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Decomposable Forms Generated by Linear Recurrences

Kálmán Győry
Institute of Mathematics
University of Debrecen
Egyetem tér 1
Debrecen H4032
Hungary
gyory@science.unideb.hu

Attila Pethő
Institute of Computer Sciences
University of Debrecen
Kassai út 26
Debrecen H4028
Hungary
petho.attila@unideb.hu

László Szalay
Department of Mathematics
J. Selye University
Hradná 21
Komárno 945 01
Slovakia
and
Institute of Informatics and Mathematics
University of Sopron
Bajcsy Zs. utca 4
Sopron H9400
Hungary
szalay.laszlo@uni-sopron.hu

Abstract

Consider $k \geq 2$ distinct, linearly independent, homogeneous linear recurrences of order k satisfying the same recurrence relation. We prove that the recurrences are related to a decomposable form of degree k, and there is a general identity with a suitable exponential expression depending on the recurrences. This identity is a common and very broad generalization of several known identities. Further, if the recurrences are integer sequences, then the diophantine equation associated with the decomposable form and the exponential term has infinitely many integer solutions generated by the terms of the recurrences. We describe a method for the complete factorization of the decomposable form. Both the form and its decomposition are explicitly given if $k = 2$, and we present a typical example for $k = 3$. The basic tool we use is the matrix method.

1 Introduction

Let k be an integer with $k \geq 2$. A sequence (G_n) of complex numbers is called a *linear recurrence of order k* if there exist $\gamma_0, \ldots, \gamma_{k-1} \in \mathbb{C}$ such that

$$G_{n+k} = \gamma_{k-1}G_{n+k-1} + \cdots + \gamma_0 G_n \tag{1}$$

holds for all $n \geq 0$, where k is the smallest integer with this property. The polynomial

$$P(X) = X^k - \gamma_{k-1}X^{k-1} - \cdots - \gamma_0 \tag{2}$$

is the *characteristic polynomial* of (G_n). Note that the definition of the order implies that $\gamma_0 \neq 0$.

Dividing (1) by $\gamma_0 \neq 0$ and rearranging the equality, we obtain

$$G_n = -\frac{\gamma_1}{\gamma_0}G_{n+1} - \cdots - \frac{\gamma_{k-1}}{\gamma_0}G_{n+k-1} + \frac{1}{\gamma_0}G_{n+k}.$$

Now applying the substitution $n \to -n-k$ we find

$$G_{-(n+k)} = -\frac{\gamma_1}{\gamma_0}G_{-(n+k-1)} - \cdots - \frac{\gamma_{k-1}}{\gamma_0}G_{-(n+1)} + \frac{1}{\gamma_0}G_{-n},$$

which is a linear recursion for the negative branch of (G_n). The results of the present paper hold for this two-sided infinite sequence, which we can also consider as a number-theoretic function $G_n : \mathbb{Z} \mapsto \mathbb{C}$.

There exist a huge number of relations involving linear recurrences in the corresponding literature, in particular for $k = 2$. Identities with Fibonacci numbers (F_n) (see The On-Line Encyclopedia of Integer Sequences [10], sequence A000045), or Lucas numbers (L_n) (A000032) numbers play a distinguished role among them. The *Fibonacci Quarterly* is devoted mainly to such work. The sequences $(F_n),(L_n)$ have the initial terms $F_0=0,L_0=2,F_1=L_1=1$, and both satisfy the recursion

$$x_n = Ax_{n-1}+Bx_{n-2} \tag{3}$$

with $A=B=1$. From this rich variety of identities we quote three, namely

$$\begin{aligned}
L_n^2-5F_n^2 &= (L_n-\sqrt{5}F_n)(L_n+\sqrt{5}F_n)=4(-1)^n, & (4)\\
F_{n+1}^2-F_nF_{n+1}-F_n^2 &= \left(F_{n+1}-\frac{1+\sqrt{5}}{2}F_n\right)\left(F_{n+1}-\frac{1-\sqrt{5}}{2}F_n\right)=(-1)^n, & (5)\\
F_{n-1}F_{n+1}-F_n^2 &= (-1)^n. & (6)
\end{aligned}$$

All three belong to the folklore and can be easily generalized to all second-order recurrences. Indeed, assume that (G_n), and $(\widehat{G}_n)$ satisfy recursion (3). If their initial terms are $G_0,G_1,\widehat{G}_0=2G_1-AG_0,\widehat{G}_1=AG_1+2BG_0$, then we say that $(\widehat{G}_n)$ is *associated* with (G_n). Here A,B,G_0,G_1 denote complex numbers with the conditions $B\neq 0$ and $|G_0|+|G_1|\neq 0$. Assume that α and β are the (not necessarily different) zeros of the characteristic polynomial $P(X)=X^2-AX-B$. Put $D=A^2+4B$, which is the discriminant of P, and introduce the notation $C_G=G_1^2-AG_0G_1-BG_0^2$. Then the identities

$$\begin{aligned}
\widehat{G}_n^2-DG_n^2 &= (\widehat{G}_n-\sqrt{D}G_n)(\widehat{G}_n+\sqrt{D}G_n)=4C_G(-B)^n, & (7)\\
G_{n+1}^2-AG_nG_{n+1}-BG_n^2 &= (G_{n+1}-\alpha G_n)(G_{n+1}-\beta G_n)=C_G(-B)^n, & (8)\\
G_{n-1}G_{n+1}-G_n^2 &= -C_G(-B)^{n-1} & (9)
\end{aligned}$$

hold, and it is a simple exercise to see that they are direct generalizations of (4)–(6) respectively. Note that the second one is equivalent to the third one by $(G_{n+1}-AG_n)G_{n+1}-BG_n^2=BG_{n-1}G_{n+1}-BG_n^2$, and this phenomenon is obviously true for (5) and (6) too.

Assuming $|B|=1$ and using (7), Pethő [11] characterized the polynomials whose values appear infinitely many times in (G_n). One can reverse identity (5) such that the only positive integer solutions to the equation

$$x^2-xy-y^2=\pm 1$$

are (F_n,F_{n+1}). Jones [5] gave a good overview of this equation and its relation to the solution of Hilbert's tenth problem.

For recurrences of higher order there are only a few and complicated identities available. In particular, we do not know any generalizations of (7). Recently, Craveiro et al. [2] explored a nice generalization of the so-called Cassini identity (6) for recurrences of arbitrary order k. They investigated also analytical and combinatorial properties of their result. Corollary 3 of Theorem 1 of this paper provides a new contribution to this question. In fact, we consider the problem with arbitrary set of initial values apart from singular cases. Corollary 3 via Theorem 1 makes it possible to determine a Cassini-like identity if the basic recurrence of order k is given. However such an identity is expected to be rather compound.

