Journal of Integer Sequences, Vol. 27 (2024),
Article 24.8.5

On the Asymptotic Density of k-tuples of Positive Integers with Pairwise Non-Coprime Components

László Tóth
Department of Mathematics
University of Pécs
Ifjúság útja 6
7624 Pécs
Hungary
ltoth@gamma.ttk.pte.hu

Abstract

We use the convolution method for arithmetic functions of several variables to deduce an asymptotic formula for the number of k-tuples of positive integers with components which are pairwise non-coprime and $\leq x$. More generally, we obtain asymptotic formulas on the number of k-tuples $(n_1, \ldots, n_k) \in \mathbb{N}^k$ such that at least r pairs (n_i, n_j), respectively exactly r pairs are coprime. Our results answer the questions raised by Moree (2005, 2014), and generalize and refine related results obtained by Heyman (2014) and Hu (2014).

1 Introduction and motivation

Let $\mathbb{N} = \{1, 2, \ldots\}$ and let $k \in \mathbb{N}$, $k \geq 2$. It is well-known that the asymptotic density of the k-tuples $(n_1, \ldots, n_k) \in \mathbb{N}^k$ having relatively prime (coprime) components is $1/\zeta(k)$. This result goes back to the work of Cesàro, Dirichlet, Mertens and others. See, e.g., [4, 6, 12, 14, 15]. More exactly, one has the asymptotic estimate

$$\sum_{\substack{n_1,\ldots,n_k \leq x \\ \gcd(n_1,\ldots,n_k)=1}} 1 = \frac{x^k}{\zeta(k)} + \begin{cases} O(x \log x), & \text{if } k = 2; \\ O(x^{k-1}), & \text{if } k \geq 3. \end{cases} \tag{1}$$

The asymptotic density of the k-tuples $(n_1,\ldots,n_k)\in \mathbb{N}^k$ with pairwise coprime components is

$$A_k=\prod_p \left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right)=\prod_p \left(1+\sum_{j=2}^k (-1)^{j-1}(j-1)\binom{k}{j}\frac{1}{p^j}\right), \tag{2}$$

and we have the asymptotic formula

$$\sum_{\substack{n_1,\ldots,n_k\le x\\ \gcd(n_i,n_j)=1\\ 1\le i<j\le k}} 1 = A_k x^k + O(x^{k-1}(\log x)^{k-1}), \tag{3}$$

valid for every fixed $k\ge 2$, proved by the author [15] using an inductive process on k. The value of A_k was also deduced by Cai and Bach [3, Thm. 3.3] using probabilistic arguments. Formula (3) has been reproved by the author [17], in a more general form, namely by investigating m-wise relatively prime integers (that is, every m of them are relatively prime) and by using the convolution method for functions of several variables. Note that the asymptotic formula for m-wise coprime integers was first proved by Hu [8] by the inductive method with a weaker error term.

Now consider pairwise non-coprime positive integers $n_1,\ldots,n_k$, satisfying $\gcd(n_i,n_j)\ne 1$ for all $1\le i<j\le k$. Let $\beta=\beta_k:\mathbb{N}^k\to \{0,1\}$ denote the characteristic function of k-tuples having this property, that is,

$$\beta(n_1,\ldots,n_k)=\begin{cases} 1, & \text{if } n_1,\ldots,n_k \text{ are pairwise non-coprime;}\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

Moree [10, 11] and Heyman [7] raised the question of finding the asymptotic density C_k of k-tuples with pairwise non-coprime components. If $k=2$, then the answer is immediate by (1): $C_2=1-1/\zeta(2)$. Heyman [7] obtained the value C_3 and deduced an asymptotic formula for the sum $\sum_{n_1,n_2,n_3\le x}\beta(n_1,n_2,n_3)$ by using functions of one variable and the inclusion-exclusion principle. The method in [7] cannot be applied if $k\ge 4$. Using the inductive approach of the author [15] and the inclusion-exclusion principle, Hu [9] gave a formula for the asymptotic density C_k ($k\ge 3$), with an incomplete proof.

Moree [10, 11] also formulated as an open problem to compute the density of k-tuples $(n_1,\ldots,n_k)\in\mathbb{N}^k$ such that at least (respectively, exactly) r pairs (n_i,n_j) are coprime. A correct answer to this problem, but with some incomplete arguments has been given by Hu [9, Cor. 3]. In fact, Hu [9, Thm. 1] also deduced a related asymptotic formula with remainder term concerning certain arbitrary coprimality conditions. See Theorem 1. Arias de Reyna and Heyman [2] used a different method, based on certain properties of arithmetic functions of one variable, and improved the error term by Hu [9].

See Sections 2 and 4 for some more details on the above results.

In this paper we use a different approach to study these questions. Applying the convolution method for functions of several variables we first reprove Theorem 1. To do this we need a careful study of the Dirichlet series of the corresponding characteristic function. See Theorem 7. Our result concerning the related asymptotic formula, with the same error term as obtained in [2], and with new representations of the constant A_G is contained in Theorem 9. Then we deduce asymptotic formulas with remainder terms on the number of k-tuples such that at least r pairs (n_i, n_j), respectively exactly r pairs are coprime. See Theorem 11. In particular, we obtain an asymptotic formula for the function $\beta = \beta_k$, for every $k \geq 2$. See Corollary 12. Our results generalize and refine those by Heyman [7] and Hu [9].

Basic properties of arithmetic functions of k variables are presented in Section 3.1. Some lemmas related to the principle of inclusion-exclusion, used in the proofs are included in Section 3.2. The proofs of our main results are similar to those in [17], and are given in Section 5. Some numerical examples are presented in Section 6.

2 Previous results

Heyman [7] proved the asymptotic formula

$$\sum_{n_1,n_2,n_3 \leq x} \beta(n_1, n_2, n_3) = C_3 x^3 + O(x^2 (\log x)^2),$$

where the constant C_3 is

$$C_3 = 1 - 3\prod_p \left(1 - \frac{1}{p^2}\right) + 3\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) - \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right). \tag{5}$$

Hu [9] gave a formula for the asymptotic density C_k of k-tuples with pairwise non-coprime components, where $k \geq 3$. See (25). It recovers (5) for $k = 3$, and for $k = 4$ it can be written as

$$\begin{aligned} C_4 = 1 &- 6\prod_p \left(1 - \frac{1}{p^2}\right) + 3\prod_p \left(1 - \frac{1}{p^2}\right)^2 + 12\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) \\ &- 4\prod_p \left(1 - \frac{3}{p^2} + \frac{3}{p^3} - \frac{1}{p^4}\right) - 16\prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right) \\ &+ 15\prod_p \left(1 - \frac{4}{p^2} + \frac{4}{p^3} - \frac{1}{p^4}\right) - 6\prod_p \left(1 - \frac{5}{p^2} + \frac{6}{p^3} - \frac{2}{p^4}\right) \\ &+ \prod_p \left(1 - \frac{6}{p^2} + \frac{8}{p^3} - \frac{3}{p^4}\right). \end{aligned} \tag{6}$$

Related to identity (6) we note that there are two typos in [9], namely $\prod_p (1 - 1/p)^2 (1 - 2/p)$ on pages 7 and 8 should be $\prod_p (1 - 1/p)^2 (1 + 2/p)$.

For a fixed $k \geq 2$ let $V = \{1, 2, \ldots, k\}$, let E be an arbitrary subset of the set $\{(i, j) : 1 \leq i < j \leq k\}$, and let take the coprimality conditions $\gcd(n_i, n_j) = 1$ for $(i, j) \in E$.

Following Hu [9] and Arias de Reyna and Heyman [2], it is convenient and suggestive to consider the corresponding simple graph $G = (V, E)$, we call it coprimality graph, with set of vertices V and set of edges E. Therefore, we use the notation $E \subseteq V^{(2)} := \{\{i, j\} : 1 \leq i < j \leq k\}$, where the edges of G are denoted by $\{i, j\} = \{j, i\}$, and adopt some related graph terminology.

