigcup_{m\in\mathbb{Z}^+} \mathcal{G}_m$ based on the number pebbles modulo 3 on each vertex.

- $X_{aaab} = \{\Gamma(a_0, a_1, a_2, a_3) : a_i \equiv a_{i+1} \equiv a_{i+2} \not\equiv a_{i-1} \pmod 3 \text{ for some } i \in \{0,1,2,3\}\}$,
- $X_{abab} = \{\Gamma(a_0, a_1, a_2, a_3) : (a_0 \equiv a_2) \not\equiv (a_1 \equiv a_3) \pmod 3\}$,

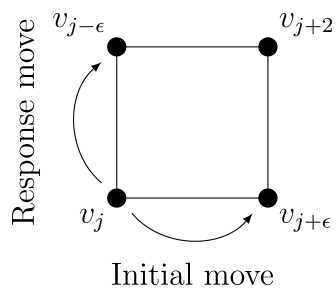

Figure 1: Reverse rotational move.

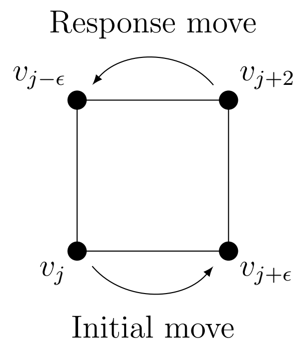

Figure 2: Rotational symmetric move.

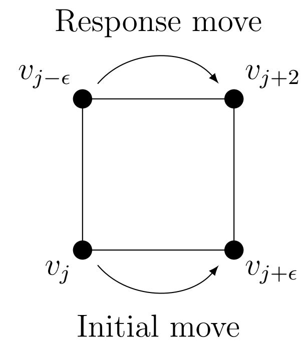

Figure 3: Reflectional move.

- $X_{aabc} = \{\Gamma(a_0, a_1, a_2, a_3) : a_i \equiv a_{i+1} \pmod 3$ and a_i, a_{i+2}, a_{i-1} are distinct modulo 3 for some $i \in \{0, 1, 2, 3\}\}$,
- $X_{abac} = \{\Gamma(a_0, a_1, a_2, a_3) : a_i \equiv a_{i+2} \pmod 3$ and a_i, a_{i+1}, a_{i-1} are distinct modulo 3 for some $i \in \{0, 1\}\}$, and
- $\hat{X}_{aabb} = \{\Gamma(a_0, a_1, a_2, a_3) : a_i \equiv a_{i+1} \pmod 3$ and $a_{i+2} \equiv a_{i-1} \pmod 3$ for some $i \in \{0, 1\}\}$. The hat notation indicates $\hat{X}_{aabb}$ includes games with $a_i \equiv a_{i+2} \pmod 3$ and those with $a_i \not\equiv a_{i+2} \pmod 3$.

We further define four subsets of $\bigcup_{m \in \mathbb{Z}^+} \mathcal{G}_m$ based on the number pebbles modulo 4 on each vertex.

- $Y_{aabc} = \{\Gamma(a_0, a_1, a_2, a_3) : a_{i+2} \equiv a_i + 2 \pmod 4$ and $a_{i+1} \equiv a_{i-1} + 1 \pmod 4$ for some $i \in \{0, 1, 2, 3\}\}$.
- $\hat{Y}_{abac} = \{\Gamma(a_0, a_1, a_2, a_3) : a_i \equiv a_{i+2} \pmod 4$ and $a_{i+1} \equiv a_{i-1} + 1 \pmod 4$ for some $i \in \{0, 1, 2, 3\}\}$. The hat indicates $\hat{Y}_{abac}$ includes games with $a_i \equiv a_{i+1} \pmod 4$, those with $a_i \equiv a_{i-1} \pmod 4$, and those with distinct a_i, a_{i+1}, a_{i-1} modulo 4.
- $\hat{Y}_{abab} = \{\Gamma(a_0, a_1, a_2, a_3) : a_0 \equiv a_2 \pmod 4$ and $a_1 \equiv a_3 \pmod 4\}$. The hat indicates $\hat{Y}_{abab}$ includes games with $a_0 \equiv a_1 \pmod 4$ and those with $a_0 \not\equiv a_1 \pmod 4$.
- $(\hat{Y}_{abab})^c = \{\Gamma(a_0, a_1, a_2, a_3) : a_0 + a_1 + a_2 + a_3$ is even and $\Gamma(a_0, a_1, a_2, a_3) \notin \hat{Y}_{abab}\}$.

