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On Sums of Practical Numbers and Polygonal Numbers

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Abstract

Practical numbers are positive integers n such that every positive integer less than or equal to n can be written as a sum of distinct positive divisors of n. We show that all positive integers can be written as a sum of a practical number and a triangular number, resolving a conjecture by Sun. We also show that all sufficiently large natural numbers can be written as a sum of a practical number and two s-gonal numbers.

1 Introduction

By a natural number, we mean a positive integer. Practical numbers, introduced by Srinivasan in [9], are natural numbers n such that every natural number less than or equal to n can be written as a sum of distinct positive divisors of n. The sequence of practical numbers can also be found on the On-Line Encyclopedia of Integer Sequences (OEIS) (see the sequence

A005153 in the OEIS [7]). Stewart [10] and Sierpiński [6] proved the characterization that a natural number $n \geq 2$ with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$, is practical if and only if $p_1 = 2$ and

$$p_j \leq \sigma\left(p_1^{a_1} p_2^{a_2} \cdots p_{j-1}^{a_{j-1}}\right) + 1$$

for all $2 \leq j \leq k$, where $\sigma(\cdot)$ denotes the sum of divisors.

There have been many works on various additive representations of natural numbers involving practical numbers. Melfi [4] showed that every even natural number is a sum of two practical numbers. Pomerance and Weingartner [5] proved that every sufficiently large odd number can be written as a sum of a practical number and a prime. Somu et al. [8] proved that all natural numbers congruent to 1 modulo 8 are expressible as a sum of a practical number and a square. In this paper, we derive results involving sums of practical and polygonal numbers.

In this article, we consider 0 as a triangular number. In Section 3, we prove that all natural numbers can be written as a sum of a practical number and a triangular number, resolving the conjecture proposed by Sun (see the sequence A208244 in the OEIS [7]). We prove the following theorem.

Theorem 1. *Every natural number can be written as a sum of a practical number and a triangular number.*

Section 4 focuses on additive representations of practical and polygonal numbers more generally. We prove that all sufficiently large natural numbers can be written as a sum of a practical number and two s-gonal numbers. We prove the following theorem.

Theorem 2. *Let s be a natural number greater than 3. Then there exists a natural number $N(s)$ such that all natural numbers greater than $N(s)$ can be written as a sum of a practical number and two s-gonal numbers.*

Finally, in Section 5, we propose two conjectures regarding additive representations of natural numbers involving practical numbers.

2 Notation

We use the following notation:

$\mathbb{N}$: the set of positive integers.

$\mathbb{N}_0$: the set of non-negative integers.

$\sigma(\cdot)$: the sum of positive divisors of a natural number.

$\equiv$: We say $a \equiv b \pmod{n}$ if n divides $a - b$.

gcd$(\cdot,\cdot)$: the greatest common divisor of two integers.

$\lfloor\cdot\rfloor$: the floor function of a real number, which is the largest integer not exceeding that real number.

$P_s(n)$: the n-th s-gonal number given by $P_s(n) = (s-2)\frac{n(n-1)}{2} + n = \frac{(s-2)n^2-(s-4)n}{2}$, for $s \geq 3$, $n \geq 0$.

$O(\cdot)$: We say $f(x) = O(g(x))$ if there exists a positive real number M such that $|f(x)| \leq Mg(x)$ for all sufficiently large x.

3 Proof of Theorem 1

In this section, we prove that all natural numbers can be written as a sum of a practical number and a triangular number, resolving the conjecture proposed by Sun (see the sequence A208244 in the OEIS [7]). We require three lemmas to prove Theorem 1.

Lemma 3. *Let m and n be natural numbers. There exists a natural number $1 \leq x \leq 2^m - 1$ such that*

$$x^2 \equiv 8n + 1 \pmod{2^{m+2}}.$$

Proof. See [8, Lemma 3.2] for proof. □

Lemma 4. *If m is a practical number and n is a natural number such that $n \leq \sigma(m) + 1$, then mn is a practical number.*

Proof. See [3, Corollary 1] for proof. □

Lemma 5. *If x is an odd natural number, then $\frac{x^2-1}{8}$ is a triangular number.*

Proof. As $x = 2k+1$ for some non-negative integer k, we have $\frac{x^2-1}{8} = \frac{k(k+1)}{2}$. Hence, we can conclude that $\frac{x^2-1}{8}$ is a triangular number. □

Now we prove Theorem 1.