Let δ_G denote the characteristic function attached to the graph G, defined by

$$\delta_G(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if } \gcd(n_i, n_j) = 1 \text{ for every } \{i, j\} \in E; \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

and note that if $E = \emptyset$, that is, the graph G has no edges, then $\delta_G(n_1, \ldots, n_k) = 1$ for every $(n_1, \ldots, n_k) \in \mathbb{N}^k$.

Furthermore, let $i_m(G)$ be the number of independent sets S of vertices in G (i.e., no two vertices of S are adjacent in G) of cardinality m. Also, for $F \subseteq E$ let $v(F)$ denote the number of distinct vertices appearing in F.

Theorem 1. *Let $G = (V, E)$ be an arbitrary graph. With the above notation,*

$$\sum_{n_1, \ldots, n_k \leq x} \delta_G(n_1, \ldots, n_k) = A_G x^k + O(x^{k-1} (\log x)^{\vartheta_G}), \tag{8}$$

where the constant A_G is given by

$$A_G = \prod_p \left(\sum_{m=0}^{k} \frac{i_m(G)}{p^m} \left(1 - \frac{1}{p}\right)^{k-m} \right) \tag{9}$$

$$= \prod_p \left(\sum_{F \subseteq E} \frac{(-1)^{|F|}}{p^{v(F)}} \right), \tag{10}$$

and $\vartheta_G = d_G := \max_{j \in V} \deg(j)$, denoting the maximum degree of the vertices of G.

Here A_G is representing the asymptotic density of k-tuples $(n_1, \ldots, n_k) \in \mathbb{N}^k$ such that $\gcd(n_i, n_j) = 1$ for $\{i, j\} \in E$. Theorem 1 was first proved by Hu [9] with the weaker exponent $\vartheta_G = k - 1$ for every subset E and with identity (9) for the constant A_G. Arias de Reyna and Heyman [2] deduced Theorem 1 by a different method, with the given exponent $\vartheta_G = d_G$ and identity (10) for the constant A_G.

Note that if we have the complete coprimality graph, namely if $E = V^{(2)}$, then δ_G is the characteristic function of the set of k-tuples with pairwise coprime components (see (12)), and (8) recovers formula (3).

3 Preliminaries

3.1 Arithmetic functions of k variables

The Dirichlet convolution of the functions $f, g : \mathbb{N}^k \to \mathbb{C}$ is defined by

$$(f * g)(n_1, \ldots, n_k) = \sum_{d_1 | n_1, \ldots, d_k | n_k} f(d_1, \ldots, d_k) g(n_1/d_1, \ldots, n_k/d_k). \tag{11}$$

Let $\mu = \mu_k : \mathbb{N}^k \to \{-1, 0, 1\}$ denote the Möbius function of k variables, defined as the inverse of the constant 1 function under the convolution (11). We have $\mu(n_1, \ldots, n_k) = \mu(n_1) \cdots \mu(n_k)$ for every $n_1, \ldots, n_k \in \mathbb{N}$, which recovers for $k = 1$ the classical (one variable) Möbius function.

The Dirichlet series of a function $f : \mathbb{N}^k \to \mathbb{C}$ is given by

$$D(f; s_1, \ldots, s_k) := \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}}.$$

If $D(f; s_1, \ldots, s_k)$ and $D(g; s_1, \ldots, s_k)$ are absolutely convergent, where $s_1, \ldots, s_k \in \mathbb{C}$, then $D(f * g; s_1, \ldots, s_k)$ is also absolutely convergent and

$$D(f * g; s_1, \ldots, s_k) = D(f; s_1, \ldots, s_k) D(g; s_1, \ldots, s_k).$$

We recall that a nonzero arithmetic function of k variables $f : \mathbb{N}^k \to \mathbb{C}$ is said to be multiplicative if

$$f(m_1 n_1, \ldots, m_k n_k) = f(m_1, \ldots, m_k) f(n_1, \ldots, n_k)$$

holds for all $m_1, n_1 \ldots, m_k, n_k \in \mathbb{N}$ such that $\gcd(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. If f is multiplicative, then it is determined by the values $f(p^{\nu_1}, \ldots, p^{\nu_k})$, where p is prime and $\nu_1, \ldots, \nu_k \in \mathbb{N} \cup \{0\}$. More exactly, $f(1, \ldots, 1) = 1$ and for all $n_1, \ldots, n_k \in \mathbb{N}$,

$$f(n_1, \ldots, n_k) = \prod_p f(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_k)}),$$

where we use the notation $n = \prod_p p^{\nu_p(n)}$ for the prime power factorization of $n \in \mathbb{N}$, the product being over the primes p and all but a finite number of the exponents $\nu_p(n)$ are zero. Examples of multiplicative functions of k variables are the GCD and LCM functions $\gcd(n_1, \ldots, n_k)$, $\operatorname{lcm}(n_1, \ldots, n_k)$ and the characteristic functions

$$\varrho(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if } \gcd(n_1, \ldots, n_k) = 1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\vartheta(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if } \gcd(n_i, n_j) = 1 \text{ for every } 1 \le i < j \le k; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

If the function f is multiplicative, then its Dirichlet series can be expanded into a (formal) Euler product, that is,

$$D(f; s_1, \ldots, s_k) = \prod_p \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\nu_1}, \ldots, p^{\nu_k})}{p^{\nu_1 s_1 + \cdots + \nu_k s_k}}, \tag{13}$$

the product being over the primes p. More exactly, for f multiplicative, the series $D(f; s_1, \ldots, s_k)$ with $s_1, \ldots, s_k \in \mathbb{C}$ is absolutely convergent if and only if

$$\sum_p \sum_{\substack{\nu_1,\ldots,\nu_k=0 \\ \nu_1+\cdots+\nu_k \geq 1}}^{\infty} \frac{|f(p^{\nu_1}, \ldots, p^{\nu_k})|}{p^{\nu_1 \Re s_1 + \cdots + \nu_k \Re s_k}} < \infty$$

and in this case equality (13) holds.

The mean value of a function $f : \mathbb{N}^k \to \mathbb{C}$ is

$$M(f) := \lim_{x_1,\ldots,x_k \to \infty} \frac{1}{x_1 \cdots x_k} \sum_{n_1 \leq x_1, \ldots, n_k \leq x_k} f(n_1, \ldots, n_k),$$

provided that this limit exists. As a generalization of Wintner's theorem (valid in the one variable case), Ushiroya [18, Thm. 1] proved the next result.

Theorem 2. *If f is a function of k variables, not necessary multiplicative, such that*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{|(\mu * f)(n_1, \ldots, n_k)|}{n_1 \cdots n_k} < \infty,$$

then the mean value $M(f)$ exists, and

$$M(f) = \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{(\mu * f)(n_1, \ldots, n_k)}{n_1 \cdots n_k}.$$

For multiplicative functions the above result can be formulated as follows. See [16, Prop. 19], [18, Thm. 4].

Theorem 3. *Let $f : \mathbb{N}^k \to \mathbb{C}$ be a multiplicative function. Assume that*

$$\sum_p \sum_{\substack{\nu_1,\ldots,\nu_k=0 \\ \nu_1+\cdots+\nu_k \geq 1}}^{\infty} \frac{|(\mu * f)(p^{\nu_1}, \ldots, p^{\nu_k})|}{p^{\nu_1 + \cdots + \nu_k}} < \infty.$$

Then the mean value $M(f)$ exists, and

$$M(f) = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\nu_1}, \ldots, p^{\nu_k})}{p^{\nu_1 + \cdots + \nu_k}}.$$

See, e.g., Delange [5] and the survey by the author [16] for these and some other related results on arithmetic functions of several variables. If $k = 1$, i.e., in the case of functions of a single variable we recover some familiar properties.

3.2 The functions δ_G and β

Consider the function δ_G defined by (7).