Remark 13. Here are some quick observations on the sets defined above.

(*a*) The sets $X_{aaab}, X_{abab}, X_{aabc}, X_{abac}, \hat{X}_{aabb}$ form a partition of $\bigcup_{m \in \mathbb{Z}^+} \mathcal{G}_m$.

(*b*) The sets Y_{aabc} and $\hat{Y}_{abac}$ form a partition of $\bigcup_{m \in \mathbb{Z}^+} \mathcal{G}_{2m-1}$, while $\hat{Y}_{abab}$ and $(\hat{Y}_{abab})^c$ form a partition of $\bigcup_{m \in \mathbb{Z}^+} \mathcal{G}_{2m}$.

The next two lemmas together show that $\hat{X}_{aabb} \cup X_{abac}$ and $X_{aaab} \cup X_{abab} \cup X_{aabc}$ are closed under pebbling moves.

Lemma 14. *Each of X_{aaab}, X_{abab}, X_{aabc}, X_{abac}, and $\hat{X}_{aabb}$ is invariant after a pebbling move by one player followed by a rotational symmetric or reflectional move by their opponent.*

Proof. Suppose one player makes a pebbling move on $\Gamma(a_0, a_1, a_2, a_3)$ to obtain $a_j - 2$ and $a_{j+\epsilon}+1$ pebbles on vertices v_j and $v_{j+\epsilon}$, respectively, for some $j \in \{0, 1, 2, 3\}$ and $\epsilon \in \{-1, 1\}$. A subsequent rotational symmetric move by their opponent results in $a_{j+2} - 2$ and $a_{j-\epsilon} + 1$ pebbles on vertices v_{j+2} and $v_{j-\epsilon}$, respectively, and a reflectional move results in $a_{j+2}+1$ and $a_{j-\epsilon} - 2$ pebbles on vertices v_{j+2} and $v_{j-\epsilon}$, respectively. After either move by the opponent, the number of pebbles on v_j is congruent to $a_j + 1$ modulo 3 for all $0 \le j \le 3$. Therefore, each of X_{aaab}, X_{abab}, X_{aabc}, X_{abac}, and $\hat{X}_{aabb}$ is invariant after the specified response move by the opponent. $\square$

Lemma 15. *Both sets $\hat{X}_{aabb} \cup X_{abac}$ and $X_{aaab} \cup X_{abab} \cup X_{aabc}$ are closed under pebbling moves.*

Proof. We first show that $\hat{X}_{aabb} \cup X_{abac}$ is closed under pebbling moves. Consider a pebbling game $\Gamma(a_0, a_1, a_2, a_3) \in \hat{X}_{aabb} \cup X_{abac}$ with one pebbling move from v_j to $v_{j+\epsilon}$ for some $j \in \{0, 1, 2, 3\}$ and $\epsilon \in \{-1, 1\}$, and let the resultant pebbling game be $\Gamma(a'_0, a'_1, a'_2, a'_3)$. Suppose that $\Gamma(a_0, a_1, a_2, a_3) \in \hat{X}_{aabb}$ with $a_i \equiv a_{i+1} \pmod 3$ and $a_{i+2} \equiv a_{i-1} \pmod 3$ for some $i \in \{0, 1\}$. If $a_j \equiv a_{j+\epsilon} \pmod 3$, then $a'_j \equiv a_j + 1 \equiv a_{j+\epsilon} + 1 \equiv a'_{j+\epsilon} \pmod 3$ and $a'_{j+2} \equiv a_{j+2} \equiv a_{j-\epsilon} \equiv a'_{j-\epsilon} \pmod 3$, thus $\Gamma(a'_0, a'_1, a'_2, a'_3) \in \hat{X}_{aabb}$. If $a_j \not\equiv a_{j+\epsilon} \pmod 3$, then Table 2 gives the remaining possible outcomes of $\Gamma(a'_0, a'_1, a'_2, a'_3)$ after one pebbling move.

