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On Residues of Rounded Shifted Fractions with a Common Numerator

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Abstract

The proportion of integers

$$\lfloor n/1 \rfloor, \lfloor n/2 \rfloor, \ldots, \lfloor n/n \rfloor$$

that are odd is asymptotically $\log 2$. If instead of the floor function, one uses the nearest-integer function, the proportion drops to $\pi/2 - 1$. In this work, we prove these facts and, more generally, give an integral formula for the proportion of

$$\lfloor (n-\nu)/1 + \alpha \rfloor,\ \lfloor (n-\nu)/2 + \alpha \rfloor,\ \ldots,\ \lfloor (n-\nu)/n + \alpha \rfloor$$

that are congruent to r modulo m. We conclude by linking this problem to the Dirichlet divisor and Gauss circle problems, and counting lattice points in other quadratic regions.

1 Introduction

For $\mathbb{N}$ the set of positive integers and $n \in \mathbb{N}$, consider the integer sequence

$$\left\lfloor \frac{n}{1} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor, \ldots, \left\lfloor \frac{n}{n} \right\rfloor, \tag{1}$$

where $\lfloor x \rfloor$ denotes the usual floor function for real x. Among the n integers in this sequence, how many are odd?

Let's start with some data. For $n = 5$, the sequence is

$$\left\lfloor \frac{5}{1} \right\rfloor, \left\lfloor \frac{5}{2} \right\rfloor, \left\lfloor \frac{5}{3} \right\rfloor, \left\lfloor \frac{5}{4} \right\rfloor, \left\lfloor \frac{5}{5} \right\rfloor = 5, 2, 1, 1, 1,$$

which contains 4 odd numbers. For $n = 10$, the sequence is $10, 5, 3, 2, 2, 1, 1, 1, 1, 1$, which contains 7 odd numbers. For $n = 13$, there are 9 odd numbers.

For any n, terms in the second half of Seq. (1) are all 1. Thus, the proportion of odd terms is at least 50%. But how much bigger will it be? Does the proportion converge to a particular number as n gets large? And what happens if we replace the floor function with the ceiling function? Or if we round to the nearest-integer?

In order to address these questions, we define sequences $(\mathcal{F}_n)_{n\geq 1}$, $(\mathcal{C}_n)_{n\geq 1}$, and $(\mathcal{R}_n)_{n\geq 1}$ by

$$\begin{aligned} \mathcal{F}_n &= \#\{k \in \mathbb{Z} : 1 \leq k \leq n,\ \lfloor n/k \rfloor \text{ is odd}\}, \\ \mathcal{C}_n &= \#\{k \in \mathbb{Z} : 1 \leq k \leq n,\ \lceil n/k \rceil \text{ is odd}\}, \\ \mathcal{R}_n &= \#\{k \in \mathbb{Z} : 1 \leq k \leq n,\ \lfloor n/k \rceil \text{ is odd}\}, \end{aligned}$$

where $\lceil x \rceil$ denotes the ceiling function and $\lfloor x \rceil$ the nearest-integer rounding function. (In this work, we use the convention for nearest-integer rounding to round half-integers up. We consider other conventions in Remark 23.) The sequences $(\mathcal{F}_n)_{n\geq 1}$, $(\mathcal{C}_n)_{n\geq 1}$, and $(\mathcal{R}_n)_{n\geq 1}$ appear in The On-Line Encyclopedia of Integer Sequences [10] as, respectively, sequences A059851, A330926, and A363341.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathcal{F}_n$	1	1	3	2	4	4	6	4	7	7	9	7	9	9	13	10	12	12	14	12
$\mathcal{C}_n$	1	1	2	1	3	2	3	2	5	3	4	3	6	5	6	3	7	6	7	6
$\mathcal{R}_n$	1	1	2	2	4	3	4	4	6	7	6	5	9	8	9	9	10	10	11	12

Figure 1: Values of $\mathcal{F}_n$, $\mathcal{C}_n$, $\mathcal{R}_n$ for $1 \leq n \leq 20$.