Lemma 4. *For every subset E, the function δ_G is multiplicative.*

Proof. This is a consequence of the fact that the gcd function $\gcd(m, n)$ is multiplicative, viewed as a function of two variables. To give a direct proof, let $m_1, n_1, \ldots, m_k, n_k \in \mathbb{N}$ such that $\gcd(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. Then we have

$$\begin{aligned}
\delta_G(m_1 n_1, \ldots, m_k n_k) &= \begin{cases} 1, & \text{if } \gcd(m_i n_i, m_j n_j) = 1 \text{ for all } \{i,j\} \in E; \\ 0, & \text{otherwise}; \end{cases} \\
&= \begin{cases} 1, & \text{if } \gcd(m_i, m_j) \gcd(n_i, n_j) = 1 \text{ for all } \{i,j\} \in E; \\ 0, & \text{otherwise}; \end{cases} \\
&= \begin{cases} 1, & \text{if } \gcd(m_i, m_j) = 1 \text{ for all } \{i,j\} \in E; \\ 0, & \text{otherwise}; \end{cases} \\
&\quad \times \begin{cases} 1, & \text{if } \gcd(n_i, n_j) = 1 \text{ for all } \{i,j\} \in E; \\ 0, & \text{otherwise}; \end{cases} \\
&= \delta_G(m_1, \ldots, m_k) \delta_G(n_1, \ldots, n_k),
\end{aligned}$$

finishing the proof. □

The function β given by (4) is not multiplicative. However, by the inclusion-exclusion principle it can be written as an alternating sum of certain multiplicative functions δ_G.

More generally, for $r \geq 0$ we define the functions $\beta_r = \beta_{k,r}$ and $\beta'_r = \beta'_{k,r}$ by

$$\beta_r(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if exactly } r \text{ pairs } (n_i, n_j) \text{ with } 1 \leq i < j \leq k \text{ are coprime}; \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

$$\beta'_r(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if at least } r \text{ pairs } (n_i, n_j) \text{ with } 1 \leq i < j \leq k \text{ are coprime}; \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

If $r = 0$, then $\beta_0 = \beta$.

Lemma 5. *Let $k \geq 2$ and $r \geq 0$. For every $n_1, \ldots, n_k \in \mathbb{N}$,*

$$\beta_r(n_1, \ldots, n_k) = \sum_{j=r}^{k(k-1)/2} (-1)^{j-r} \binom{j}{r} \sum_{\substack{E \subseteq V^{(2)} \\ |E| = j}} \delta_G(n_1, \ldots, n_k), \tag{16}$$

$$\beta(n_1, \ldots, n_k) = \sum_{j=0}^{k(k-1)/2} (-1)^{j} \sum_{\substack{E \subseteq V^{(2)} \\ |E| = j}} \delta_G(n_1, \ldots, n_k). \tag{17}$$

Proof. Given $n_1, \dots, n_k \in \mathbb{N}$, assume that for $1 \le i < j \le k$ condition $\gcd(n_i, n_j) = 1$ holds for t times, where $0 \le t \le k(k-1)/2$. Then the right hand side of (16) is

$$N_r := \sum_{j=r}^{t} (-1)^{j-r} \binom{j}{r} \binom{t}{j}.$$

If $t < r$, then this is the empty sum, and $N_r = 0 = \beta_r(n_1, \dots, n_k)$. If $t \ge r$, then

$$N_r = \binom{t}{r} \sum_{j=r}^{t} (-1)^{j-r} \binom{t-r}{j-r} = \binom{t}{r} \sum_{m=0}^{t-r} (-1)^m \binom{t-r}{m} = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t > r, \end{cases}$$

which is exactly $\beta_r(n_1, \dots, n_k)$.

In the case $r = 0$ we obtain identity (17). □

Lemma 6. *Let $k \ge 2$ and $r \ge 1$. For every $n_1, \dots, n_k \in \mathbb{N}$,*

$$\beta'_r(n_1, \dots, n_k) = \sum_{j=r}^{k(k-1)/2} (-1)^{j-r} \binom{j-1}{r-1} \sum_{\substack{E \subseteq V^{(2)} \\ |E| = j}} \delta_G(n_1, \dots, n_k). \tag{18}$$

Proof. We have by using (16),

$$\begin{aligned} \beta'_r(n_1, \dots, n_k) &= \sum_{t=r}^{k(k-1)/2} \beta_t(n_1, \dots, n_k) \\ &= \sum_{t=r}^{k(k-1)/2} \sum_{j=t}^{k(k-1)/2} (-1)^{j-t} \binom{j}{t} \sum_{\substack{E \subseteq V^{(2)} \\ |E| = j}} \delta_G(n_1, \dots, n_k) \\ &= \sum_{j=r}^{k(k-1/2} (-1)^j \sum_{\substack{E \subseteq V^{(2)} \\ |E| = j}} \delta_G(n_1, \dots, n_k) \sum_{t=r}^{j} (-1)^t \binom{j}{t}, \end{aligned}$$

where the inner sum is $(-1)^r \binom{j-1}{r-1}$, finishing the proof. □

The above identities are similar to some known generalizations of the principle of inclusion-exclusion. See, e.g., the books by Aigner [1, Sect. 5.1] and Stanley [13, Ch. 2].

4 Main results

Given a graph $G = (V, E)$, the asymptotic density A_G of the of k-tuples $(n_1, \dots, n_k) \in \mathbb{N}^k$ such that $\gcd(n_i, n_j) = 1$ for $\{i, j\} \in E$ is the mean value of the characteristic function δ_G

defined by (7). According to Theorem 2, $A_G = D(\mu * \delta_G, 1, \ldots, 1)$, provided that this series is absolutely convergent. We show this by a careful study of the Dirichlet series of the function δ_G.

To formulate our results we need the following additional notation. For a graph $G = (V, E)$ let I be the set of non-isolated vertices of G, and J be a (minimum) vertex cover of G, that is, a set of vertices that includes at least one endpoint of every edge (of smallest possible size). The notation $\sum'_{L\subseteq J}$ means the sum over independent subsets L of J (no two vertices of L are adjacent in G). Also, let $N(j)$ denote the neighbourhood of a vertex j, and for a subset L of V let $N(L) = \cup_{j\in L} N(j)$.

Theorem 7. *Let $k \geq 2$ and $G = (V, E)$ be an arbitrary graph. Then, with the above notation,*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{\delta_G(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} = \zeta(s_1)\cdots\zeta(s_k) D'_G(s_1,\ldots,s_k),$$

where

$$D'_G(s_1,\ldots,s_k) = \prod_p \left({\sum_{L\subseteq J}}' \prod_{\ell\in L} \frac{1}{p^{s_\ell}} \prod_{i\in (J\setminus L)\cup(N(L)\setminus J)} \left(1 - \frac{1}{p^{s_i}}\right)\right) \tag{19}$$

$$= \prod_p \left(1 - \sum_{\{i,j\}\in E} \frac{1}{p^{s_i+s_j}} + \sum_{j=3}^{|I|} \sum_{\substack{i_1,\ldots,i_j\in I\\ i_1<\cdots<i_j}} \frac{c(i_1,\ldots,i_j)}{p^{s_{i_1}+\cdots+s_{i_j}}}\right),$$

where $c(i_1,\ldots,i_j)$ are some integers, depending on $i_1,\ldots,i_j$, but not on p.

Furthermore, $D'(s_1,\ldots,s_k)$ with $s_1,\ldots,s_k \in \mathbb{C}$ is absolutely convergent provided that $\Re(s_{i_1} + \cdots + s_{i_j}) > 1$ for every $i_1,\ldots,i_j \in I$ with $i_1 < \cdots < i_j$, $2 \leq j \leq |I|$.

Remark 8. By choosing $J = I$ or $J = \{1,\ldots,k\}$ the sum over L in identity (19) has more terms than in the case of a minimum vertex cover J of G. However, if $J = I$, then $N(L) \setminus I = \emptyset$ for every L, and (19) takes the slightly simpler form

$$D'_G(s_1,\ldots,s_k) = \prod_p \left({\sum_{L\subseteq I}}' \prod_{\ell\in L} \frac{1}{p^{s_\ell}} \prod_{i\in I\setminus L} \left(1 - \frac{1}{p^{s_i}}\right)\right),$$

and similarly if $J = \{1,\ldots,k\}$.

Next we prove by the convolution method the asymptotic formula already given in Theorem 1. This approach leads to new representations of the constant A_G.