$\{a_i, a_{i+2}\} \pmod 3$	$(a_j, a_{j+\epsilon}) \pmod 3$	$(a'_j, a'_{j+\epsilon}, a'_{j+2}, a'_{j-\epsilon}) \pmod 3$	$\Gamma(a'_0, a'_1, a'_2, a'_3)$
$\{0, 1\}$	$(0, 1)$	$(1, 2, 1, 0)$	X_{abac}
	$(1, 0)$	$(2, 1, 0, 1)$	
$\{0, 2\}$	$(0, 2)$	$(1, 0, 2, 0)$	
	$(2, 0)$	$(0, 1, 0, 2)$	
$\{1, 2\}$	$(1, 2)$	$(2, 0, 2, 1)$	
	$(2, 1)$	$(0, 2, 1, 2)$	

Table 2: Resultant games after one pebbling move from $\hat{X}_{aabb}$ if $a_j \not\equiv a_{j+\epsilon} \pmod 3$.

Next, suppose that $\Gamma(a_0, a_1, a_2, a_3) \in X_{abac}$ with $a_i \equiv a_{i+2} \pmod 3$ and a_i, a_{i+1}, a_{i-1} being distinct modulo 3 for some $i \in \{0, 1\}$. Table 3 gives all possible outcomes of $\Gamma(a'_0, a'_1, a'_2, a'_3)$ after one pebbling move. By exhausting all possible resultant games $\Gamma(a'_0, a'_1, a'_2, a'_3)$, we see that $\hat{X}_{aabb} \cup X_{abac}$ is closed under pebbling moves.

$a_i \pmod 3$	$(a_j, a_{j+\epsilon}) \pmod 3$	$(a'_j, a'_{j+\epsilon}, a'_{j+2}, a'_{j-\epsilon}) \pmod 3$	$\Gamma(a'_0, a'_1, a'_2, a'_3)$
0	$(0,1)$ $(1,0)$	$(1,2,0,2)$ $(2,1,2,0)$	X_{abac}
	$(0,2)$ $(2,0)$	$(1,0,0,1)$ $(0,1,1,0)$	$\hat{X}_{aabb}$
1	$(1,0)$ $(0,1)$	$(2,1,1,2)$ $(1,2,2,1)$	$\hat{X}_{aabb}$
	$(1,2)$ $(2,1)$	$(2,0,1,0)$ $(0,2,0,1)$	X_{abac}
2	$(2,0)$ $(0,2)$	$(0,1,2,1)$ $(1,0,1,2)$	X_{abac}
	$(2,1)$ $(1,2)$	$(0,2,2,0)$ $(2,0,0,2)$	$\hat{X}_{aabb}$

Table 3: Resultant games after one pebbling move from X_{abac}.

To show that $X_{aaab} \cup X_{abab} \cup X_{aabc}$ is closed under pebbling moves, we suppose by way of contradiction that there exists a pebbling game $\Gamma(a_0, a_1, a_2, a_3)$ in $X_{aaab} \cup X_{abab} \cup X_{aabc}$ and a pebbling move from v_j to $v_{j+\epsilon}$ for some $j \in \{0,1,2,3\}$ and $\epsilon \in \{-1,1\}$ such that the resultant game $\Gamma(a'_0, a'_1, a'_2, a'_3)$ is not in $X_{aaab} \cup X_{abab} \cup X_{aabc}$. By Remark 13(a), we have $\Gamma(a'_0, a'_1, a'_2, a'_3)$ in $\hat{X}_{aabb} \cup X_{abac}$. Now we consider the game $\Gamma(a_0+2, a_1+2, a_2+2, a_3+2)$ in $X_{aaab} \cup X_{abab} \cup X_{aabc}$. After the pebbling move from v_j to $v_{j+\epsilon}$, a rotational symmetric response move from v_{j+2} to $v_{j-\epsilon}$ yields a resultant game in $X_{aaab} \cup X_{abab} \cup X_{aabc}$ by Lemma 14. This contradicts that $\hat{X}_{aabb} \cup X_{abac}$ is closed under pebbling moves since $\Gamma(a'_0+2, a'_1+2, a'_2+2, a'_3+2)$ is in $\hat{X}_{aabb} \cup X_{abac}$. □