The main result of this article is a common and wide extension of (7), (8), and other identities to $k \geq 2$ for linear recurrences of order k satisfying the same recursive rule, including their connection with decomposable forms and diophantine equations.

2 New results

We shall prove some general results. Note that the basic field we work in is the set $\mathbb{C}$ of complex numbers, but the machinery works even for arbitrary fields.

Theorem 1. *Let $(G_n^{(j)}), j = 1, \ldots, k$ be linear recursive sequences of complex numbers of order k with the same characteristic polynomial (2) having constant term $-\gamma_0 \neq 0$. If the sequences are $\mathbb{C}$-linearly independent, then there exists a homogeneous polynomial $F \in \mathbb{C}[X_1, X_2, \ldots, X_k]$ of degree k such that*

$$F(G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(k)}) = ((-1)^{k+1}\gamma_0)^n$$

holds for all $n \in \mathbb{Z}$.

A homogeneous polynomial $Q \in \mathbb{C}[X_1, X_2, \ldots, X_k]$ is called *decomposable* if it splits completely into linear factors. Hence a decomposable form is necessarily homogeneous, but for $k \geq 3$ the converse is not true in general. The next theorem is a remarkable completion of Theorem 1.

Theorem 2. *Under the assumptions of Theorem 1 the homogeneous polynomial F is decomposable.*

Clearly, Theorem 2 is a continuation of Theorem 1, and we could have presented them together. However their proofs differ significantly, which explains their separation.

We emphasize that the proofs are constructive, and following their arguments one can compute the polynomial F and its decomposition into linear factors. Unfortunately, F has no nice general form if $k \geq 3$, so we will omit its explicit constitution. On the other hand, F is given precisely when $k = 2$ (see Theorem 9 and the description after). Furthermore, we will show some representative examples in Section 6.

A simple consequence of the main theorems is the following generalization of (8).

Corollary 3. *Let (G_n) be a linear recursive sequence of complex numbers of order k satisfying (1). If the vectors $\bar{g}_j = (G_j, G_{j+1}, \ldots, G_{j+k-1}), j = 0, \ldots, k-1$ are $\mathbb{C}$-linearly independent, then there exists a decomposable form $F \in \mathbb{C}[X_1, X_2, \ldots, X_k]$ of degree k such that*

$$F(G_n, G_{n+1}, \ldots, G_{n+k-1}) = ((-1)^{k+1}\gamma_0)^n$$

holds for all $n \in \mathbb{Z}$.

The linear recursive sequences with initial values $G_0 = \cdots = G_{k-2} = 0, G_{k-1} = 1$ satisfy always the assumption of this corollary, hence the statement too. The k-generalized Fibonacci sequences $(F_n^{(k)})$ (or, in other words, k-step Fibonacci numbers), which are defined by the initial values $F_0^{(k)} = \cdots = F_{k-2}^{(k)} = 0, F_{k-1}^{(k)} = 1$ and by the recursion $F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)}$ are important examples. Here the upper index (k) in $F_n^{(k)}$ means traditionally the parameter k, while in Theorem 1 the upper index (j) denotes the jth from the k given sequences. Clearly, the case $k = 2$ is the Fibonacci sequence.

In fact, the sequence (G_n) is a complex-valued number-theoretic function. Another direct consequence of Theorem 1 is as follows. The conditions of Theorem 1 are valid here, too.

Corollary 4. *Let $(G_n^{(j)}), j = 1, \dots, k$ be linear recursive sequences of complex numbers of order k with the same characteristic polynomial (2). Then these sequences and the sequence (γ_0^n) are algebraically dependent.*

It is possible to extend (G_n) to a complex function $G(z)$ such that $G_n = G(n)$ for all $n \in \mathbb{Z}$. Indeed, assume that $P(X)$ in (2) has the factorization

$$P(X) = (X - \alpha_1)^{m_1}(X - \alpha_2)^{m_2} \cdots (X - \alpha_h)^{m_h},$$

where $\alpha_1, \dots, \alpha_h$ denote pairwise distinct complex numbers and m_j, $j = 1, \dots, h$ are positive integers. Then, see e.g., [13], there exist polynomials $g_j \in \mathbb{C}[X]$ of degree at most $m_j - 1$ such that

$$G_n = g_1(n)\alpha_1^n + g_2(n)\alpha_2^n + \cdots + g_h(n)\alpha_h^n$$

holds for all $n \in \mathbb{Z}$. Hence $G(z) = g_1(z)\alpha_1^z + g_2(z)\alpha_2^z + \cdots + g_h(z)\alpha_h^z$ is a complex function with $G(n) = G_n$ for all $n \in \mathbb{Z}$. For example, the extensions of the Fibonacci and the Lucas sequences are the complex functions

$$F(z) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^z - \left(\frac{1-\sqrt{5}}{2}\right)^z\right), \quad L(z) = \left(\frac{1+\sqrt{5}}{2}\right)^z + \left(\frac{1-\sqrt{5}}{2}\right)^z,$$

respectively. One can easily verify that identities (4)–(6) remain true if we replace n, F_n, L_n, $(-1)^n$ by z, $F(z)$, $L(z)$, $(-1)^z$, respectively. An interpretation of the generalized identities is the fact that the corresponding complex functions are algebraically dependent.

It is well known, see e.g., [7, Chapter 10.2], that a function $f(z)$ satisfies the linear differential equation with characteristic polynomial $P(X)$ if and only if it is identical with one of the above defined $G(z)$. Our next theorem is a generalization of Corollary 4.