Theorem 9. *Asymptotic formula (8) holds with the exponent $\vartheta_G = d_G$ in the error term, and with the constant*

$$A_G = \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{(\mu * \delta_G)(n_1,\ldots,n_k)}{n_1\cdots n_k}$$

$$= \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{\delta_G(p^{\nu_1},\ldots,p^{\nu_k})}{p^{\nu_1+\cdots+\nu_k}}$$

$$= \prod_p \left({\sum_{L\subseteq J}}' \frac{1}{p^{|L|}} \left(1-\frac{1}{p}\right)^{|J\setminus L|+|N(L)\setminus J|} \right). \tag{20}$$

Remark 10. If $J = I$, then $N(L)\setminus I = \emptyset$ for every L, and (20) gives

$$A_G = \prod_p \left({\sum_{L\subseteq I}}' \frac{1}{p^{|L|}} \left(1-\frac{1}{p}\right)^{|I\setminus L|} \right)$$

$$= \prod_p \left(\sum_{m=0}^{|I|} \frac{i_m(G,I)}{p^m} \left(1-\frac{1}{p}\right)^{|I|-m} \right),$$

where $i_m(G,I)$ denotes the number of independent subsets of I of cardinality m in the graph G. Similarly, by choosing $J = \{1,\ldots,k\}$, (20) reduces to identity (9) by Hu [9].

Now consider the functions β_r and β'_r defined by (14) and (15).

Theorem 11. *Let $k \ge 2$. Then for $r \ge 0$ we have*

$$\sum_{n_1,\ldots,n_k\le x} \beta_r(n_1,\ldots,n_k) = C_r x^k + O(x^{k-1}(\log x)^{k-1}), \tag{21}$$

and for $r \ge 1$,

$$\sum_{n_1,\ldots,n_k\le x} \beta'_r(n_1,\ldots,n_k) = C'_r x^k + O(x^{k-1}(\log x)^{k-1}), \tag{22}$$

where

$$C_r = C_{k,r} = \sum_{j=r}^{k(k-1)/2} (-1)^{j-r}\binom{j}{r} \sum_{\substack{E\subseteq V^{(2)}\\ |E|=j}} A_G, \tag{23}$$

and

$$C'_r = C'_{k,r} = \sum_{j=r}^{k(k-1)/2} (-1)^{j-r}\binom{j-1}{r-1} \sum_{\substack{E\subseteq V^{(2)}\\ |E|=j}} A_G \tag{24}$$

are the asymptotic densities of the k-tuples $(n_1,\ldots,n_k)\in\mathbb{N}^k$ such that $\gcd(n_i,n_j)=1$ occurs exactly r times, respectively at least r times, with A_G given in Theorems 1 and 9.

We remark that identities (23) and (24) have been obtained by Hu [9, Cor. 3] with an incomplete proof.

Corollary 12. $(r = 0)$ *We have*

$$\sum_{n_1,\dots,n_k \le x} \beta(n_1,\dots,n_k) = C_k x^k + O(x^{k-1}(\log x)^{k-1}),$$

where

$$\begin{aligned} C_k = C_{k,0} &= \sum_{n_1,\dots,n_k=1}^{\infty} \frac{(\mu * \beta)(n_1,\dots,n_k)}{n_1 \cdots n_k} \\ &= \sum_{j=0}^{k(k-1)/2} (-1)^j \sum_{\substack{E \subseteq V^{(2)} \\ |E|=j}} A_G. \end{aligned} \tag{25}$$

Identity (25) has been obtained by Hu [9, Cor. 3].

Note that if G and G' are isomorphic graphs then the corresponding densities A_G and $A_{G'}$ are equal. The asymptotic densities $C_{k,r}$, $C'_{k,r}$ and C_k can be computed for given values of k and r from identities (23), (24) and (25), respectively by determining the cardinalities of the isomorphism classes of graphs G with k vertices and j edges ($0 \le j \le k(k-1)/2$) and by computing the corresponding values of A_G. In particular, C_3 and C_4 given by (5) and (6) can be obtained in this way.

5 Proofs

We first prove the key result of our treatment.

Proof of Theorem 7. We have

$$D(\delta_G; s_1,\dots,s_k) = \sum_{n_1,\dots,n_k=1}^{\infty} \frac{\delta_G(n_1,\dots,n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \sum_{\substack{n_1,\dots,n_k=1 \\ \gcd(n_{i_1},n_{i_2})=1 \\ \{i_1,i_2\} \in E}}^{\infty} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Let I denote the set of non-isolated vertices of G. Then

$$\begin{aligned} D(\delta_G; s_1,\dots,s_k) &= \prod_{i \notin I} \zeta(s_i) \sum_{\substack{n_i \ge 1,\, i \in I \\ \gcd(n_{i_1},n_{i_2})=1,\, \{i_1,i_2\} \in E}} \prod_{i \in I} \frac{1}{n_i^{s_i}} \\ &= \prod_{i \notin I} \zeta(s_i) \prod_p \left(\sum_{\substack{\nu_i \ge 0,\, i \in I \\ \nu_{i_1}\nu_{i_2}=0,\, \{i_1,i_2\} \in E}} \frac{1}{p^{\sum_{i \in I} \nu_i s_i}} \right) \\ &=: \prod_{i \notin I} \zeta(s_i) \prod_p S_p, \end{aligned}$$

say, using that the function δ_G is multiplicative by Lemma 4.

Now choose a (minimum) vertex cover J. Then ν_j ($j \in J$) cover all the conditions $\nu_{i_1}\nu_{i_2} = 0$ with $\{i_1, i_2\} \in E$, that is, for every $\{i_1, i_2\} \in E$ there is $j \in J$ such that $j = i_1$ or $j = i_2$. Group the terms of the sum S_p according to the subsets $L = \{\ell \in J : \nu_\ell \geq 1\}$ of J. Here $\nu_j = 0$ for every $j \in J \setminus L$. Note that L cannot contain any two adjacent vertices. Also, for such a fixed subset $L \subseteq J$ let M be the set of indexes m such that ν_m is forced to be zero by L. More exactly, let $M = \{m \in I \setminus J : \text{there is } \ell \in L \text{ with } \{m, \ell\} \in E\}$. If $m \in M$, then $\nu_m \nu_\ell = 0$ for some $\ell \in L$. Since $\nu_\ell \geq 1$, we obtain $\nu_m = 0$. Here $M = N(L) \setminus J$, where $N(L)$ is set of vertices adjacent to vertices in L.

Let $\sum'_{L\subseteq J}$ denote the sum over subsets L of J that have no adjacent vertices. We obtain

$$\begin{aligned}
S_p &= \sideset{}{'}\sum_{L\subseteq J} \sum_{\substack{\nu_\ell\geq 1,\, \ell\in L\\ \nu_j=0,\, j\in J\setminus L\\ \nu_m=0,\, m\in M\\ \nu_i\geq 0,\, i\in I\setminus(J\cup M)}} \frac{1}{p^{\sum_{i\in I}\nu_i s_i}} \\
&= \sideset{}{'}\sum_{L\subseteq J} \sum_{\nu_\ell\geq 1,\, \ell\in L} \frac{1}{p^{\sum_{\ell\in L}\nu_\ell s_\ell}} \sum_{\nu_i\geq 0,\, i\in I\setminus(J\cup M)} \frac{1}{p^{\sum_{i\in I\setminus(J\cup M)}\nu_i s_i}} \\
&= \sideset{}{'}\sum_{L\subseteq J} \prod_{\ell\in L} \frac{1}{p^{s_\ell}} \left(1-\frac{1}{p^{s_\ell}}\right)^{-1} \prod_{i\in I\setminus(J\cup M)} \left(1-\frac{1}{p^{s_i}}\right)^{-1} \\
&= \prod_{i\in I}\left(1-\frac{1}{p^{s_i}}\right)^{-1} \sideset{}{'}\sum_{L\subseteq J} \prod_{\ell\in L}\frac{1}{p^{s_\ell}} \prod_{i\in(J\setminus L)\cup M}\left(1-\frac{1}{p^{s_i}}\right) \\
&=: \prod_{i\in I}\left(1-\frac{1}{p^{s_i}}\right)^{-1} T_p,
\end{aligned}$$

say. We deduce that

$$\prod_p S_p = \prod_{i\in I}\zeta(s_i)\prod_p T_p,$$

which shows that

$$D(\delta_G; s_1, \ldots, s_k) = \Big(\prod_{i=1}^k \zeta(s_i)\Big) D'(s_1, \ldots, s_k),$$

where

$$D'(s_1, \ldots, s_k) = \prod_p T_p = \prod_p \left(\sideset{}{'}\sum_{L\subseteq J} \prod_{\ell\in L}\frac{1}{p^{s_\ell}} \prod_{i\in(J\setminus L)\cup M}\left(1-\frac{1}{p^{s_i}}\right)\right). \tag{26}$$