We now begin classifying all $(2:1)$-pebbling games on C_4 as P-games and N-games.

Theorem 16. *A game $\Gamma(a_0, a_1, a_2, a_3) \in \hat{X}_{aabb} \cup X_{abac}$ is a P-game if and only if $a_0 + a_1 + a_2 + a_3$ is even.*

Proof. If $a_0 + a_1 + a_2 + a_3 = 0$, then $\Gamma(0,0,0,0)$ is the only possibility and is a P-game. There is no game in $\hat{X}_{aabb} \cup X_{abac}$ with $a_0 + a_1 + a_2 + a_3 = 1$. If $a_0 + a_1 + a_2 + a_3 = 2$, then $(a_i, a_{i+1}, a_{i+2}, a_{i-1}) = (1,1,0,0)$ for some $i \in \{0,1,2,3\}$, and $\Gamma(a_0, a_1, a_2, a_3)$ is a P-game. If $a_0 + a_1 + a_2 + a_3 = 3$, then $(a_i, a_{i+1}, a_{i+2}, a_{i-1}) \in \{(3,0,0,0), (0,1,0,2)\}$ for some $i \in \{0,1,2,3\}$, and $\Gamma(a_0, a_1, a_2, a_3)$ is an N-game. If $a_0 + a_1 + a_2 + a_3 = 4$, then $(a_i, a_{i+1}, a_{i+2}, a_{i-1}) \in \{(2,2,0,0), (1,1,1,1), (1,0,1,2)\}$ for some $i \in \{0,1,2,3\}$, and it is not difficult to verify that $\Gamma(a_0, a_1, a_2, a_3)$ is a P-game. Now assume that for some integer $k \geq 4$, $\Gamma(a_0, a_1, a_2, a_3)$ is a P-game if and only if $a_0+a_1+a_2+a_3 = k$ is even. If $a_0+a_1+a_2+a_3 = k+1$, then since $k+1 \geq 5$, there exists $j \in \{0,1,2,3\}$ such that $a_j \geq 2$. Hence, there is at least one pebbling move on $\Gamma(a_0, a_1, a_2, a_3)$, and the resultant pebbling game $\Gamma(a'_0, a'_1, a'_2, a'_3)$ is in $\hat{X}_{aabb} \cup X_{abac}$ by Lemma 15. By the induction hypothesis, $\Gamma(a'_0, a'_1, a'_2, a'_3)$ is a P-game if and only if $a'_0 + a'_1 + a'_2 + a'_3 = k$ is even, which shows that $\Gamma(a_0, a_1, a_2, a_3)$ is an N-game if and only if $k+1$ is odd. This completes our proof by induction. □

Theorem 17. *Every game* $\Gamma(a_0, a_1, a_2, a_3) \in \hat{Y}_{abab}$ *is a* P*-game.*

Proof. For $\Gamma(a_0, a_1, a_2, a_3) \in \hat{Y}_{abab}$, if there does not exist $j \in \{0, 1, 2, 3\}$ such that $a_j \geq 2$, then player A does not have a move and thus $\Gamma(a_0, a_1, a_2, a_3)$ is a P-game. Otherwise, assume that player A made a pebbling move from v_j to $v_{j+\epsilon}$ for some $j \in \{0, 1, 2, 3\}$ and $\epsilon \in \{-1, 1\}$, where $a_j \geq 2$. Since $a_{j+2} \equiv a_j \pmod 4$, we have $\max(a_j - 2, a_{j+2}) \geq 2$. As a result, if $a_j - 2 \geq 2$, then player B can apply a reverse rotational move from v_j to $v_{j-\epsilon}$; if $a_{j+2} \geq 2$, then player B can apply a rotational symmetric move from v_{j+2} to $v_{j-\epsilon}$. In both cases, player B has a strategy to ensure that the resultant game returns to $\hat{Y}_{abab}$. Therefore, $\Gamma(a_0, a_1, a_2, a_3)$ can be proved to be a P-game inductively. $\square$