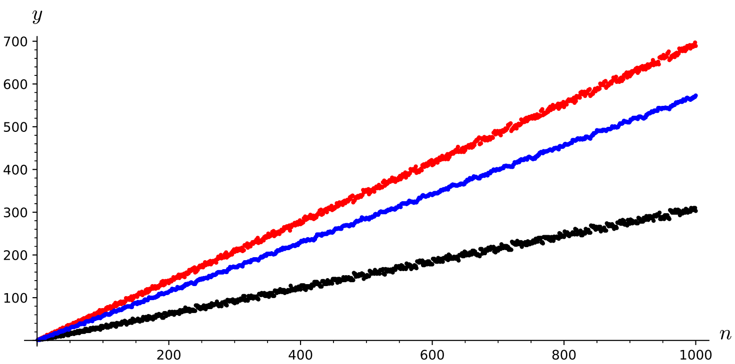

Figure 2: Plots of $y = \mathcal{F}_n$ (red, top-most), $y = \mathcal{R}_n$ (blue, middle), and $y = \mathcal{C}_n$ (black, bottom-most), for $1 \le n \le 1000$.

From the data generated above, we have $\mathcal{F}_5 = 4$, $\mathcal{F}_{10} = 7$, and $\mathcal{F}_{13} = 9$. The first 20 terms of each sequence are in Figure 1. Plots of the first 1000 terms of each sequence appear in Figure 2.

The plots in Figure 2 suggest that each sequence is roughly linear. We will see that this is true asymptotically, with slopes given by

$$\lim_{n\to\infty} \frac{\mathcal{F}_n}{n} = \log 2, \quad \lim_{n\to\infty} \frac{\mathcal{C}_n}{n} = 1 - \log 2, \quad \text{and } \lim_{n\to\infty} \frac{\mathcal{R}_n}{n} = \frac{\pi}{2} - 1.$$

In particular, since $\log 2 \approx 0.693147$, the proportion of odd terms in Seq. (1) is asymptotically approximately 69.3%.

Our results follow from two classical results in analytic number theory about counting lattice points in the plane: the *Dirichlet divisor problem* (Theorem 6), which involves counting lattice points beneath the hyperbola $xy = n$; and the *Gauss circle problem* (Theorem 20), which involves counting lattice points within a circle of radius $\sqrt{n}$.

These classical results lead to geometric interpretations for our sequences. In Proposition 4, we find that $\mathcal{F}_n$ is the difference of the numbers of lattice points in two hyperbolic regions in the plane. In Proposition 17, we show that the number of lattice points contained in a circle of radius $\sqrt{2n}$ is equal to $4\mathcal{R}_n + 4n + 1$.

We now consider more general sequences. Given that $\lfloor x \rceil = \lfloor x + 1/2 \rfloor$, we can think of the nearest-integer rounding function as the usual floor function shifted by $1/2$. If we replace this $1/2$ with an arbitrary real number α and also incorporate a real number ν to offset the numerator n, we obtain an *α-shifted, ν-offset, floor sequence of length n*

$$\left\lfloor \frac{n-\nu}{1} + \alpha \right\rfloor, \left\lfloor \frac{n-\nu}{2} + \alpha \right\rfloor, \ldots, \left\lfloor \frac{n-\nu}{n} + \alpha \right\rfloor. \tag{2}$$

As before, we can ask about the number of odd terms in such a sequence. With an answer to that, we can then find α and ν for which (asymptotically) half of these integers are odd and half are even. We show in Section 5.1 that there is a unique value for $\alpha \in [0,1]$, independent of ν, for which this occurs, and numerically approximate it.

Another problem, given integers r and m, is to compute $\mathcal{N}_{n,\alpha,\nu,r,m}$, the number of integers in Seq. (2) that are congruent to r modulo m. Our main result is Theorem 37, which states that for any $m \in \mathbb{N}$ and $\alpha \in [0,1)$,

$$\mathcal{N}_{n,\alpha,\nu,1,m} = \frac{-\alpha n}{1-\alpha} + n \int_0^1 \frac{(1-x)x^{-\alpha}}{1-x^m} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big) \tag{3}$$

and, for $2 \leq r \leq m$,

$$\mathcal{N}_{n,\alpha,\nu,r,m} = n \int_0^1 \frac{(1-x)x^{r-1-\alpha}}{1-x^m} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big). \tag{4}$$

Finally, we look more closely at connections between the number of integers in a certain congruence class in an α-shifted, ν-offset, floor sequence of length n and the number of lattice points in a certain region of the plane. We demonstrate the connections by obtaining formulas in terms of these counts for the number of lattice points in the following elliptical regions: $x^2+y^2 \leq n$; $x^2+xy+y^2 \leq n$; and $x^2+2y^2 \leq n$, each for any $n \in \mathbb{N}$. Our results here follow from the theory of binary quadratic forms. In particular, counting lattice points in the elliptical region $x^2+xy+y^2 \leq n$ gives us a way to count the number of lattice points in a hexagonal lattice contained in a disc, which we describe in Corollary 47.