Let us investigate the terms of the sum $\sum'_{L\subseteq J}$ in (26). If $L = \emptyset$, that is, $\nu_j = 0$ for every $j \in J$, then $M = \emptyset$ and we have

$$\prod_{i\in J}\left(1-\frac{1}{p^{s_i}}\right) = 1 - \sum_{i\in J}\frac{1}{p^{s_i}} + \sum_{i,j\in J,\, i<j}\frac{1}{p^{s_i+s_j}} - \cdots. \tag{27}$$

If $L=\{i_0\}$ for some fixed $i_0 \in J$, then obtain, with $M=M_{i_0}:=N(i_0)\setminus J$,

$$\frac{1}{p^{s_{i_0}}}\prod_{\substack{i\in J\cup M_{i_0}\\ i\neq i_0}}\left(1-\frac{1}{p^{s_i}}\right)=\frac{1}{p^{s_{i_0}}}-\sum_{\substack{i\in J\cup M_{i_0}\\ i\neq i_0}}\frac{1}{p^{s_{i_0}+s_i}}+\sum_{\substack{i,j\in J\cup M_{i_0}\\ i_0\neq i<j\neq i_0}}\frac{1}{p^{s_{i_0}+s_i+s_j}}-\cdots.$$

Here if $i_0=t$ runs over J, then we have the terms

$$\sum_{t\in J}\frac{1}{p^{s_t}}-\sum_{\substack{i\in J\cup M_t\\ t\in J\\ i\neq t}}\frac{1}{p^{s_t+s_i}}+\sum_{\substack{i,j\in J\cup M_t\\ t\in J, i\neq t, j\neq t\\ i<j}}\frac{1}{p^{s_t+s_i+s_j}}-\cdots. \tag{28}$$

If $L=\{i_0,i_0'\}$ with some fixed $i_0,i_0'\in J$, $i_0<i_0'$, which are not adjacent, then obtain, with $M=M_{i_0,i_0'}:=N(\{i_0,i_0'\})\setminus J$,

$$\frac{1}{p^{s_{i_0}+s_{i_0'}}}\prod_{\substack{i\in J\cup M_{i_0,i_0'}\\ i\neq i_0,i_0'}}\left(1-\frac{1}{p^{s_i}}\right)=\frac{1}{p^{s_{i_0}+s_{i_0'}}}-\sum_{\substack{i\in J\cup M_{i_0,i_0'}\\ i\neq i_0,i_0'}}\frac{1}{p^{s_{i_0}+s_{i_0'}+s_i}}+\cdots. \tag{29}$$

If $i_0=t, i_0'=v$ run over J, then we obtain from (29),

$$\sum_{\substack{t,v\in J\\ t<v\\ t,v \text{ not adjacent}}}\frac{1}{p^{s_t+s_v}}-\sum_{\substack{i\in J\cup M_{t,v}\\ t,v\in J \text{ not adjacent}\\ i\neq t,v}}\frac{1}{p^{s_t+s_v+s_i}}+\cdots. \tag{30}$$

Putting together (27), (28) and (30) we obtain the sum S, where

$$\begin{aligned}S&=1-\sum_{i\in J}\frac{1}{p^{s_i}}+\sum_{i,j\in J, i<j}\frac{1}{p^{s_i+s_j}}+\sum_{t\in J}\frac{1}{p^{s_t}}-\sum_{\substack{i\in J\cup M_t\\ t\in J\\ i\neq t}}\frac{1}{p^{s_t+s_i}}\\&\quad+\sum_{\substack{t,v\in J\\ t<v\\ t,v \text{ not adjacent}}}\frac{1}{p^{s_t+s_v}}\pm \text{ other terms}\\&=1-\sum_{\substack{i,t\in J\\ i,t \text{ adjacent}}}\frac{1}{p^{s_t+s_i}}\pm \text{ other terms},\end{aligned}$$

where the terms $\pm 1/p^i$ with $i\in J$ cancel out. Also the terms $\pm 1/p^{s_i+s_j}$ with $i,j\in J$ (each appearing twice) cancel out, excepting when i,j are adjacent. Here for the "other terms", including the terms obtained if L has at least three elements, the exponents of p are sums of at least three distinct values s_i, s_j, s_ℓ with $i,j,\ell\in I$.

Hence the infinite product (26) is absolutely convergent provided the given condition. □

Remark 13. It turns out that the function $\mu * \delta_G$ is multiplicative (in general not symmetric in the variables) and for all prime powers $p^{\nu_1}, \ldots, p^{\nu_k}$,

$$(\mu * \delta_G)(p^{\nu_1}, \ldots, p^{\nu_k}) = \begin{cases} 1, & \text{if } \nu_1 = \cdots = \nu_k = 0; \\ c(\nu_1, \ldots, \nu_k), & \text{if } \nu_1, \ldots, \nu_k \in \{0,1\},\ j := \nu_1 + \cdots + \nu_k \geq 2; \\ 0, & \text{otherwise,} \end{cases} \tag{31}$$

where $c(\nu_1, \ldots, \nu_k)$ are some integers, depending on $\nu_1, \ldots, \nu_k$, but not on p.

Note that $(\mu * \delta_G)(p^{\nu_1}, \ldots, p^{\nu_k}) = 0$ provided that $\nu_i \geq 2$ for at least one $1 \leq i \leq k$, or $\nu_1, \ldots, \nu_k \in \{0,1\}$ and $\nu_1 + \cdots + \nu_k = 1$. If $\nu_1, \ldots, \nu_k \in \{0,1\}$ and $\nu_1 + \cdots + \nu_k = 2$, say $\nu_{i_0} = \nu_{i_0'} = 1$ and $\nu_i = 0$ for $i \neq i_0, i_0'$, then $(\mu * \delta_G)(p^{\nu_1}, \ldots, p^{\nu_k}) = -1$ if i_0 and i_0' are adjacent in the graph G and 0 otherwise.

Proof of Theorem 9. Write

$$\begin{aligned} \sum_{n_1, \ldots, n_k \leq x} \delta_G(n_1, \ldots, n_k) &= \sum_{n_1, \ldots, n_k \leq x} \sum_{d_1 \mid n_1, \ldots, d_k \mid n_k} (\mu * \delta_G)(d_1, \ldots, d_k) \\ &= \sum_{d_1, \ldots, d_k \leq x} (\mu * \delta_G)(d_1, \ldots, d_k) \left\lfloor \frac{x}{d_1} \right\rfloor \cdots \left\lfloor \frac{x}{d_k} \right\rfloor \\ &= \sum_{d_1, \ldots, d_k \leq x} (\mu * \delta_G)(d_1, \ldots, d_k) \left(\frac{x}{d_1} + O(1) \right) \cdots \left(\frac{x}{d_k} + O(1) \right) \\ &= x^k \sum_{d_1, \ldots, d_k \leq x} \frac{(\mu * \delta_G)(d_1, \ldots, d_k)}{d_1 \cdots d_k} + R_k(x), \end{aligned} \tag{32}$$

with

$$R_k(x) \ll \sum_{u_1, \ldots, u_r} x^{u_1 + \cdots + u_k} \sum_{d_1, \ldots, d_k \leq x} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_1^{u_1} \cdots d_k^{u_k}},$$

where the first sum is over $u_1, \ldots, u_k \in \{0,1\}$ such that at least one u_i is 0. Let $u_1, \ldots, u_k$ be fixed and assume that $u_{i_0} = 0$. Since $(x/d_i)^{u_i} \leq x/d_i$ for every i $(1 \leq i \leq k)$ we have