Theorem 5. *Let $P(X) \in \mathbb{C}[X]$ be a polynomial of degree k with simple zeros, and assume $P(0) \neq 0$. Let $k_0 \geq k$ and denote by $G_j(z)$, $1 \leq j \leq k_0$ pairwise different solutions of a homogeneous linear differential equation with characteristic polynomial $P(X)$. Then $G_j(z)$, $1 \leq j \leq k_0$ and the function $((-1)^k P(0))^z$ are algebraically dependent.*

As we noted, Theorem 2 and its constructive proof have a remarkable consequence in the theory of diophantine equations. This is presented as Theorem 6 in Section 3, where it is more suitable to formulate the precise statement. Indeed, if the $(G_n^{(j)})$'s are integer sequences, then it is possible to have infinitely many integer solutions to the polynomial-exponential diophantine equation

$$\tilde{F}(x_1, x_2, \ldots, x_k) = pq^n \tag{10}$$

in integers $x_1, x_2, \ldots, x_k$ and $n \geq 0$, where $\tilde{F}$ is a decomposable form of degree k depending on the parameters of the sequences, and p, q are suitable integers (see (24) below). If $\tilde{F}$ is irreducible over $\mathbb{Q}[X_1, \ldots, X_k]$, then we may assume that $\tilde{F}$ is, up to a constant factor, a full norm form. It follows from a result of Borevich and Safarevich [1] that if the splitting field of $\tilde{F}$ differs from $\mathbb{Q}$ and from the imaginary quadratic number fields, then (10) has infinitely many solutions for $q = 1$ and for some integer p. More generally, by a celebrated theorem of Schmidt [12] on norm form equations, the set of solutions is a union of a finite set and finitely many families of solutions. For a further generalization for arbitrary decomposable form equations see Győry [4]. The families mentioned are sums of products of powers of appropriate algebraic numbers. Fixing all but one exponent and letting the remaining exponents run through non-negative integers we see that each family includes linear recursive sequences.

Our result shows that (10) may have infinitely many solutions in the case when $\tilde{F}$ is reducible over $\mathbb{Q}[X_1, \ldots, X_k]$, too.

In our paper, we apply a linear algebraic approach. We were also motivated to contribute to the development of the so-called matrix method often used in the theory of linear recurrences. At the early 80's Gould [3] presented a survey on the Q-matrices, and he refers to Simson who first gave the formula (6). This identity, together with its generalization (9) is an easy consequence of our Theorem 9 in Section 6 with the notation $H_n = G_{n+1}$ (for H_n again see Theorem 9). Theorem 9 is a common extension of identities (7)–(9), and provides the background for the case of $k \geq 2$ linear recurrences of order k in general appearing in Theorem 1.

We mention that recently Craviero et al. [2] have given a generalization of Cassini identity for the k-generalized Fibonacci numbers with specific initial vector set. The basic approach of [2] coincides with the idea of Lemma 2.1 of [9], and we exploit the advantages of this argument in the present paper.

The paper is organized as follows. In Section 3, we give two proofs for Theorem 1. In the next section, we prove Theorem 2, while Section 5 is devoted to the proof of corollaries 3 and 4 and of Theorem 5. Section 6 specializes the results of Theorem 1 and 2 to the cases $k = 2$ and $k = 3$. In Subsection 6.1, we present explicit formula in full generality if $k = 2$, while in Subsection 6.2, we compute the appropriate formula only for three given third order recursive sequence. Other examples for larger k values might be easily presented by following the method of the proofs.

3 The homogeneous polynomials

Before starting the first proof of Theorem 1, we introduce some notation and summarize well known facts.

Recall that $k \geq 2$ is an integer, and $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ denote arbitrary complex numbers. Consider the set of recurrent sequences

$$\Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})} = \{(x_n)_{n\in\mathbb{Z}} \in \mathbb{C}^\infty \mid x_n = \gamma_{k-1}x_{n-1} + \gamma_{k-2}x_{n-2} + \cdots + \gamma_0 x_{n-k}\}\,. \tag{11}$$

Clearly, sequences (11) satisfy a common recurrence rule, but they differ in their initial values. They constitute a $\mathbb{C}$-vector space with respect to coordinate-wise addition and multiplication by scalar. The dimension of the vector space is k if and only if $\gamma_0 \neq 0$. To be able to apply linear algebraic tools we assume $\gamma_0 \neq 0$ in the sequel. Throughout this paper we deal with vector spaces over the field of complex numbers $\mathbb{C}$, therefore we do not always mention the ground field.

3.1 First proof of Theorem 1

Proof. Assume that the recurrences

$$(G_n^{(1)}), (G_n^{(2)}), \ldots, (G_n^{(k)}) \in \Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})} \tag{12}$$

are $\mathbb{C}$-linearly independent. Then, equivalently, the vectors

$$\bar{g}_0 = \begin{bmatrix} G_0^{(1)} \\ \vdots \\ G_0^{(k)} \end{bmatrix},\ \bar{g}_1 = \begin{bmatrix} G_1^{(1)} \\ \vdots \\ G_1^{(k)} \end{bmatrix}, \ldots,\ \bar{g}_{k-1} = \begin{bmatrix} G_{k-1}^{(1)} \\ \vdots \\ G_{k-1}^{(k)} \end{bmatrix} \tag{13}$$

are also linearly independent. Define the matrices $\mathbf{G}, \mathbf{G}^\star \in \mathbb{C}^{k\times k}$ as follows. The column vectors of $\mathbf{G}$ are $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_{k-1}$, respectively, while $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_k$ admit the column vectors of $\mathbf{G}^\star$ in this order. The k recurrences in (12), together with their initial values build up the vector recurrence

$$\bar{g}_n = \gamma_{k-1}\bar{g}_{n-1} + \gamma_{k-2}\bar{g}_{n-2} + \cdots + \gamma_0\bar{g}_{n-k}, \quad n \geq k \tag{14}$$

with initial vectors (13). In particular, (14) provides the last column vector $\bar{g}_k$ of $\mathbf{G}^\star$. The linear independence of the column vectors of $\mathbf{G}$ guarantees that

$$\Delta = \det(\mathbf{G}) \neq 0. \tag{15}$$

Let $\bar{r}_i$ denote the ith column vector of the transposition $\mathbf{G}^\top$ of $\mathbf{G}$ for each $i = 1, 2, \ldots, k$. That is $\bar{r}_i = [G_0^{(i)}, G_1^{(i)}, \ldots, G_{k-1}^{(i)}]^\top$, the entries are exactly the initial values of the ith sequence we fixed in (12). Clearly, the vectors

$$\bar{r}_1, \bar{r}_2 \ldots, \bar{r}_k \tag{16}$$

form a basis of the vector space $\mathbb{C}^k$.