1.1 Motivation

This paper grew out of a problem in a recent paper by Shor [9, Sec. 5.2] on numerical semigroups. Obtaining an asymptotic formula for the number of reflective numerical semigroups with a given Frobenius number amounted to finding an asymptotic formula for $(\mathcal{F}_n)_{n\geq 1}$.

1.2 Organization

In Section 2, we find exact and asymptotic formulas for $\mathcal{F}_n$ and $\mathcal{C}_n$. In Section 3, we do the same for $\mathcal{R}_n$. Our main result appears in Section 4, where we find exact (Theorem 37) and asymptotic (Corollary 38) formulas for $\mathcal{N}_{n,\alpha,\nu,r,m}$, generalizing the results of the previous sections. Finally, in Section 5, we have two tasks. First, we compute the (unique) shift $\alpha = \alpha_0$ for which (asymptotically) the α-shifted, ν-offset, floor sequence of length n contains as many odd terms as even terms. Second, we determine the number of lattice points in certain elliptical regions of the plane in terms of the sequences $\mathcal{N}_{n,\alpha,\nu,r,m}$.

1.3 A question for further study

Our error term for $\mathcal{N}_{n,\alpha,\nu,r,m}$, (Eqs. (3) and (4), from Theorem 37) which is $\mathcal{O}(\sqrt{n})$, can almost certainly be improved upon.

The sequences $\mathcal{F}_n$ and $\mathcal{R}_n$ are special cases of $\mathcal{N}_{n,\alpha,\nu,r,m}$. The error term for $\mathcal{F}_n$ is the same as the error term for the Dirichlet divisor problem, and the error term for $\mathcal{R}_n$ is the same as the error term for the Gauss circle problem. At present, the best error bounds known for these two problems are the same: $\mathcal{O}(n^{131/416})$, as shown by Huxley [5]. It would be interesting to see if that error bound holds for $\mathcal{N}_{n,\alpha,nu,r,m}$ in general, or at least for other particular parameter values.

2 The floor and ceiling sequences

The quantities $\mathcal{F}_n$ and $\mathcal{C}_n$ are very closely related, so we investigate both of them in this section.

2.1 The floor sequence

As was described in the introduction, the number of odd integers in the floor sequence

$$\left\lfloor \frac{n}{1} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor, \ldots, \left\lfloor \frac{n}{n} \right\rfloor \tag{5}$$

is $\mathcal{F}_n$. In this subsection, we find an exact formula for $\mathcal{F}_n$. Then, with that formula and the solution to Dirichlet's divisor problem, we obtain an asymptotic formula for $\mathcal{F}_n$.

We begin by counting the number of integers in Seq. (5) that are greater than or equal to a given integer.

Lemma 1. *Fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, Seq. (5) contains $\lfloor n/k \rfloor$ terms that are greater than or equal to k.*

Proof. Via the division algorithm, there are $q, r \in \mathbb{Z}$ for which $n = kq + r$ with $0 \leq r < k$. Then $\lfloor n/k \rfloor = q$. Our approach is to show that the first q terms in Seq. (5) are greater than or equal to k.

Note that

$$\left\lfloor \frac{n}{q} \right\rfloor = \left\lfloor \frac{kq + r}{q} \right\rfloor = \left\lfloor k + \frac{r}{q} \right\rfloor \geq k.$$

Now consider an arbitrary term $\lfloor n/d \rfloor$ in Seq. (5). Then $1 \leq d \leq n$. If $d < q$, then $n/d > n/q$ and hence $\lfloor n/d \rfloor \geq \lfloor n/q \rfloor \geq k$. On the other hand, if $d > q$, then $d \geq q + 1$. This means $kd \geq k(q + 1) > n$ and therefore $k > n/d \geq \lfloor n/d \rfloor$. We have already shown that $\lfloor n/d \rfloor \geq k$ if $d = q$. We conclude that $\lfloor n/d \rfloor \geq k$ if and only if $d \in \{1, 2, \ldots, q\}$. $\square$

We can now find an exact formula for $\mathcal{F}_n$.