$$\begin{aligned} A &:= x^{u_1 + \cdots + u_k} \sum_{d_1, \ldots, d_k \leq x} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_1^{u_1} \cdots d_k^{u_k}} \\ &\leq x^{k-1} \sum_{d_1, \ldots, d_k \leq x} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{\prod_{1 \leq i \leq k, i \neq i_0} d_i} \\ &\leq x^{k-1} \prod_{p \leq x} \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{|(\mu * \delta_G)(p^{\nu_1}, \ldots, p^{\nu_k})|}{p^{\sum_{1 \leq i \leq k, i \neq i_0} \nu_i}} \\ &= x^{k-1} \prod_{p \leq x} \left(1 + \frac{c_{i_0,1}}{p} + \frac{c_{i_0,2}}{p^2} + \cdots + \frac{c_{i_0,k-1}}{p^{k-1}} \right), \end{aligned} \tag{33}$$

cf. (31), where $c_{i_0,j}$ $(1 \le j \le k-1)$ are certain non-negative integers. Here $c_{i_0,1} = \deg(i_0)$, the degree of i_0, according to Remark 13. We obtain that

$$A \ll x^{k-1} \prod_{p \le x} \left(1 + \frac{1}{p}\right)^{\deg(i_0)} \ll x^{k-1} (\log x)^{\deg(i_0)}$$

by Mertens' theorem. This shows that

$$R_k(x) \ll x^{k-1} (\log x)^{\max \deg(i_0)}. \tag{34}$$

Furthermore, for the main term of (32) we have

$$\sum_{d_1,\ldots,d_k \le x} \frac{(\mu * \delta_G)(d_1,\ldots,d_k)}{d_1 \cdots d_k}$$

$$= \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{(\mu * \delta_G)(d_1,\ldots,d_k)}{d_1 \cdots d_k} - \sum_{\emptyset \ne I \subseteq \{1,\ldots,k\}} \sum_{\substack{d_i > x,\, i \in I \\ d_j \le x,\, j \notin I}} \frac{(\mu * \delta_G)(d_1,\ldots,d_k)}{d_1 \cdots d_k}, \tag{35}$$

where the series is convergent by Theorem 7 and its sum is $D(\mu * \delta_G; 1, \ldots, 1) = A_G$.

Let I be fixed with $|I| = t$. We estimate the sum

$$B := \sum_{\substack{d_i > x,\, i \in I \\ d_j \le x,\, j \notin I}} \frac{|(\mu * \delta_G)(d_1,\ldots,d_k)|}{d_1 \cdots d_k}.$$

Case I. Assume that $|I| = t \ge 3$. If $0 < \varepsilon < 1/2$, then

$$\begin{aligned} B &= \sum_{\substack{d_i > x,\, i \in I \\ d_j \le x,\, j \notin I}} \frac{|(\mu * \delta_G)(d_1,\ldots,d_k)| \prod_{i \in I} d_i^{\varepsilon - 1/2}}{\prod_{i \in I} d_i^{1/2+\varepsilon} \prod_{j \notin I} d_j} \\ &\le x^{t(\varepsilon - 1/2)} \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{|(\mu * \delta_G)(d_1,\ldots,d_k)|}{\prod_{i \in I} d_i^{1/2+\varepsilon} \prod_{j \notin I} d_j} \\ &\ll x^{t(\varepsilon - 1/2)}, \end{aligned}$$

since the series is convergent (for $t \ge 1$). Using that $t(\varepsilon - 1/2) < -1$ for $0 < \varepsilon < (t-2)/(2t)$, here we need $t \ge 3$, we obtain $B \ll \frac{1}{x}$.

Case II. $t = 1$: Let $d_{i_0} > x$, $d_i \le x$ for $i \ne i_0$, and consider a prime p. If $p \mid d_i$ for some $i \ne i_0$, then $p \le x$. If $p \mid d_{i_0}$ and $p > x$, then $p \nmid d_i$ for every $i \ne i_0$, and $(\mu * \delta_G)(d_1, \ldots, d_k) = 0$, cf.

Remark 13. Hence it is enough to consider the primes $p \le x$. We deduce

$$\begin{aligned} B &< \frac{1}{x} \sum_{\substack{d_{i_0} > x \\ d_i \le x, i \ne i_0}} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{\prod_{i \ne i_0} d_i} \\ &\le \frac{1}{x} \prod_{p \le x} \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{|(\mu * \delta_G)(p^{\nu_1}, \ldots, p^{\nu_k})|}{p^{\sum_{i \ne i_0} \nu_i}} \\ &\ll \frac{1}{x} (\log x)^{\max \deg(i_0)}, \end{aligned}$$

similar to the estimate (34).

Case III. $t = 2$: Let $d_{i_0} > x$, $d_{i_0'} > x$. We split the sum B into two sums, namely

$$\begin{aligned} B &= \sum_{\substack{d_{i_0} > x, d_{i_0'} > x \\ d_i \le x, i \ne i_0, i_0'}} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_1 \cdots d_k} \\ &= \sum_{\substack{d_{i_0} > x^{3/2}, d_{i_0'} > x \\ d_i \le x, i \ne i_0, i_0'}} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_1 \cdots d_k} + \sum_{\substack{x^{3/2} \ge d_{i_0} > x, d_{i_0'} > x \\ d_i \le x, i \ne i_0, i_0'}} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_1 \cdots d_k} \\ &=: B_1 + B_2, \end{aligned}$$

say, where

$$\begin{aligned} B_1 &= \sum_{\substack{d_{i_0} > x^{3/2}, d_{i_0'} > x \\ d_i \le x, i \ne i_0, i_0'}} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_{i_0}^{1/3} \prod_{i \ne i_0} d_i} \frac{1}{d_{i_0}^{2/3}} \\ &< \frac{1}{x} \sum_{d_1, \ldots, d_k = 1}^{\infty} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{d_{i_0}^{1/3} \prod_{i \ne i_0} d_i} \\ &\ll \frac{1}{x}, \end{aligned}$$

since the latter series is convergent. Furthermore,

$$B_2 < \frac{1}{x} \sum_{\substack{x^{3/2} \ge d_{i_0}, d_{i_0'} > x \\ d_i \le x, i \ne i_0, i_0'}} \frac{|(\mu * \delta_G)(d_1, \ldots, d_k)|}{\prod_{i \ne i_0'} d_i},$$

and consider a prime p. For the last sum, if $p \mid d_i$ for some $i \ne i_0'$ then $p \le x^{3/2}$. If $p \mid d_{i_0'}$ and $p > x^{3/2}$, then $p \nmid d_i$ for every $i \ne i_0'$ and $(\mu * \delta_G)(d_1, \ldots, d_r) = 0$, cf. Remark 13. Hence it is enough to consider the primes $p \le x^{3/2}$. We deduce, similar to (33), (34) that

$$B_2 < \frac{1}{x} \prod_{p\le x^{3/2}} \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{|(\mu * \delta_G)(p^{\nu_1},\ldots,p^{\nu_k})|}{p^{\sum_{i\ne i_0'} \nu_i}} \ll \frac{1}{x}(\log x^{3/2})^{d_G} \ll \frac{1}{x}(\log x)^{d_G},$$

with $d_G = \max_{i\in G} \deg(i)$.