Put

$$\mathbf{M} = \mathbf{G}^\star \mathbf{G}^{-1} = \frac{1}{\Delta} \cdot \mathbf{G}^\star \cdot \operatorname{adj}(\mathbf{G}).$$

The extension of Lemma 2.1 of [9] from $\mathbb{R}$ to $\mathbb{C}$ shows that the vector sequence $(\bar{g}_n)$ generated by the initial vectors (13) and by the vector recurrence (14) satisfies

$$\bar{g}_{n+1} = \mathbf{M}\bar{g}_n, \qquad n \geq 0. \tag{17}$$

The proof of this lemma even shows that the characteristic polynomial of matrix $\mathbf{M}$ is

$$k_{\mathbf{M}}(x) = \det(x\mathbf{I} - \mathbf{M}) = x^k - \gamma_{k-1}x^{k-1} - \cdots - \gamma_1 x - \gamma_0, \tag{18}$$

which is also the characteristic polynomial of each recurrence of $\Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})}$ ($\mathbf{I}$ denotes the $k \times k$ unit matrix). Thus $k_{\mathbf{M}}(0) = \det(-\mathbf{M}) = -\gamma_0$, and then $\det(\mathbf{M}) = (-1)^k(-\gamma_0)$, which is denoted in the sequel by δ. Hence

$$\det(\mathbf{M}^n) = (\det(\mathbf{M}))^n = \delta^n. \tag{19}$$

This observation gives later the right-hand side of the principal identity of this article (with a suitable homogeneous polynomial of degree k on the left-hand side).

Now we turn our attention to the matrices

$$\mathbf{M}^n = \left[m_{i,j}^{(n)}\right]_{i,j=1,2,\ldots,k}, \qquad n \in \mathbb{N}.$$

Since $\mathbf{M}^n = \gamma_{k-1}\mathbf{M}^{n-1} + \gamma_{k-2}\mathbf{M}^{n-2} + \cdots + \gamma_0\mathbf{M}^{n-k}$ holds for $n \geq k$, the same recurrence rule is valid for each element-wise sequence $(m_{i,j}^{(n)})$ of the matrix sequence $(\mathbf{M}^n)$. That is,

$$m_{i,j}^{(n)} = \gamma_{k-1}m_{i,j}^{(n-1)} + \gamma_{k-2}m_{i,j}^{(n-2)} + \cdots + \gamma_0 m_{i,j}^{(n-k)}.$$

Such a sequence is determined by the initial values $m_{i,j}^{(0)}, m_{i,j}^{(1)}, \ldots, m_{i,j}^{(k-1)}$. Collect them into the initial column vector $\overline{m}_{i,j} \in \mathbb{C}^k$ (see (20)), which can be given as a linear combination of the basis vectors (16). More precisely, there exist uniquely determined coordinates $c_{i,j}^{(u)} \in \mathbb{C}$, $u = 1, 2, \ldots, k$, such that

$$\overline{m}_{i,j} = \begin{bmatrix} m_{i,j}^{(0)} \\ m_{i,j}^{(1)} \\ \vdots \\ m_{i,j}^{(k-1)} \end{bmatrix} = \sum_{u=1}^{k} c_{i,j}^{(u)} \bar{r}_u. \tag{20}$$

This is a system of k linear equations in the k unknowns $c_{i,j}^{(1)}, c_{i,j}^{(2)}, \ldots, c_{i,j}^{(k)}$ (if i, j are fixed), and the system can be solved, for instance, by Cramer's rule. Assume that $\mathbf{G}_u$ is the matrix derived by replacing the uth column vector of $\mathbf{G}^\top$ by the vector $\overline{m}_{i,j}$. Then, using $\det(\mathbf{M}^\top) = \det(\mathbf{M}) = \Delta$, clearly

$$c_{i,j}^{(u)} = \frac{\det(\mathbf{G}_u)}{\Delta}. \tag{21}$$

Since the initial values determine a complete sequence of $\Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})}$, the coefficients (21) appearing in (20) descend for the whole sequence $(m_{i,j}^{(n)})$. Thus the main consequence of the previous arguments is the equality

$$m_{i,j}^{(n)} = \sum_{u=1}^{k} c_{i,j}^{(u)} G_n^{(u)}.$$

In this manner, we are able to represent any entry of the matrix $\mathbf{M}^n$ as a linear combination of the nth terms of the linear recurrences $(G_n^{(1)}), (G_n^{(2)}), \ldots, (G_n^{(k)}) \in \Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})}$. Subsequently, the determinant of $\mathbf{M}^n$ is a homogeneous polynomial of degree k in $G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(k)}$. We let F denote this form. Combining it with (19) we obtain

$$F(G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(k)}) = \delta^n, \tag{22}$$

and the proof is complete. □

Now assume that the coefficients γ_i, and the initial terms of the sequences $G_n^{(j)}$ are integers. Then $(G_n^{(j)})$ is a sequence of integers (j is fixed). Multiply both sides of (22) by Δ^k, because (21) is now a rational number with denominator dividing Δ. Then $\tilde{F} = \Delta^k F$ has integer coefficients. Consequently, the polynomial-exponential diophantine equation

$$\tilde{F}(x_1, x_2, \ldots, x_k) = \Delta^k \delta^n \tag{23}$$

possesses infinitely many integer solutions in integers $x_1, x_2, \ldots, x_k$ and n. Now we record the results in

Theorem 6. *Let (12) be recursive sequences of integers with initial values (13) such that (15) holds. Then for $n \geq 0$*

$$(x_1, x_2, \ldots, x_k) = (G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(k)})$$

is the solution to the diophantine equation

$$\tilde{F}(x_1, x_2, \ldots, x_k) = \Delta^k \delta^n, \tag{24}$$

where the coefficients of the homogeneous form $\tilde{F}$ depend on the parameters of the sequences.

Proof. The proof is given before the enunciation of the theorem. □

Note that the proof is constructive, in the sense that following the method we can compute $\tilde{F}$, Δ, and δ if the initial values and the recurrence rule is fixed. This will be illustrated later, in Section 6 for binary recurrences in general, and for a triple of ternary recurrences.

3.2 Alternative proof of Theorem 1

Proof. Here we provide an alternative proof for the existence of the form F on the left-hand side of (22). The notation is taken from the preceding parts of Section 3. Observe first that

$$\mathbf{G}^* = \mathbf{G}\mathbf{T}$$

with the matrix

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \gamma_0 \\ 1 & 0 & \cdots & 0 & \gamma_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma_{k-2} \\ 0 & 0 & \cdots & 1 & \gamma_{k-1} \end{bmatrix}.$$

Thus we have

$$\mathbf{M} = \mathbf{G}^*\mathbf{G}^{-1} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1}.$$

This matrix equality immediately implies

- $\det(\mathbf{M}) = \det(\mathbf{T}) = (-1)^{k-1}\gamma_0$, and
- $\mathbf{M}^n = \mathbf{G}\mathbf{T}^n\mathbf{G}^{-1}$ for $n \geq 0$.