Theorem 18. *Every game* $\Gamma(a_0, a_1, a_2, a_3) \in (X_{abab} \cup X_{aabc} \cup X_{aaab}) \cap \hat{Y}_{abac}$ *is a* P*-game.*

Proof. For $\Gamma(a_0, a_1, a_2, a_3) \in (X_{abab} \cup X_{aabc} \cup X_{aaab}) \cap \hat{Y}_{abac}$, if there does not exist $j \in \{0, 1, 2, 3\}$ such that $a_j \geq 2$, then player A does not have a move and thus $\Gamma(a_0, a_1, a_2, a_3)$ is a P-game. Otherwise, assume that player A made a pebbling move from v_j to $v_{j+\epsilon}$ for some $j \in \{0, 1, 2, 3\}$ and $\epsilon \in \{-1, 1\}$, where $a_j \geq 2$.

First, consider the case when $a_{j+2} \equiv a_j \pmod 4$. In this case, $\max(a_j - 2, a_{j+2}) \geq 2$. If $a_j - 2 \geq 2$, then player B can apply the strategy of a reverse rotational move from v_j to $v_{j-\epsilon}$; if $a_{j+2} \geq 2$, then player B can apply the strategy of a rotational symmetric move from v_{j+2} to $v_{j-\epsilon}$. In both cases, player B has a strategy to ensure that the resultant game returns to $\hat{Y}_{abac}$.

Next, consider the case when $a_{j+2} \equiv a_j \pm 1 \pmod 4$. If $\max(a_j - 2, a_{j+2}) \geq 2$, then player B has the same strategy as above to obtain a resultant game in $\hat{Y}_{abac}$. Otherwise, $(a_j, a_{j+2}) \in \{(2, 1), (3, 0)\}$ and $a_{j+1} \equiv a_{j-1} \pmod 4$. Note that $\Gamma(2, 0, 1, 0)$ and $\Gamma(3, 0, 0, 0)$ are not in $X_{abab} \cup X_{aabc} \cup X_{aaab}$, so $(a_{j+1}, a_{j-1}) \neq (0, 0)$. If $a_{j+\epsilon} \geq 1$, then when $(a_j, a_{j+2}) = (2, 1)$, player B can apply an echo pebbling move from $v_{j+\epsilon}$ to v_j, and when $(a_j, a_{j+2}) = (3, 0)$, player B can make a pebbling move from $v_{j+\epsilon}$ to v_{j+2}. Otherwise, if $a_{j+\epsilon} = 0$, then $a_{j-\epsilon} \geq 4$, so when $(a_j, a_{j+2}) = (2, 1)$, player B can make a pebbling move from $v_{j-\epsilon}$ to v_j, and when $(a_j, a_{j+2}) = (3, 0)$, player B can apply a reflectional move from $v_{j-\epsilon}$ to v_{j+2}. In all cases, player B has a strategy to ensure that the resultant game returns to $\hat{Y}_{abac}$. By Lemma 15, this resultant game is in $(X_{abab} \cup X_{aabc} \cup X_{aaab}) \cap \hat{Y}_{abac}$. Therefore, the game $\Gamma(a_0, a_1, a_2, a_3)$ can be proved to be a P-game inductively. $\square$

Theorems 16, 17, and 18 are summarized in Table 4. Recall from Remark 13(b) that $|\gamma| = a_0 + a_1 + a_2 + a_3$ is even if $\gamma \in \hat{Y}_{abab} \cup (\hat{Y}_{abab})^c$ and $|\gamma|$ is odd if $\gamma \in \hat{Y}_{abac} \cup Y_{aabc}$.