Hence given any $|I| = t \ge 1$ we have $B \ll \frac{1}{x}(\log x)^{d_G}$. Therefore, by (35),

$$\sum_{d_1,\ldots,d_r\le x} \frac{(\mu * \delta_G)(d_1,\ldots,d_k)}{d_1\cdots d_k} = A_G + O(x^{-1}(\log x)^{d_G}). \tag{36}$$

The proof is complete by putting together (32), (34) and (36). □

Proof of Theorem 11. According to identities (16) and (18) we have

$$\sum_{n_1,\ldots,n_k\le x} \beta_r(n_1,\ldots,n_k) = \sum_{j=r}^{k(k-1)/2} (-1)^{j-r} \binom{j}{r} \sum_{\substack{E\subseteq V^{(2)}\\|E|=j}} \sum_{n_1,\ldots,n_k\le x} \delta_G(n_1,\ldots,n_k), \tag{37}$$

and

$$\sum_{n_1,\ldots,n_k\le x} \beta_r'(n_1,\ldots,n_k) = \sum_{j=r}^{k(k-1)/2} (-1)^{j-r} \binom{j-1}{r-1} \sum_{\substack{E\subseteq S\\|E|=j}} \sum_{n_1,\ldots,n_k\le x} \delta_G(n_1,\ldots,n_k). \tag{38}$$

Now for the inner sums $\sum_{n_1,\ldots,n_k\le x} \delta_G(n_1,\ldots,n_k)$ of identities (37) and (38) use asymptotic formula (8). For the complete coprimality graph with $E = V^{(2)}$, corresponding to all coprimality conditions, the error term is $O(x^{k-1}(\log x)^{k-1})$, and this is the final error term in both cases. This proves asymptotic formulas (21) and (22). □

Proof of Corollary 12. Apply formula (21) for $r = 0$, with the constant $C_{k,0}$ given by (23). □

6 Examples

To illustrate identities (19) and (20) let us work out the following examples.

Example 14. Let $k = 4$ and $G = (V, E)$ with $V = \{1,2,3,4\}$, $E = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$, that is, $\gcd(n_1,n_2) = 1$, $\gcd(n_2,n_3) = 1$, $\gcd(n_3,n_4) = 1$, $\gcd(n_4,n_1) = 1$. See Figure 1.

Here $I = \{1,2,3,4\}$ and choose the minimum vertex cover $J = \{1,3\}$. According to (19),

$$D_G'(s_1,s_2,s_3,s_4) = \prod_p \left(\sum_{L\subseteq J}{}' \prod_{\ell\in L} \frac{1}{p^{s_\ell}} \prod_{i\in(J\setminus L)\cup(N(L)\setminus J)} \left(1 - \frac{1}{p^{s_i}}\right)\right) \tag{39}$$

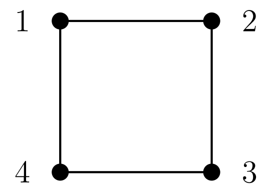

Figure 1: Graph of Example 14

L	$N(L)$	$(J \setminus L) \cup (N(L) \setminus J)$	S_L
$\emptyset$	$\emptyset$	$\{1,3\}$	$(1-x_1)(1-x_3)$
$\{1\}$	$\{2,4\}$	$\{2,3,4\}$	$x_1(1-x_2)(1-x_3)(1-x_4)$
$\{3\}$	$\{2,4\}$	$\{1,2,4\}$	$x_3(1-x_1)(1-x_2)(1-x_4)$
$\{1,3\}$	$\{2,4\}$	$\{2,4\}$	$x_1x_3(1-x_2)(1-x_4)$

Table 1: Terms of the sum in Example 14

Write the terms of the sum in (39), see Table 1, where $x_i = 1/p^{s_i}$ $(1 \le i \le 4)$. Note that all subsets of J are independent.

We obtain

$$\begin{aligned} D'_G(s_1,\ldots,s_4) &= \prod_p \left(S_\emptyset + S_{\{1\}} + S_{\{3\}} + S_{\{1,3\}}\right) \\ &= \prod_p \Bigg(1 - \frac{1}{p^{s_1+s_2}} - \frac{1}{p^{s_1+s_4}} - \frac{1}{p^{s_2+s_3}} - \frac{1}{p^{s_3+s_4}} + \frac{1}{p^{s_1+s_2+s_3}} \\ &\quad + \frac{1}{p^{s_1+s_2+s_4}} + \frac{1}{p^{s_1+s_3+s_4}} + \frac{1}{p^{s_2+s_3+s_4}} - \frac{1}{p^{s_1+s_2+s_3+s_4}}\Bigg). \end{aligned}$$

Observe that the terms $\pm 1/p^i$ with $i \in J = \{1,3\}$ cancel out, and we have the terms $-1/p^{s_i+s_j}$ with $\{i,j\} \in E$, according to the edges of G. Hence the infinite product is absolutely convergent provided that $\Re(s_{i_1} + \cdots + s_{i_j}) > 1$ for every $i_1,\ldots,i_j \in \{1,2,3,4\}$ with $i_1 < \cdots < i_j$, $2 \le j \le 4$.

The asymptotic density of 4-tuples $(n_1,\ldots,n_4) \in \mathbb{N}^4$ such that $\gcd(n_i,n_j) = 1$ with $\{i,j\} \in E$ is

$$D'_G(1\ldots,1) = \prod_p \left(1 - \frac{4}{p^2} + \frac{4}{p^3} - \frac{1}{p^4}\right).$$

This asymptotic density has been obtained using identity (10) by de Reyna and Heyman [2, Sect. 4].

Example 15. Now let $k = 7$ and $G = (V,E)$ with $V = \{1,2,3,4,5,6,7\}$ and

$$E = \{\{1,2\},\{1,3\},\{2,4\},\{2,5\},\{3,4\},\{4,5\}\},$$

that is, $\gcd(n_1, n_2) = 1$, $\gcd(n_1, n_3) = 1$, $\gcd(n_2, n_4) = 1$, $\gcd(n_2, n_5) = 1$, $\gcd(n_3, n_4) = 1$, $\gcd(n_4, n_5) = 1$. See Figure 2.

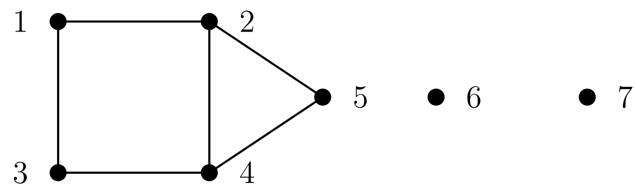

Figure 2: Graph of Example 15

Here $I = \{1, 2, 3, 4, 5\}$, since the variables n_6, n_7 do not appear in the constraints. Choose the minimum vertex cover $J = \{1, 2, 4\}$. Consider the subsets L of J and write the corresponding terms S_L of the sum in (19), see Table 2, where $x_i = p^{-s_i}$ $(1 \leq i \leq 5)$. The subsets $L = \{1, 2\}$ and $L = \{2, 4\}$ do not appear in the sum, since $1, 2$ and $2, 4$ are adjacent vertices.

L	$N(L)$	$(J \setminus L) \cup (N(L) \setminus J)$	S_L
$\emptyset$	$\emptyset$	$\{1, 2, 4\}$	$(1 - x_1)(1 - x_2)(1 - x_4)$
$\{1\}$	$\{3\}$	$\{2, 3, 4\}$	$x_1(1 - x_2)(1 - x_3)(1 - x_4)$
$\{2\}$	$\{5\}$	$\{1, 4, 5\}$	$x_2(1 - x_1)(1 - x_4)(1 - x_5)$
$\{4\}$	$\{3, 5\}$	$\{1, 2, 3, 5\}$	$x_4(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_5)$
$\{1, 4\}$	$\{3, 5\}$	$\{2, 3, 5\}$	$x_1 x_4(1 - x_2)(1 - x_3)(1 - x_5)$

Table 2: Terms of the sum in Example 15

It follows that

$$
\begin{aligned}
D'_G(s_1, \ldots, s_7) &= \prod_p \left(S_\emptyset + S_{\{1\}} + S_{\{2\}} + S_{\{4\}} + S_{\{1,4\}} \right) \\
&= \prod_p \Bigg(1 - \frac{1}{p^{s_1+s_2}} - \frac{1}{p^{s_1+s_3}} - \frac{1}{p^{s_2+s_4}} - \frac{1}{p^{s_2+s_5}} - \frac{1}{p^{s_3+s_4}} - \frac{1}{p^{s_4+s_5}} \\
&\quad + \frac{1}{p^{s_1+s_2+s_3}} + \frac{1}{p^{s_1+s_2+s_4}} + \frac{1}{p^{s_1+s_2+s_5}} + \frac{1}{p^{s_1+s_3+s_4}} + \frac{1}{p^{s_2+s_3+s_4}} \\
&\quad + \frac{2}{p^{s_2+s_4+s_5}} + \frac{1}{p^{s_3+s_4+s_5}} - \frac{1}{p^{s_1+s_2+s_3+s_4}} - \frac{1}{p^{s_1+s_2+s_4+s_5}} - \frac{1}{p^{s_2+s_3+s_4+s_5}} \Bigg).
\end{aligned}
$$

Observe that the terms $\pm 1/p^i$ with $i, j \in \{1, 2, 4\}$ cancel out, and we have the terms $-1/p^{s_i+s_j}$ with $\{i, j\} \in E$, according to the edges of G. Here the infinite product is absolutely convergent provided that $\Re(s_{i_1} + \cdots + s_{i_j}) > 1$ for every $i_1, \ldots, i_j \in \{1, 2, 3, 4, 5\}$ with $i_1 < \cdots < i_j$, $2 \leq j \leq 5$.