Proof of Theorem 1. Let n be a natural number and $m = \lfloor \log_2 \sqrt{8n+1} \rfloor$. By Lemma 3, there exists a natural number x such that $1 \leq x \leq 2^m - 1$ and $x^2 \equiv 8n + 1 \pmod{2^{m+2}}$. Since $x \leq 2^m - 1$ and $m = \lfloor \log_2 \sqrt{8n+1} \rfloor$, we have $x^2 < 2^{2m} \leq 8n + 1$.

As $x^2 < 8n + 1$ and $x^2 \equiv 8n + 1 \pmod{2^{m+2}}$, we have $8n + 1 - x^2 = 2^{m+2}s$ for some natural number s. As $m = \lfloor \log_2 \sqrt{8n+1} \rfloor$, we have $2^{m+2}s \leq 8n + 1 \leq 2^{2m+2}$, which implies that $s \leq 2^m$.

Since 2^{m-1} is a practical number and $s \leq 2^m = \sigma(2^{m-1})+1$, by Lemma 4, we can conclude that $2^{m-1}s$ is a practical number. Notice that $\frac{x^2-1}{8}$ is a triangular number, as x is an odd natural number because $x^2 \equiv 8n + 1 \pmod{2^{m+2}}$. Now, we have $8n + 1 - x^2 = 2^{m+2}s$, or equivalently $n = 2^{m-1}s + \frac{x^2-1}{8}$. Since $2^{m-1}s$ is a practical number and $\frac{x^2-1}{8}$ is a triangular number, we can conclude that n is a sum of a practical number and a triangular number. Therefore, all natural numbers can be written as a sum of a practical number and a triangular number. $\square$

4 Proof of Theorem 2

In this section, we prove some results regarding additive representations of natural numbers involving practical and polygonal numbers more generally. We require five lemmas to prove Theorem 2.

Lemma 6. *Let p be an odd prime, let n be a natural number, and let s be a natural number greater than 3. There exist natural numbers x and y such that*

$$P_s(x) + P_s(y) \equiv n \pmod{p}.$$

Proof. If $p \mid (s-2)$, then $x = n$ and $y = p$ satisfy

$$P_s(x) + P_s(y) \equiv n \pmod{p}.$$

If $p \nmid (s-2)$, then

$$\left|\{P_s(i) \bmod p : 1 \le i \le p\}\right| = \left|\{(n - P_s(j)) \bmod p : 1 \le j \le p\}\right| = \frac{p+1}{2}.$$

This implies that

$$\{P_s(i) \bmod p : 1 \le i \le p\} \cap \{(n - P_s(j)) \bmod p : 1 \le j \le p\} \ne \varnothing.$$

Thus, there exist natural numbers x and y such that

$$P_s(x) + P_s(y) \equiv n \pmod{p}.$$

$\square$

Lemma 7. *Let n be a natural number and s be a natural number greater than 3. There exist natural numbers x and y such that*

$$P_s(x_1) + P_s(y_1) \equiv n \pmod{2}$$

for all natural numbers $x_1 \equiv x \pmod{4}$ and $y_1 \equiv y \pmod{4}$.

Proof. If n is even, then $(x, y) = (4, 4)$ satisfies the condition above. If n is odd, then $(x, y) = (4, 1)$ satisfies the condition above. $\square$

Lemma 8. *Let p be a prime congruent to 1 modulo 4. For all natural numbers n and k, there exist natural numbers x and y such that $x^2 + y^2 \equiv n \pmod{p^k}$ and $p \nmid y$.*

Proof. We will prove the lemma using mathematical induction on k. Let us first prove the lemma for $k = 1$. Let n be any natural number. By [1, Theorem 84], we have

$$|\{i^2 \bmod p : 1 \leq i \leq p\}| = |\{(n - j^2) \bmod p : 1 \leq j \leq p\}| = \frac{p+1}{2}.$$