Here $\mathbf{G}\mathbf{T}^n$ is a matrix with the column vectors $\bar{g}_n, \bar{g}_{n+1}, \ldots, \bar{g}_{n+k-1}$. On the other hand, from (17) we know $\bar{g}_{n+j} = \mathbf{M}^j\bar{g}_n$. Hence all entries of $\mathbf{G}\mathbf{T}^n$ are linear combinations of the terms $G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(k)}$, where the coefficients are the elements of $\mathbf{M}^j, j = 0, \ldots, k-1$, and they are independent from n. Subsequently, the determinant of $\mathbf{G}\mathbf{T}^n$ is a signed sum of the products of k linear forms in $G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(k)}$ i.e., a homogeneous form of degree k. Multiplying this by $\det(\mathbf{G}^{-1})$, which is the determinant of a constant matrix, we obtain that $\det(\mathbf{M}^n)$ is also a homogeneous form of degree k. □

4 Decomposability of F

This section is devoted to the study and proof of the decomposability of the homogeneous polynomial F provided in Section 3. Recall that $F(x_1, x_2, \ldots, x_k) = \tilde{F}(x_1, x_2, \ldots, x_k)/\Delta^k$ (see (22) and (23)).

Proof of Theorem 2. The proof consists of two main parts. In the first part, we show that a specific homogeneous polynomial, say F_S, is decomposable if the initial k sequences from $\Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})}$ are chosen advantageously. Then, in the second part, we find a connection between these favorable sequences and arbitrary packages of k sequences.

Part 1. Recall that the characteristic polynomial $k_{\mathbf{M}}(x)$ of the family $\Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})}$ of homogeneous linear recurrences (11) we investigate is (18). Assume that it has the decomposition

$$k_{\mathbf{M}}(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_h)^{m_h} \tag{25}$$

into linear factors, where $\alpha_i \in \mathbb{C}$ $(i = 1, 2, \ldots, h)$ are distinct, and $m_1 + m_2 + \cdots + m_h = k$. For each pair (i, j), where $i \in \{1, 2, \ldots, h\}$ and $j \in \{0, 1, \ldots, m_i - 1\}$, consider the sequences

$$S_n^{(i,j)} = n^j \alpha_i^n, \quad (n \geq 0). \tag{26}$$

Note that the range for j depends on the value of i, but we do not indicate this fact in the notation.

The sequences (26) belong to $\Gamma_k^{(\gamma_0, \gamma_1, \ldots, \gamma_{k-1})}$; this fact is based on the following two observations, or simply we may refer to the theory of difference equations, see e.g., the classical book of Milne-Thomson [8, Chapter XIII]. Firstly, if the multiplicity of an α_i is m_i, then α_i is also a zero of the derivatives $k_{\mathbf{M}}^{(j)}(x)$ for $j = 0, 1, \ldots, m_i - 1$. Secondly, from the theory of Stirling number of second kind we know that $n^j = \sum_{v=0}^{j} \left\{ {j \atop v} \right\} (n)_v$, where $\left\{ {j \atop v} \right\}$ denotes a Stirling numbers of second kind, and $(n)_v = n(n-1) \cdots (n - v + 1)$ is a falling factorial.

Another important argument is that the sequences in (26) are $\mathbb{C}$-linearly independent, consequently they form a basis of $\Gamma_k^{(\gamma_0, \gamma_1, \ldots, \gamma_{k-1})}$. This follows again from [8, Chapter XIII]. (A similar result is given in [6].)

Analogously to $\mathbf{G}$, we define the matrix $\mathbf{G}_S$ by the initial values of the sequences $S_n^{(i,j)}$ such that the first row contains the initial values of $S_n^{(1,0)}$, etc. The matrix $\mathbf{G}_S$ can be split into horizontal stripes such that stripe $i \in \{1, 2, \ldots, h\}$ contains m_i rows. The enumeration of stripes begins at the top of the matrix, and the entries of stripe i are given as follows:

$$\mathbf{G}_S = \left[\begin{array}{ccccccc}
 & & & \vdots & & & \\
\hdashline
 & & & \vdots & & & \\
\hdashline
1 & \alpha_i & \alpha_i^2 & \alpha_i^3 & \ldots & \ldots & \alpha_i^{k-1} \\
0 & \alpha_i & 2\alpha_i^2 & 3\alpha_i^3 & \ldots & \ldots & (k-1)\alpha_i^{k-1} \\
0 & \alpha_i & 2^2\alpha_i^2 & 3^2\alpha_i^3 & \ldots & \ldots & (k-1)^2\alpha_i^{k-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\
0 & \alpha_i & 2^{m_i-1}\alpha_i^2 & 3^{m_i-1}\alpha_i^3 & \ldots & \ldots & (k-1)^{m_i-1}\alpha_i^{k-1} \\
\hdashline
 & & & \vdots & & & \\
\hdashline
 & & & \vdots & & &
\end{array} \right].$$

Let $M_i = m_1 + m_2 + \cdots + m_{i-1}$. The numbering of rows of stripe i are $M_i + (j + 1)$, where $j = 0, \ldots, m_i - 1$. The matrix $\mathbf{G}_S^\star$ is analogous to $\mathbf{G}^\star$ too. We define the block diagonal matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{B}_h \end{bmatrix} \in \mathbb{C}^{k \times k} \text{ with } \mathbf{B}_i = \begin{bmatrix} \alpha_i & 0 & \cdots & 0 & \cdots & 0 \\ \alpha_i & \alpha_i & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_i & \cdots & \cdots & \alpha_i & \cdots & 0 \\ \vdots & \vdots & b_{u,v}^{(i)} & \vdots & \ddots & \vdots \\ \alpha_i & \cdots & \cdots & \cdots & \cdots & \alpha_i \end{bmatrix} \in \mathbb{C}^{m_i \times m_i}$$

for $i = 1, 2, \dots, h$ such that the general term $b^{(i)}_{u,v} = \binom{u}{v}\alpha_i$ is the entry of the $(u+1)$th row and $(v+1)$th column of $\mathbf{B}_i$, with the conditions $0 \le v \le u \le m_i - 1$. Here the minors $\mathbf{O}$ are suitable zero matrices. Since $\mathbf{B}_i$ is a lower diagonal matrix then $\mathbf{B}$ is so.