Proposition 2. *For $n \in \mathbb{N}$,*

$$\mathcal{F}_n = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor (-1)^{d+1}.$$

Proof. We wish to count the number of odd terms in Seq. (5). Here we can use Lemma 1 to see that for a given k there are $\lfloor n/k \rfloor$ terms whose value is at least k, and thus $\lfloor n/k \rfloor - \lfloor n/(k+1) \rfloor$ terms whose value is exactly k. To compute $\mathcal{F}_n$, we sum for all odd k. We find

$$\mathcal{F}_n = \sum_{\substack{1 \le k \le n \\ k \text{ odd}}} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n}{k+1} \right\rfloor \right) = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor (-1)^{d+1},$$

as desired. □

With this exact formula, we now work toward an asymptotic formula for $\mathcal{F}_n$. For $\tau(n)$ the number of positive divisors of a positive integer n, consider the *divisor summatory function*

$$D(x) = \sum_{m \le x} \tau(m),$$

where the summation is taken over positive integers m. We have the following for all $n \in \mathbb{N}$:

$$D(n) = \sum_{m=1}^{n} \tau(m) = \sum_{m=1}^{n} \sum_{d|m} 1 = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor.$$

Viewed this way, we see that $D(n)$ is equal to the number of lattice points in the interior and boundary of the region in the xy-plane bounded by the graphs of $x = 1$, $y = 1$, and the hyperbola $xy = n$.

Corollary 3. *For all $n \in \mathbb{N}$, we have $\mathcal{F}_n = D(n) - 2D(n/2)$.*

Proof. We manipulate the formula for $\mathcal{F}_n$ obtained in Proposition 2, considering odd and even d (with $1 \le d \le n$) separately:

$$\begin{aligned}
\mathcal{F}_n &= \sum_{d \text{ odd}} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d \text{ even}} \left\lfloor \frac{n}{d} \right\rfloor \\
&= \sum_{d \text{ odd}} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d \text{ even}} \left\lfloor \frac{n}{d} \right\rfloor + \sum_{d \text{ even}} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d \text{ even}} \left\lfloor \frac{n}{d} \right\rfloor \\
&= \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - 2 \sum_{b=1}^{\lfloor n/2 \rfloor} \left\lfloor \frac{n}{2b} \right\rfloor \\
&= \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - 2 \sum_{b=1}^{\lfloor n/2 \rfloor} \left\lfloor \frac{n/2}{b} \right\rfloor.
\end{aligned}$$

The result follows. □

The geometric interpretation for $D(n)$ gives a geometric interpretation for $\mathcal{F}_n$. For $n \in \mathbb{N}$, we define two regions. Let $H_{1,n}$ be the hyperbolic region in the xy-plane with $x, y \geq 1$ bounded by the graphs of the hyperbola $xy = n$ and the hyperbola $xy = n/2$. Let $H_{2,n}$ be the region in the xy-plane bounded by the graphs of $x = 1$, $y = 1$, and the hyperbola $xy = n/2$. Then $D(n)$ is equal to the number of lattice points in $H_{1,n} \cup H_{2,n}$, and $D(n/2)$ is equal to the number of lattice points in $H_{2,n}$.

Proposition 4. *For all $n \in \mathbb{N}$, the number of lattice points in $H_{1,n}$ minus the number of lattice points in $H_{2,n}$ is equal to $\mathcal{F}_n$. For $H_{1,n}$, we include all points on the boundary except for those on the boundary curve $xy = n/2$. For $H_{2,n}$, we include all points on the boundary.*