Hence

$$\{i^2 \bmod p : 1 \leq i \leq p\} \cap \{(n - j^2) \bmod p : 1 \leq j \leq p\} \neq \varnothing.$$

Thus there exist natural numbers x and y such that

$$x^2 + y^2 \equiv n \pmod{p}.$$

If $n \not\equiv 0 \pmod{p}$, then x and y cannot both be multiples of p. Without loss of generality, we can let $p \nmid y$. Since $p \equiv 1 \pmod{4}$, there exists a natural number a such that $a^2 + 1 \equiv 0 \pmod{p}$ (see [1, Theorem 86]). So, we can conclude that $x = a$ and $y = 1$ is a solution to $x^2 + y^2 \equiv 0 \pmod{p}$ with $p \nmid y$.

Now suppose that there exist natural numbers x and y_s such that

$$x^2 + y_s^2 \equiv n \pmod{p^s}$$

and $p \nmid y_s$, where $s \geq 1$. Let l be any natural number satisfying

$$l \equiv \left(\frac{n - x^2 - y_s^2}{p^s}\right)(2y_s)^{-1} \pmod{p},$$

and let $y_{s+1} = y_s + p^s l$. Now, as $y_{s+1} \equiv y_s \pmod{p^s}$ and $p \nmid y_s$, we have $p \nmid y_{s+1}$. As $\frac{x^2+y_s^2-n}{p^s} + 2ly_s \equiv 0 \pmod{p}$, we have

$$\begin{aligned}
x^2 + y_{s+1}^2 &= x^2 + (y_s + p^s l)^2 \\
&= x^2 + y_s^2 + 2p^s l y_s + p^{2s} l^2 \\
&\equiv x^2 + y_s^2 + 2p^s l y_s \pmod{p^{s+1}} \\
&\equiv n + p^s\left(\frac{x^2 + y_s^2 - n}{p^s} + 2ly_s\right) \pmod{p^{s+1}} \\
&\equiv n \pmod{p^{s+1}}.
\end{aligned}$$

Hence, by mathematical induction, for all natural numbers k, there exist natural numbers x and y such that

$$x^2 + y^2 \equiv n \pmod{p^k}$$

and $p \nmid y$. □

Lemma 9. *Let s be a natural number greater than 3. There exists an odd prime p not dividing $s - 2$ such that for all $k, n \in \mathbb{N}$, there exist $x, y \in \mathbb{N}$ such that*

$$P_s(x) + P_s(y) \equiv n \pmod{p^k}.$$

Proof. Let k and n be any natural numbers, and let p be a prime congruent to 1 modulo 4 such that $p \nmid (s-2)$. Note that

$$8(s-2)P_s(x) = (2(s-2)x - (s-4))^2 - (s-4)^2.$$

Since $p \equiv 1 \pmod 4$, by Lemma 8, there exist natural numbers x_0 and y_0 such that

$$x_0^2 + y_0^2 \equiv 8(s-2)n + 2(s-4)^2 \pmod{p^k}.$$

Let x and y be natural numbers satisfying the congruences

$$x \equiv 2^{-1}(s-2)^{-1}(x_0 + s - 4) \pmod{p^k}$$

and

$$y \equiv 2^{-1}(s-2)^{-1}(y_0 + s - 4) \pmod{p^k}.$$

We have

$$\begin{aligned} 8(s-2)P_s(x) + 8(s-2)P_s(y) &\equiv x_0^2 + y_0^2 - 2(s-4)^2 \pmod{p^k} \\ &\equiv 8(s-2)n \pmod{p^k}. \end{aligned}$$

Since $\gcd(p^k, 8(s-2)) = 1$, we have $P_s(x) + P_s(y) \equiv n \pmod{p^k}$. □

Lemma 10. *Let s be a natural number greater than 3, and let $p_{i(s)}$ be the smallest prime for which Lemma 9 holds. There exists a real number $A(s)$ such that for all $x \geq 1$, we have*

$$\frac{2P_s(2p_{i(s)}x)}{x^2} \leq A(s).$$

Proof. Since $2P_s(2p_{i(s)}x)$ is a quadratic polynomial, we have

$$\frac{2P_s(2p_{i(s)}x)}{x^2} = O(1).$$

Hence, there exists a real number $A(s)$ such that

$$\frac{2P_s(2p_{i(s)}x)}{x^2} \leq A(s)$$

for all real numbers $x \geq 1$. □

Now we are ready to give a proof of Theorem 2.