Using the notation above we will show that

$$\mathbf{G}^*_S = \mathbf{B}\mathbf{G}_S.$$

Note that block $\mathbf{B}_i$ has influence only on stripe i of $\mathbf{G}_S$ when carrying out the matrix multiplication $\mathbf{B}\mathbf{G}_S$. In stripe i we have exactly the rows M_i+1+u, where $u = 0, 1, \dots, m_i-1$. Our method is to calculate the dot product[1] of row $M_i + 1 + u$ from $\mathbf{B}$ and column ℓ from $\mathbf{G}_S$, where $\ell = 1, 2, \dots, k$. This is given in detail by

$$\begin{bmatrix} 0 & \cdots & 0 & \binom{u}{0}\alpha_i & \binom{u}{1}\alpha_i & \cdots & \binom{u}{u}\alpha_i & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \hdashline \alpha_i^{\ell-1} \\ (\ell-1)\alpha_i^{\ell-1} \\ \vdots \\ (\ell-1)^u\alpha_i^{\ell-1} \\ \vdots \\ (\ell-1)^{m_i-1}\alpha_i^{\ell-1} \\ \hdashline \vdots \end{bmatrix}.$$

Hence the dot product equals

$$\binom{u}{0}\alpha_i^\ell + \binom{u}{1}(\ell-1)\alpha_i^\ell + \cdots + \binom{u}{u}(\ell-1)^u\alpha_i^\ell = ((l-1)+1)^u\,\alpha_i^\ell = \ell^u\alpha_i^\ell = S^{(i,u)}_\ell.$$

Recall that $S^{(i,u)}_\ell$ is the general term of the matrix $\mathbf{G}^*_S$ such that it is the element of the $(u+1)$th row of stripe i in the column ℓ ($\ell = 1, 2, \dots, k$).

Subsequently, we have

$$\mathbf{M}_S = \mathbf{G}^*_S\mathbf{G}_S^{-1} = (\mathbf{B}\mathbf{G}_S)\,\mathbf{G}_S^{-1} = \mathbf{B},$$

and then $\mathbf{M}^n_S = \mathbf{B}^n$ follows. We know that $\mathbf{B}$ is a lower diagonal matrix. Thus $\det(\mathbf{B})$ is the product of the elements lying in the main diagonal, i.e.,

$$\det(\mathbf{B}) = \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_h^{m_h}.$$

Finally,

$$\det(\mathbf{M}^n_S) = (\det(\mathbf{B}))^n = \alpha_1^{m_1 n}\alpha_2^{m_2 n}\cdots\alpha_h^{m_h n} = \left(S^{(1,0)}_n\right)^{m_1}\left(S^{(2,0)}_n\right)^{m_2}\cdots\left(S^{(h,0)}_n\right)^{m_h}.$$

[1] The dot product of the vectors $\bar{\mu} = [\mu_1, \dots, \mu_k]$ and $\bar{\nu}^\top = [\nu_1, \dots, \nu_k]^\top$ is the complex number $\sum_{j=1}^k \mu_j\nu_j$.

So we have found that $\det(\mathbf{M}_S^n)$ is a corresponding value of the decomposable form

$$F_S(y_1, y_2, \ldots, y_h) = y_1^{m_1} y_2^{m_2} \cdots y_h^{m_h}$$

at the point $\left((S_n^{(1,0)}), (S_n^{(2,0)}), \cdots, (S_n^{(h,0)})\right)$. Thus the assertion of the theorem holds for this particular choice of the recurrences.

Part 2. Recall that sequences (12) form a basis in $\Gamma_k^{(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})}$ because $\Delta \neq 0$. Hence there exists a unique matrix $\mathbf{A} = [a_{i,j}] \in \mathbb{C}^{h\times k}$ such that

$$\begin{bmatrix} S_n^{(1,0)} & S_n^{(2,0)} & \cdots & S_n^{(h,0)} \end{bmatrix}^\top = \mathbf{A} \begin{bmatrix} G_n^{(1)} & G_n^{(2)} & \cdots & G_n^{(k)} \end{bmatrix}^\top. \tag{27}$$

Now consider the substitution $\bar{y} = [y_1, y_2, \ldots, y_h]^\top = \mathbf{A}[x_1, x_2, \ldots, x_k]^T = \mathbf{A}\bar{x}$. Denoting by $\bar{a}_i$ the ith row vector of $\mathbf{A}$ we compute the dot product

$$y_i = \bar{a}_i[x_1, x_2, \ldots, x_k]^\top$$

for every $i = 1, 2, \ldots, h$. Now we can define $F(x_1, \ldots, x_k) = F(\bar{x})$ as follows:

$$F_S(\bar{y}) = y_1^{m_1} y_2^{m_2} \cdots y_h^{m_h} = \prod_{i=1}^{h} \left(\bar{a}_i[x_1, x_2 \ldots, x_k]^\top\right)^{m_i} = F(\bar{x}).$$

With the result from Part 1 we know that F is a decomposable form of degree k with the property

$$F(G_n^{(1)}, \ldots, G_n^{(k)}) = ((-1)^k k_{\mathbf{M}}(0))^n = \delta^n, \quad (n = 0, 1, \ldots).$$

Thus the proof is complete. □

Remark 7. If every zero of the characteristic polynomial $k_{\mathbf{M}}(x)$ is simple (i.e., $h = k$), then the matrix $\mathbf{G}_S$ simplifies to

$$\mathbf{G}_S = \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{k-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} \end{bmatrix} = \mathbf{AG}$$

(the second equality is implied by (27)). Consequently, $\mathbf{A} = \mathbf{G}_S\mathbf{G}^{-1}$, and then

$$\bar{y} = \mathbf{G}_S\mathbf{G}^{-1}\bar{x}.$$

Finally, we obtain $F_S(\bar{y}) = F_S(\mathbf{G}_S\mathbf{G}^{-1}\bar{x}) = F(\bar{x})$.

We conclude this section by giving a general example to demonstrate the power of our results.

Example 8. Consider the sequences of (12) such that the characteristic polynomial has factorization (25). Assume that we have the specific initial vectors

$$\bar{g}_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{g}_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \ \bar{g}_{k-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{28}$$

Using the advantageous properties of this orthonormal basis, we can easily show that

$$S_n^{(j,0)} = 1 \cdot G_n^{(1)} + \alpha_j \cdot G_n^{(2)} + \cdots + \alpha_j^{k-1} \cdot G_n^{(k)}, \quad (j = 1, 2, \dots, h).$$

Consequently,

$$y_j = x_1 + \alpha_j x_2 + \cdots + \alpha_j^{k-1} x_k, \quad (j = 1, 2, \dots, h),$$

and then

$$F(x_1, x_2, \dots, x_k) = \prod_{i=1}^{h} (x_1 + \alpha_j x_2 + \cdots + \alpha_j^{k-1} x_k)^{m_j}$$

follows. Hence the complete factorization of the form F is determined. Note that the simplicity depends on the features of the orthonormal basis (28).