The asymptotic density of 7-tuples $(n_1, \ldots, n_7) \in \mathbb{N}^7$ with the corresponding constraints $\gcd(n_i, n_j) = 1$ with $\{i, j\} \in E$ is

$$D'_G(1 \ldots, 1) = \prod_p \left(1 - \frac{6}{p^2} + \frac{8}{p^3} - \frac{3}{p^4}\right).$$

Application of identity (10) by de Reyna and Heyman [2] is more laborious here, since G has six edges and there are $2^6 = 64$ subsets of E.

Example 16. Now consider the case of pairwise coprime integers with $E = \{\{i, j\} : 1 \leq i < j \leq k\}$. For $k = 4$ the graph is in Figure 3.

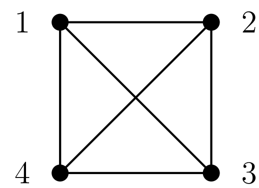

Figure 3: Graph to Example 16

Here $I = \{1, \ldots, k\}$ and choose the minimum vertex cover $J = \{1, \ldots, k-1\}$. The only independent subsets L of J are $L = \emptyset$ and $L = \{1\}$, ..., $L = \{k-1\}$ having one single element.

If $L = \emptyset$, then $N(L) = \emptyset$, $(J \setminus L) \cup (N(L) \setminus J) = J$ and obtain, with $x_i = p^{-s_i}$ $(1 \leq i \leq k)$,

$$S_\emptyset = (1 - x_1) \cdots (1 - x_{k-1}).$$

If $L = \{\ell\}$, $\ell \in J$, then $N(L) = \{k\}$, $(J \setminus L) \cup (N(L) \setminus J) = \{1, \ldots, k\} \setminus \{\ell\}$, and have

$$S_{\{\ell\}} = x_\ell \prod_{\substack{j=1 \\ j \neq \ell}}^{k} (1 - x_j).$$

We need to evaluate the sum

$$S := S_\emptyset + \sum_{\ell=1}^{k-1} S_{\{\ell\}}. \tag{40}$$

Let $e_j(x_1, \ldots, x_k) = \sum_{1 \leq i_1 < \ldots < i_j \leq k} x_{i_1} \cdots x_{i_j}$ denote the elementary symmetric polynomials in $x_1, \ldots, x_k$ of degree j $(j \geq 0)$. By convention, $e_0(x_1, \ldots, x_k) = 1$.

Consider the polynomial

$$P(x) = \prod_{j=1}^{k} (x - x_j) = \sum_{j=0}^{k} (-1)^j e_j(x_1, \ldots, x_k) x^{k-j}.$$

Its derivative is

$$P'(x) = \sum_{j=0}^{k-1} (-1)^j (k-j) e_j(x_1, \ldots, x_k) x^{k-j-1},$$

and on the other hand

$$P'(x) = \sum_{j=1}^{k} \prod_{\substack{i=1 \\ i \neq j}}^{k} (x - x_i).$$

We obtain that the sum (40) is

$$\begin{aligned} S &= \prod_{j=1}^{k-1} (1 - x_j) + \sum_{j=1}^{k-1} x_j \prod_{\substack{i=1 \\ i \neq j}}^{k} (1 - x_i) \\ &= \sum_{j=1}^{k} \prod_{\substack{i=1 \\ i \neq j}}^{k} (1 - x_i) - (k-1) \prod_{j=1}^{k} (1 - x_j) \\ &= P'(1) - (k-1) P(1) \\ &= 1 + \sum_{j=2}^{k} (-1)^{j-1} (j-1) e_j(x_1, \ldots, x_k), \end{aligned}$$

that is,

$$\sum_{\substack{n_1, \ldots, n_k = 1 \\ \gcd(n_i, n_j) = 1,\, 1 \le i < j \le k}}^{\infty} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} = \prod_p \left(1 + \sum_{j=2}^{k} (-1)^{j-1} (j-1) e_j(p^{-s_1}, \ldots, p^{-s_k}) \right).$$

For $s_1 = \cdots = s_k = 1$ this gives identity (2), representing the asymptotic density of k-tuples with pairwise relatively prime components.

References

[1] M. Aigner, *A Course in Enumeration*, Graduate Texts in Mathematics, Vol. 238, Springer, 2007.

[2] J. Arias de Reyna and R. Heyman, Counting tuples restricted by pairwise coprimality conditions, *J. Integer Sequences* **18** (2015), Article 15.10.4.

[3] J.-Y. Cai and E. Bach, On testing for zero polynomials by a set of points with bounded precision, *Theoret. Comp. Sci.* **296** (2003), 15–25.

[4] J. Christopher, The asymptotic density of some k-dimensional sets, *Amer. Math. Monthly* **63** (1956), 399–401.

[5] H. Delange, On some sets of pairs of positive integers, *J. Number Theory* **1** (1969), 261–279.

[6] J. L. Fernández and P. Fernández, Divisibility properties of random samples of integers, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **115** (2021), no. 1, Paper No. 26.

[7] R. Heyman, Pairwise non-coprimality of triples, arXiv preprint arXiv:1309.5578 [math.NT], May 7 2014. Available at https://arxiv.org/abs/1309.5578.

[8] J. Hu, The probability that random positive integers are k-wise relatively prime, *Int. J. Number Theory* **9** (2013), 1263–1271.

[9] J. Hu, Pairwise relative primality of positive integers, ArXiv preprint, arXiv:1406.3113 [math.NT], June 12 2014. Available at https://arxiv.org/abs/1406.3113.

[10] P. Moree, Counting carefree couples, ArXiv preprint arXiv:math/0510003 [math.NT], September 30 2005. Revision, July 4 2014. Available at https://arxiv.org/abs/math/0510003.

[11] P. Moree, Counting carefree couples, *Mathematics Newsletter* **24** (2014), no. 4, 103–110.

[12] J. E. Nymann, On the probability that k positive integers are relatively prime, *J. Number Theory* **4** (1972), 469–473.

[13] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge Univ. Press, Cambridge, 1997.

[14] L. Tóth, On the asymptotic densities of certain subsets of $\mathbb{N}^k$, *Riv. Mat. Univ. Parma* **(6) 4** (2001), 121–131.

[15] L. Tóth, The probability that k positive integers are pairwise relatively prime, *Fibonacci Quart.* **40** (2002), 13–18.

[16] L. Tóth, Multiplicative arithmetic functions of several variables: a survey, in Th. M. Rassias and P. Pardalos, eds., *Mathematics Without Boundaries, Surveys in Pure Mathematics*, Springer, 2014, pp. 483–514.

[17] L. Tóth, Counting r-tuples of positive integers with k-wise relatively prime components, *J. Number Theory* **166** (2016), 105–116.

[18] N. Ushiroya, Mean-value theorems for multiplicative arithmetic functions of several variables, *Integers* **12** (2012), 989–1002.

2020 Mathematics Subject Classification: Primary 11A25; Secondary 11N25, 11N37, 05A15, 05C07.
Keywords: pairwise coprime integers, pairwise non-coprime integers, asymptotic density, asymptotic formula, multiplicative function of several variables, inclusion-exclusion principle, simple graph, vertex cover.

(Concerned with sequences A229099, A256390, A256391, and A256392.)

Received March 19 2024; revised version received September 11 2024. Published in *Journal of Integer Sequences*, November 29 2024.

Return to Journal of Integer Sequences home page.