Proof of Theorem 2. Let p_i denote the i-th prime, and let $p_{i(s)}$ be the smallest prime for which Lemma 9 holds. By Lemma 10, there exists a real number $A(s)$ such that

$$\frac{2P_s(2p_{i(s)}x)}{x^2} \leq A(s)$$

for all real numbers $x \geq 1$. Let r be the smallest natural number such that $r \geq i(s)$ and

$$\frac{\sigma(p_1 p_2 \cdots p_r)}{p_1 p_2 \cdots p_r} \geq A(s).$$

Such an r is well-defined, as the product $\prod_{p \text{ prime}}(1+\frac{1}{p})$ diverges (see [2, Chapter 7, Theorem 3] and [1, Theorem 19]). Let

$$N(s) = 2P_s(2p_1 p_2 \cdots p_r).$$

Consider any natural number n greater than $N(s)$. Let k be the largest natural number such that

$$2P_s\big(2p_1 p_2 \cdots p_{i(s)-1} p_{i(s)}^k p_{i(s)+1} \cdots p_r\big) < n.$$

Let $n_k = 2p_1 p_2 \cdots p_{i(s)-1} p_{i(s)}^k p_{i(s)+1} \cdots p_r$. From the definition of k, we have

$$2P_s(n_k) < n \leq 2P_s(p_{i(s)} n_k).$$

From Lemma 6, there exists a solution $x \bmod p_i$, $y \bmod p_i$ to the equation $P_s(x) + P_s(y) \equiv n \pmod{p_i}$ for $2 \leq i \leq r$ and $i \neq i(s)$. From Lemma 7, there exists a solution $x \bmod 2p_1$, $y \bmod 2p_1$ to the equation $P_s(x) + P_s(y) \equiv n \pmod{p_1}$. From Lemma 9, there exists a solution $x \bmod p_{i(s)}^k$, $y \bmod p_{i(s)}^k$ to $P_s(x) + P_s(y) \equiv n \pmod{p_{i(s)}^k}$. Hence, by the Chinese remainder theorem, there exists a solution $x \bmod n_k$, $y \bmod n_k$ to the equation

$$P_s(x) + P_s(y) \equiv n \ (\mathrm{mod}\ \frac{n_k}{2}).$$

Thus, there exist natural numbers $x, y \leq n_k$ such that

$$P_s(x) + P_s(y) \equiv n \ (\mathrm{mod}\ \frac{n_k}{2}).$$

This, together with the fact that $n > 2P_s(n_k)$, implies that

$$\frac{2(n - P_s(x) - P_s(y))}{n_k} \in \mathbb{N}.$$

Note that

$$\frac{2(n - P_s(x) - P_s(y))}{n_k} \leq \frac{2n}{n_k} \leq \frac{4P_s(p_{i(s)} n_k)}{n_k}.$$

By Lemma 10, we have

$$\frac{2P_s(p_{i(s)} n_k)}{\frac{n_k^2}{4}} \leq A(s).$$

Therefore, we have

$$\frac{2(n - P_s(x) - P_s(y))}{n_k} \leq \frac{4P_s(p_{i(s)} n_k)}{n_k} \leq \frac{A(s) n_k}{2}.$$

Also, we have

$$\frac{\sigma(\frac{n_k}{2})}{\frac{n_k}{2}} = \frac{\sigma\left(p_1p_2\cdots p_{i(s)-1}p_{i(s)}^kp_{i(s)+1}\cdots p_r\right)}{p_1p_2\cdots p_{i(s)-1}p_{i(s)}^kp_{i(s)+1}\cdots p_r} \geq \frac{\sigma(p_1p_2\cdots p_r)}{p_1p_2\cdots p_r} \geq A(s).$$