Theorem 19. *Every game* $\Gamma(a_0, a_1, a_2, a_3) \in Y_{aabc}$ *is an* N*-game.*

Proof. For $\Gamma(a_0, a_1, a_2, a_3) \in Y_{aabc}$, let $i \in \{0, 1, 2, 3\}$ and $\epsilon \in \{-1, 1\}$ such that

$$a_i \equiv a_{i+\epsilon} \pmod 4 \text{ and } a_{i+2} \equiv a_i + 2 \pmod 4.$$

	$\hat{X}_{aabb}$	X_{abac}	X_{abab}	X_{aabc}	X_{aaab}
$\hat{Y}_{abab}$	P	P	P	P	P
$(\hat{Y}_{abab})^c$	P	P			
$\hat{Y}_{abac}$	N	N	P	P	P
Y_{aabc}	N	N			

Table 4: Classification of C_4 $(2:1)$-pebbling games on C_4 due to Theorems 16, 17, and 18.

By the definition of Y_{aabc}, we have $a_{i-\epsilon} \equiv a_{i+\epsilon} \pm 1 \pmod 4$. Note that $\max(a_i, a_{i+2}) \geq 2$. If $a_{i-\epsilon} \equiv a_{i+\epsilon} + 1 \pmod 4$, then player A can make a pebbling move from v_i or v_{i+2} to $v_{i+\epsilon}$; otherwise, if $a_{i-\epsilon} \equiv a_{i+\epsilon} - 1 \pmod 4$, then player A can make a pebbling move from v_i or v_{i+2} to $v_{i-\epsilon}$. In both cases, player A has a strategy to obtain a resultant game in $\hat{Y}_{abab}$, which is a P-game by Theorem 17. Therefore, the game $\Gamma(a_0, a_1, a_2, a_3)$ is an N-game. □

Theorem 20. *Every game* $\Gamma(a_0, a_1, a_2, a_3) \in (X_{abab} \cup X_{aabc} \cup X_{aaab}) \cap (\hat{Y}_{abab})^c$ *is an* N*-game.*

Proof. Let $\Gamma(a_0, a_1, a_2, a_3) \in (X_{abab} \cup X_{aabc} \cup X_{aaab}) \cap (\hat{Y}_{abab})^c$. By the definition of $(\hat{Y}_{abab})^c$, there exists $i \in \{0, 1\}$ such that $a_i \not\equiv a_{i+2} \pmod 4$. If $a_{i+2} \equiv a_i + 2 \pmod 4$, then $\max(a_i, a_{i+2}) \geq 2$, and player A can make a pebbling move from v_i to $v_{i+\epsilon}$ or from v_{i+2} to $v_{i+\epsilon}$. Otherwise, we have $a_{i+2} \equiv a_i \pm 1 \pmod 4$. This implies that $a_{i-\epsilon} \equiv a_{i+\epsilon} + 1 \pmod 4$ for some $\epsilon \in \{-1, 1\}$ since $a_0 + a_1 + a_2 + a_3$ is even. If $\max(a_0, a_1, a_2, a_3) \leq 1$, then $(a_i, a_{i+\epsilon}, a_{i+2}, a_{i-\epsilon})$ is in $\{(0, 0, 1, 1), (1, 0, 0, 1)\}$, contradicting that $\Gamma(a_0, a_1, a_2, a_3) \in X_{abab} \cup X_{aabc} \cup X_{aaab}$. Hence, either $\max(a_i, a_{i+2}) \geq 2$ or $\max(a_{i+\epsilon}, a_{i-\epsilon}) \geq 2$. Without loss of generality, assume that $\max(a_i, a_{i+2}) \geq 2$. Then player A can make a pebbling move from v_i to $v_{i+\epsilon}$ or from v_{i+2} to $v_{i+\epsilon}$. In both cases, player A has a strategy to obtain a resultant game in $\hat{Y}_{abac}$. By Lemma 15, this resultant game is in $(X_{abab} \cup X_{aabc} \cup X_{aaab}) \cap \hat{Y}_{abac}$, which is a P-game by Theorem 18. Therefore, the game $\Gamma(a_0, a_1, a_2, a_3)$ is an N-game. □

Theorems 16 through 20 completely classify all $(2:1)$-pebbling games on C_4 into P-games and N-games. To conclude, we summarize the results in Table 5.