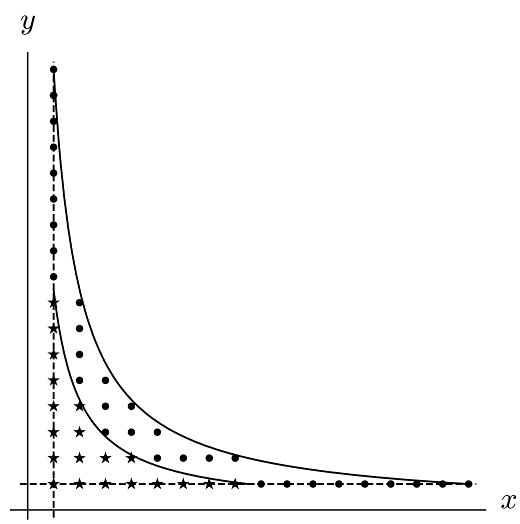

Figure 3: For $n = 17$, the graphs of $xy = 17$ and $xy = 17/2$ along with lattice points in the hyperbolic regions $H_{1,17}$ (circles) and in $H_{2,17}$ (stars).

Example 5. To illustrate Proposition 4, we have drawn the regions $H_{1,n}$ and $H_{2,n}$ for $n = 17$ in Figure 3. Since $H_{1,17}$ contains 32 points and $H_{2,17}$ contains 20 points, we find $\mathcal{F}_{17} = 32 - 20 = 12$.

The following theorem of Dirichlet gives an asymptotic formula for $D(x)$. (For details, see, e.g., Apostol's textbook [1, Theorem 3.3]. Later in this paper, we give a modified version of the proof of this theorem. See Proposition 32.)

Theorem 6. *For all $x \geq 1$,*

$$D(x) = x \log x + (2\gamma - 1)x + \mathcal{O}(\sqrt{x}),$$

where $\gamma = \lim_{n\to\infty} \left(-\log n + \sum_{k=1}^{n} 1/k \right) \approx 0.577216$ *is the Euler-Mascheroni constant.*

Finally, let $F(x) = D(x) - 2D(x/2)$. Then $\mathcal{F}_n = F(n)$ and, via Theorem 6 we have $F(x) = x \log 2 + \mathcal{O}(\sqrt{x})$. Restricting to integers n, we get an asymptotic result for $\mathcal{F}_n$.

Proposition 7. *Let $n \in \mathbb{N}$. Then $\mathcal{F}_n = n \log 2 + \mathcal{O}\big(\sqrt{n}\big)$ and $\lim_{n\to\infty} \frac{1}{n}\mathcal{F}_n = \log 2 \approx 0.693147$.*

Remark 8. The error bound for $D(x)$ in Theorem 6 has been improved over the years since Dirichlet's original result. The *Dirichlet divisor problem* is to give the best error term. The best error bound currently known is $\mathcal{O}\big(x^{131/416}\big)$, due to Huxley [5]. This gives us the following improved error bound for $\mathcal{F}_n$:

$$\mathcal{F}_n = n \log 2 + \mathcal{O}\big(n^{131/416}\big).$$

2.2 The ceiling sequence

As was mentioned in the introduction, for $n \in \mathbb{N}$, the number of odd integers in the ceiling sequence

$$\left\lceil \frac{n}{1} \right\rceil, \left\lceil \frac{n}{2} \right\rceil, \ldots, \left\lceil \frac{n}{n} \right\rceil \tag{6}$$

is $\mathcal{C}_n$. In this subsection, we find a relation between $\mathcal{C}_n$ and $\mathcal{F}_{n-1}$ which, along with results from the previous subsection, leads to exact and asymptotic formulas for $\mathcal{C}_n$.

Proposition 9. *For $n \in \mathbb{N}$,*

$$\mathcal{C}_n = n - \mathcal{F}_{n-1} = \sum_{d=2}^{n} \left\lfloor \frac{n}{d} \right\rfloor (-1)^d.$$

Proof. We begin by showing that $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$ for all $k \in \mathbb{N}$. Taking the same approach as with Lemma 1, via the division algorithm, there are $q, r \in \mathbb{Z}$ for which $n = kq + r$ with $0 \le r < k$.

If $r = 0$, then $n/k = q$ and so $\lceil n/k \rceil = \lceil q \rceil = q$. We also have

$$\left\lfloor \frac{n-1}{k} \right\rfloor = \left\lfloor \frac{n}{k} - \frac{1}{k} \right\rfloor = \left\lfloor q - \frac{1}{k} \right\rfloor = q - 1.$$

In this case, $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$.