5 Proof of the corollaries and Theorem 5

Proof of Corollary 3. The entries of the vectors $\bar{g}_j$, $j = 0, \dots, k-1$ are the initial values of the sequences (G_{n+j}), $j = 0, \dots, k-1$. At the beginning of the proof of Theorem 1 we pointed out that the linear independence of the vectors $\bar{g}_j$ and the linear independence of the sequences (G_{n+j}) is equivalent. Thus the sequences (G_{n+j}), $j = 0, \dots, k-1$ are linearly independent and Theorem 1 implies the existence of a homogeneous form F, while Theorem 2 justifies the decomposability of F. □

Proof of Corollary 4. If the sequences $(G_n^{(j)})$, $j = 1, \dots, k$ are linearly dependent, then upon enlarging this set with the sequence (γ_0^n), the linear dependent property remains true, and this is a very special algebraic dependence. Otherwise, if $(G_n^{(j)}), j = 1, \dots, k$ are linearly independent, then Theorem 1 (more precisely equation (22)) verifies the algebraic dependence. □

Proof of Theorem 5. Assume that $P(X) = z_k X^k + z_{k-1} X^{k-1} + \cdots + z_0$, where $z_0 \neq 0$. The differential equation with characteristic polynomial $P(X)$ has the form

$$z_k f(z)^{(k)} + z_{k-1} f(z)^{(k-1)} + \cdots + z_0 = 0. \tag{29}$$

Here $f^{(\ell)}$ denotes the ℓth derivative of f. The pair-wisely distinct zeros of $P(X)$ are denoted by $\alpha_1, \alpha_2, \dots, \alpha_k$. The function $f(z)$ is a solution of (29) if and only if it has the form

$$f(z) = f_1 \alpha_1^z + f_2 \alpha_2^z + \cdots + f_k \alpha_k^z,$$

with complex numbers $f_1, f_2, \dots, f_k$.

The set of solutions of (29) form a $\mathbb{C}$-vector space with respect to the addition of functions and multiplication with complex numbers. We let $S(P)$ denote this set. The functions $\alpha_1^z, \alpha_2^z, \dots, \alpha_k^z$ are linearly independent, thus the dimension of $S(P)$ is k.

Assume that $G_j(z), j \le k_0$ are pairwise different solutions of (29). If they are $\mathbb{C}$-linearly dependent, which always holds when $k_0 > k$, then the same is true if we enlarge their set with the function $((-1)^k P(0))^z$. It remains to examine the case when $k_0 = k$ and the functions $G_j(z), j \le k_0$ are linearly independent. Then they form a basis of $S(P)$. The functions $\alpha_1^z, \alpha_2^z, \dots, \alpha_k^z$ belong to $S(P)$, thus there exist complex numbers $a_{ij}, 1 \le i, j \le k$ such that

$$\alpha_i^z = \sum_{j=1}^{k} a_{ij} G_j(z). \tag{30}$$

On the other hand, we have

$$\alpha_1^z \alpha_2^z \cdots \alpha_k^z = (\alpha_1 \alpha_2 \cdots \alpha_k)^z = ((-1)^k P(0))^z.$$

Inserting the linear relations of Eq. (30) here we obtain the statement. □

6 Binary and ternary recurrences

6.1 Binary recurrences, a general identity

In this subsection, we present the general identity (33) below with two binary recurrences satisfying the same recurrence relation. This is a special case of (22). On the other hand it provides a common generalization of (7)–(9). At the end of this subsection a few examples will illustrate (33). In the computations, we will follow the arguments of the previous sections.

Assume $k = 2$. For simplicity suppose that the two recurrent sequences are (G_n) and (H_n), their initial values are G_0, G_1 and H_0, H_1 respectively. Furthermore set $\gamma_1 = A$ and $\gamma_0 = B$. Then the two sequences above belong to

$$\Gamma_2^{(B,A)} = \{(x_n)_{n \in \mathbb{Z}} \in \mathbb{C}^\infty \mid x_n = Ax_{n-1} + Bx_{n-2},\ n \ge 2\}.$$

Now

$$\mathbf{G} = \begin{bmatrix} G_0 & G_1 \\ H_0 & H_1 \end{bmatrix}, \qquad \mathbf{G}^\star = \begin{bmatrix} G_1 & AG_1 + BG_0 \\ H_1 & AH_1 + BH_0 \end{bmatrix},$$

and assume that $\Delta = \det(\mathbf{G}) = G_0 H_1 - G_1 H_0 \ne 0$. Then

$$\mathbf{G}^{-1} = \frac{1}{\Delta} \begin{bmatrix} H_1 & -G_1 \\ -H_0 & G_0 \end{bmatrix}.$$

We introduce the notation $C_G = G_1^2 - AG_0G_1 - BG_0^2$ (also see before the equations (7)–(9)), and analogously $C_H = H_1^2 - AH_0H_1 - BH_0^2$. We further define $E_{10} = G_1H_1 - AG_1H_0 - BG_0H_0$, and $E_{01} = G_1H_1 - AG_0H_1 - BG_0H_0$. One can easily see that

$$\mathbf{M} = \mathbf{G}^{\star}\mathbf{G}^{-1} = \frac{1}{\Delta}\left[\begin{array}{cc} E_{10} & -C_G \\ C_H & -E_{01} \end{array}\right]. \tag{31}$$

Clearly, $k_{\mathbf{M}}(x) = x^2 - Ax - B = (x-\alpha)(x-\beta)$ and $\det(\mathbf{M}) = -B$. Obviously, $\det(\mathbf{M}^n) = (-B)^n$, $A = \alpha + \beta$ and $B = -\alpha\beta$. (Note that α and β are not necessarily distinct.)