This implies $\frac{A(s)n_k}{2} \leq \sigma(\frac{n_k}{2})$. Therefore, we have

$$\frac{2(n-P_s(x)-P_s(y))}{n_k} \leq \frac{A(s)n_k}{2} \leq \sigma\left(\frac{n_k}{2}\right).$$

Note that $\frac{n_k}{2} = p_1p_2\cdots p_{i(s)-1}p_{i(s)}^kp_{i(s)+1}\cdots p_r$ is a practical number by the characterization of practical numbers (see [10, Section 3]). Thus, by Lemma 4, we can conclude that

$$\frac{2(n-P_s(x)-P_s(y))}{n_k}\left(\frac{n_k}{2}\right) = n-P_s(x)-P_s(y)$$

is a practical number. Therefore, the natural number n can be written as a sum of a practical number and two s-gonal numbers. $\square$

In Theorem 1, we have proved that all natural numbers can be written as a sum of a triangular number and a practical number. In Theorem 2, we have proved that for all $s > 3$, all sufficiently large natural numbers can be written as a sum of a practical number and two s-gonal numbers. Now we show that there are infinitely many $s > 3$ for which we cannot write all sufficiently large natural numbers as a sum of a practical number and an s-gonal number. We also show that as s tends to infinity, the number of natural numbers that cannot be written as a sum of a practical number and two s-gonal numbers tends to infinity. Hence, we cannot drop "sufficiently large" from the statement of Theorem 2. We will require one lemma to prove these claims.

Lemma 11. *If q is a practical number such that q is neither divisible by 3 nor by 4, then $q = 1$ or $q = 2$.*

Proof. For the sake of contradiction, assume that $q > 2$. Since $q > 2$ and $4 \nmid q$, the natural number q should have at least one odd prime divisor. Let p be the smallest odd prime divisor of q. As $3 \nmid q$, we have $p \geq 5$. As $p \geq 5 > \sigma(2) + 1$, by the characterization of practical numbers, we can conclude that q is not practical (see [10, Section 3]). This is a contradiction. $\square$

Proposition 12.

(a) If $s \equiv 0 \pmod{12}$ or $s \equiv 4 \pmod{12}$, then infinitely many natural numbers cannot be written as a sum of a practical number and an s-gonal number.

(b) Let $E(s)$ be the number of natural numbers that cannot be written as a sum of a practical number and two s-gonal numbers. Then we have

$$\lim_{s\to\infty} E(s) = \infty.$$

Proof.

(a) If $s \equiv 0 \pmod{12}$ or $s \equiv 4 \pmod{12}$, then

$$P_s(n) = \frac{s-2}{2}n^2 + \frac{s-4}{2}n = an^2 + bn,$$

where $a = \frac{s-2}{2}$ and $b = \frac{s-4}{2}$. Note that a is odd and is not divisible by 3. Also, note that b is even. Hence, as

$$P_s(n) = \frac{1}{a}\left(\left(an + \frac{b}{2}\right)^2 - \frac{b^2}{4}\right),$$

for all $n \in \mathbb{N}_0$, we have

$$P_s(n) \not\equiv a^{-1}\left(2 - \frac{b^2}{4}\right) \pmod{3},$$

and

$$P_s(n) \not\equiv a^{-1}\left(2 - \frac{b^2}{4}\right) \pmod{4}.$$

This is because 2 is a quadratic non-residue modulo 3 and modulo 4. Let r be a natural number such that

$$r \equiv a^{-1}\left(2 - \frac{b^2}{4}\right) \pmod{12},$$

and let x be any positive real number. Let k be a natural number congruent to r modulo 12 and not exceeding x such that $k = P_s(m) + q$ for some $m \in \mathbb{N}_0$ and some practical number q. Since $k \equiv r \pmod{12}$, we have $3 \nmid (k - P_s(m))$ and $4 \nmid (k - P_s(m))$. Hence, by Lemma 11, we can conclude that $q = 1$ or $q = 2$, and thus

$$k \in \{P_s(n) + 1 : n \in \mathbb{N}_0 \text{ and } P_s(n) + 1 \leq x\} \cup \{P_s(n) + 2 : n \in \mathbb{N}_0 \text{ and } P_s(n) + 2 \leq x\}.$$