	$\hat{X}_{aabb}$	X_{abac}	X_{abab}	X_{aabc}	X_{aaab}
$\hat{Y}_{abab}$	P	P	P	P	P
$(\hat{Y}_{abab})^c$	P	P	N	N	N
$\hat{Y}_{abac}$	N	N	P	P	P
Y_{aabc}	N	N	N	N	N

Table 5: Complete classification of $(2:1)$-pebbling games on C_4.

4 Future directions

As mentioned in Section 1, among the computational results given in Table 1, the data in the first column and the first row are now fully justified by mathematical proofs. It should be noted that different strategies were used for these proofs. It will be interesting if one can develop a unified technique to prove the patterns in additional rows or columns in the table.

A Appendix: Computer code

```
(*n represents the number of vertices in the complete graph. Each
pebbling move removes k+1 pebbles from a vertex and adds k pebbles to
an adjacent vertex.*)

Do[(*Given n and m, list all possible assignments with m pebbles.*)
 alltuples[n_, m_] := IntegerPartitions[m + n, {n}] - 1;

 (*Given an assignment, list all resultant assignments after one
 pebbling move; only works for n>=3.*)
 pebblemoves[config_] :=
  Block[{n, temp}, n = Length[config];
   temp = Table[config, {i, n (n - 1)}] +
     Permutations[Join[{-(k + 1), k}, Table[0, {i, n - 2}]]];
   temp = Select[temp, Min[#] >= 0 &];
   temp = ReverseSort[DeleteDuplicates[ReverseSort /@ temp]]];

 (*Given n and m, list all assignments that are P-games.*)
 Plist = {};
 plist[n_, m_] :=
  Block[{index, tuples},
   While[Length[Plist] < n, index = Length[Plist];
    AppendTo[Plist, {{Join[{1}, Table[0, {i, index}]]}}]];
   Do[AppendTo[Plist[[n]], {}]; tuples = alltuples[n, i];
    Do[If[
      Not[IntersectingQ[pebblemoves[tuples[[j]]], Plist[[n, i - 1]]]],
      AppendTo[Plist[[n, i]], tuples[[j]]]], {j, Length[tuples]}],
    {i, Length[Plist[[n]]] + 1, m}];
   Plist[[n, m]]];

 (*Given n, print out the minimum m such that there are no P-games
 with m pebbles*)
 Do[m = 1; While[plist[n, m] != {}, m++];
```

```
Print["k=", k, " n=", n, " m=", m], {n, 5, 10}], {k, 1, 6}]
```

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays, Vol. 1*, CRC Press, 2001.

[2] E. Fiorini, M. Lind, A. Woldar, and T. W. H. Wong, Characterizing winning positions in the impartial two-player pebbling game on complete graphs, *J. Integer Sequences* **24** (2021), Article 21.6.4.

[3] M. Lind, E. Fiorini, and A. Woldar, On properties of pebble assignment graphs, *Graphs Combin.* **38** (2022), Article 45.

[4] G. Isaak and M. Prudente, Two-player pebbling on diameter 2 graphs, *Int. J. Game Theory* (2021), `https://doi.org/10.1007/s00182-021-00766-0`.

[5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`, 2024.

2020 *Mathematics Subject Classification*: Primary 91A46; Secondary 91A43, 91A05.
Keywords: pebbling game, impartial game, pebble assignment, N-game, P-game.

(Concerned with sequences A016921, A340631, A346197, A347637.)

Received June 4 2023; revised versions received June 9 2023; May 14 2024. Published in *Journal of Integer Sequences*, June 3 2024.

Return to Journal of Integer Sequences home page.