The element sequences of the powers of matrix $\mathbf{M}$ satisfy

$$m_{i,j}^{(n)} = Am_{i,j}^{(n-1)} + Bm_{i,j}^{(n-2)}, \qquad (n \geq 2;\ 1 \leq i,j \leq 2).$$

The initial values are clear from $\mathbf{M}^0 = \mathbf{I}$ and from $\mathbf{M}$ (here $\mathbf{I}$ is the 2×2 unit matrix). Now

$$\left[\begin{array}{c} m_{i,j}^{(0)} \\ m_{i,j}^{(1)} \end{array}\right] = c_{i,j}^{(1)}\left[\begin{array}{c} G_0 \\ G_1 \end{array}\right] + c_{i,j}^{(2)}\left[\begin{array}{c} H_0 \\ H_1 \end{array}\right], \qquad (1 \leq i,j \leq 2). \tag{32}$$

Once we have the solutions to (32) in $c_{i,j}^{(1)}$ and $c_{i,j}^{(2)}$, then $m_{i,j}^{(n)} = c_{i,j}^{(1)}G_n + c_{i,j}^{(2)}H_n$ holds. A straightforward calculation shows that

$$\mathbf{M}^n = \frac{1}{\Delta}\left[\begin{array}{cc} (-H_0m_{1,1}^{(1)} + H_1)G_n + (G_0m_{1,1}^{(1)} - G_1)H_n & -H_0m_{1,2}^{(1)}G_n + G_0m_{1,2}^{(1)}H_n \\ -H_0m_{2,1}^{(1)}G_n + G_0m_{2,1}^{(1)}H_n & (-H_0m_{2,2}^{(1)} + H_1)G_n + (G_0m_{2,2}^{(1)} - G_1)H_n \end{array}\right],$$

where we used the values $m_{1,1}^{(0)} = m_{2,2}^{(0)} = 1$, $m_{1,2}^{(0)} = m_{2,1}^{(0)} = 0$. In fact, to determine $\det(\mathbf{M}^n)$ we do not need the exact values $m_{i,j}^{(1)}$ given in (31), only the identities

$$m_{1,1}^{(1)}m_{2,2}^{(1)} - m_{1,2}^{(1)}m_{2,1}^{(1)} = \det(\mathbf{M}) = -B$$

and

$$m_{1,1}^{(1)} + m_{2,2}^{(1)} = \operatorname{tr}(\mathbf{M}) = A.$$

Indeed, if we figure out simply the determinant of the matrix $\mathbf{M}^n$, and collect the coefficient of the terms G_n^2, G_nH_n, and H_n^2 respectively, then the key moment of the simplification is the application of these identities. Finally, the computations lead to

$$(-B)^n = \det(\mathbf{M}^n) = \frac{C_H}{\Delta^2}G_n^2 + \frac{C_{GH}}{\Delta^2}G_nH_n + \frac{C_G}{\Delta^2}H_n^2,$$

where $C_{GH} = -(E_{10} + E_{01})$. Moreover, one can show that C_{GH} can be given by using the corresponding associated sequences as follows: $C_{GH} = G_0\widehat{H}_1 - G_1\widehat{H}_0 = H_0\widehat{G}_1 - H_1\widehat{G}_0$. Then we have the desired equality given in

Theorem 9. *The terms of the recurrences above satisfy*

$$C_HG_n^2 + C_{GH}G_nH_n + C_GH_n^2 = (-B)^n\Delta^2. \tag{33}$$

This is a nice common generalization of (7)–(9). Indeed, observe that Theorem 9 leads to (7) whenever (H) is the associated sequence of (G). Really, it is easy to show that $C_{\widehat{G}} = -DC_G$, $C_{G\widehat{G}} = 0$ (i.e., C_{GH} vanishes if $(H) = (\widehat{G})$), and in this particular case $\Delta = -2C_G$ holds. Insert these values into (33) to immediately obtain (7). For (9), let $H_n = G_{n+1}$; the details here are omitted.

The binary quadratic form on the left-hand side of (33) is trivially decomposable. The decomposition is formulated by

$$((H_1 - \alpha H_0)G_n - (G_1 - \alpha G_0)H_n)) \cdot ((H_1 - \beta H_0)G_n - (G_1 - \beta G_0)H_n) = (-B)^n\Delta^2.$$

Now we illustrate Theorem 9 by five examples given in Table 1. One or two coefficients from

(A, B)	**G**	**M**	$(-B)^n\Delta^2 = C_H G_n^2 + C_{GH} G_n H_n + C_G H_n^2$
$(0, 4)$	$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 7 & -2 \end{bmatrix}$	$(-4)^n(-1)^2 = -7G_n^2 + 4G_nH_n = G_n(4H_n - 7G_n)$
$(2, -1)$	$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$	$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 3/2 \end{bmatrix}$	$1^n(-2)^2 = G_n^2 - 2G_nH_n + H_n^2 = (G_n - H_n)^2$
$(7, -10)$	$\begin{bmatrix} 0 & 1 \\ 2 & 7 \end{bmatrix}$	$\begin{bmatrix} 7/2 & 1/2 \\ 9/2 & 7/2 \end{bmatrix}$	$10^n(-2)^2 = -9G_n^2 + H_n^2$
$(7, -10)$	$\begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$	$10^n 3^2 = 9G_nH_n$
$(4, -1)$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 13/2 & -3/2 \\ 23/2 & -5/2 \end{bmatrix}$	$1^n(-2)^2 = -23G_n^2 + 18G_nH_n - 3H_n^2$

Table 1: Binary recurrence examples.

$\{C_G, C_H, C_{GH}\}$ may vanish in (33), which provides a large variety of identities.

6.2 Ternary recurrences

Instead of pointing out the complicated general formula it is better to present an example which is typical of the ternary forms we are studying. Examples for higher-order linear recurrences can be produced analogously.

There are only a few papers in the literature which work with three different ternary recurrences satisfying the same recurrence rule. An example is [14], which studies the Narayana sequence (A_n) (A000930), Narayana-Lucas sequence (B_n), (A001609), and Narayana-Perrin sequence (C_n). These recurrences satisfy

$$x_n = x_{n-1} + x_{n-3}$$

with initial values given by

$$A_0 = 0, A_1 = 1, A_2 = 1; \quad B_0 = 3, B_1 = 1, B_2 = 1; \quad C_0 = 3, C_1 = 0, C_2 = 2,$$

respectively. Let α_i $(i = 1, 2, 3)$ denote the simple zeros of the characteristic polynomial $k(x) = x^3 - x^2 - 1$ such that $\alpha_1 \in \mathbb{R}$ and α_3 is the complex conjugate of α_2. Following the the method we described in detail leads to the diophantine equation

$$-187x^3+159x^2y-45x^2z-189xy^2+306xyz-117xz^2+y^3-45y^2z+63yz^2-27z^3 = -216, \quad (34)$$

which possesses infinitely many integer solutions $x = A_n$, $y = B_n$, $z = C_n$. In this case, $\Delta = -6$, and the base of the exponential term is 1. Note that all the 10 coefficients in the corresponding $\tilde{F}$ are non-zero. The ternary form of (34) can be decomposed in the algebraic number field $\mathbb{Q}[\alpha_1, \alpha_2]$ as

$$-\prod_{i=1}^{3} \left((3\alpha_i^2 + 3\alpha_i - 2)x + (-3\alpha_i^2 + 3\alpha_i + 2)y + (3\alpha_i^2 - 3\alpha_i)z\right).$$

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