Since

$$|\{P_s(n) + 1 : n \in \mathbb{N}_0 \text{ and } P_s(n) + 1 \leq x\}| + |\{P_s(n) + 2 : n \in \mathbb{N}_0 \text{ and } P_s(n) + 2 \leq x\}| = O\big(\sqrt{x}\big),$$

there are at most $O(\sqrt{x})$ natural numbers not exceeding x that are congruent to r modulo 12 and expressible as a sum of a practical number and an s-gonal number. Since there are $\frac{x}{12} + O(1)$ natural numbers not exceeding x that are congruent to r modulo 12, there are at least $\frac{x}{12} + O(\sqrt{x})$ natural numbers not exceeding x that are congruent to r modulo 12 and not expressible as a sum of a practical number and an s-gonal number. Therefore, infinitely many natural numbers cannot be written as a sum of a practical number and an s-gonal number if $s \equiv 0 \pmod{12}$ or $s \equiv 4 \pmod{12}$.

(b) Let us count the number of natural numbers less than s that can be represented as a sum of a practical number and two s-gonal numbers. Only 0 and 1 are s-gonal numbers less than s. Thus, if $n < s$ is a sum of a practical number and two s-gonal numbers, then

$$n \in \{P + i : P \text{ is practical}, P \leq s, \text{ and } i \in \{0, 1, 2\}\}.$$

From a result of Weingartner [12, Theorem 1], it follows that

$$\left|\{P + i : P \text{ is practical}, P \leq s, \text{ and } i \in \{0, 1, 2\}\}\right| = O\left(\frac{s}{\log s}\right).$$

Hence, at least $s + O(\frac{s}{\log s})$ natural numbers less than s cannot be written as a sum of a practical number and two s-gonal numbers. Therefore, we have

$$\lim_{s \to \infty} E(s) = \infty.$$

$\square$

5 Conjectures on sums of practical and polygonal numbers

In this section, we propose a few conjectures on some additive representations involving practical and polygonal numbers based on numerical computations. All of the code for the conjectures below, written in Python, can be found on Github [11].

For each natural number $s > 3$, let n_s be the number of natural numbers below 10^8 that cannot be written as a sum of a practical number and an s-gonal number, and let N_s be the largest number below 10^8 that cannot be written as a sum of a practical number and an s-gonal number. We have the following conjecture based on Table 1.

s	n_s	N_s
4	17929061	99999998
5	13	2671
6	101	1332329
7	73	79445
8	414	4005819

s	n_s	N_s
9	186	325808
10	341	13613213
11	68	105712
12	16663689	99999998
13	609	1612172

s	n_s	N_s
14	79	106878
15	767	1486748
16	16665797	99999998
17	106	9314
18	1020	8541224

Table 1: Data regarding the sum of a practical number and an s-gonal number.

Conjecture 13. If $s > 3$, $s \not\equiv 0 \pmod{12}$, and $s \not\equiv 4 \pmod{12}$, then all sufficiently large natural numbers can be written as a sum of a practical number and an s-gonal number.

For $s \in \{4, 5, 6, 7, 8, 10\}$, we also computationally verified that all natural numbers below 10^8 are expressible as a sum of a practical number and two s-gonal numbers. We have the following conjecture, which is a stronger version of Theorem 2, for the cases $s \in \{4, 5, 6, 7, 8, 10\}$.

Conjecture 14. For $s \in \{4, 5, 6, 7, 8, 10\}$, all natural numbers can be written as a sum of a practical number and two s-gonal numbers.

The conjecture above does not hold for other values of s. One can verify that 23 cannot be written as a sum of a practical number and two nonagonal numbers and that 11 cannot be written as a sum of a practical number and two s-gonal numbers if $s \geq 11$.

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