If $r \ne 0$, then $1 \le r < k$. We have

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{kq + r}{k} \right\rceil = \left\lceil q + \frac{r}{k} \right\rceil = q + 1.$$

and

$$\left\lfloor \frac{n-1}{k} \right\rfloor = \left\lfloor \frac{kq + r - 1}{k} \right\rfloor = \left\lfloor q + \frac{r-1}{k} \right\rfloor = q.$$

In this case, $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$.

Since we have $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$ for all $k \in \mathbb{N}$, for each pair of integers $(\lceil n/k \rceil, \lfloor (n-1)/k \rfloor)$, exactly one integer is odd. Hence, the total number of odd integers in the two sequences

$$\left\lceil \frac{n}{1} \right\rceil, \left\lceil \frac{n}{2} \right\rceil, \ldots, \left\lceil \frac{n}{n} \right\rceil \text{ and } \left\lfloor \frac{n-1}{1} \right\rfloor, \left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, \left\lfloor \frac{n-1}{n} \right\rfloor$$

is n. The number of odd integers in the first sequence is $\mathcal{C}_n$. Since the last term in the second sequence is $\lfloor (n-1)/n \rfloor = 0$, the number of odd integers in the second sequence is $\mathcal{F}_{n-1}$. This means $\mathcal{F}_{n-1} + \mathcal{C}_n = n$. The stated result then follows from Proposition 2. □

We immediately obtain the following asymptotic formula for $\mathcal{C}_n$.

Corollary 10. *Let $n \in \mathbb{N}$. Then $\mathcal{C}_n = (1 - \log 2)n + \mathcal{O}(\sqrt{n})$ and $\lim_{n\to\infty} \frac{1}{n}\mathcal{C}_n = 1 - \log 2 \approx 0.306853$.*

Proof. Combining Proposition 9 with the asymptotic formula for $\mathcal{F}_n$ in Proposition 7, we find

$$\mathcal{C}_n = n - \mathcal{F}_{n-1} = n - (n-1)\log 2 + \mathcal{O}\big(\sqrt{n-1}\big) = n - n\log 2 + \mathcal{O}\big(\sqrt{n}\big).$$

The limit result follows. □

Remark 11. Alternatively, if we just want the asymptotic formula for $\mathcal{C}_n$, we can get it via the asymptotic formula for $\mathcal{F}_n$ (Proposition 7) and the following lemma.

Lemma 12. *For $n \in \mathbb{N}$, the number of positive divisors of n is at most $2\sqrt{n}$.*

Proof. Suppose $n = de$ for positive integers $d \le e$. Then $1 \le d \le \sqrt{n}$. It follows that there are at most $\sqrt{n}$ pairs of divisors d, e, and thus at most $2\sqrt{n}$ positive divisors of n. □

For any real number x, if x is an integer then $\lceil x \rceil = \lfloor x \rfloor$, and otherwise $\lceil x \rceil = \lfloor x \rfloor + 1$. Taking $x = n/k$, this means

$$\lceil n/k \rceil = \begin{cases} \lfloor n/k \rfloor, & \text{if } k \mid n; \\ \lfloor n/k \rfloor + 1, & \text{if } k \nmid n. \end{cases}$$

When k does not divide n, exactly one of $\lfloor n/k \rfloor$ and $\lceil n/k \rceil$ is odd. When k does divide n, then either both $\lfloor n/k \rfloor$ and $\lceil n/k \rceil$ are odd, or neither is. Since there are at most $2\sqrt{n}$ divisors d of n, we have

$$\mathcal{F}_n + \mathcal{C}_n \in [\, n - 2\sqrt{n}, n + 2\sqrt{n}\,].$$

Therefore $\mathcal{F}_n + \mathcal{C}_n = n + \mathcal{O}(\sqrt{n})$. We conclude that $\mathcal{C}_n = n - \mathcal{F}_n + \mathcal{O}(\sqrt{n}) = (1 - \log 2)n + \mathcal{O}(\sqrt{n})$. This gives an alternate proof of Corollary 10.

Finally, we can use the improved error bound for $\mathcal{F}_n$ (mentioned in Remark 8) together with Proposition 9 to get an improved error bound for $\mathcal{C}_n$ as well:

$$\mathcal{C}_n = (1 - \log 2)n + \mathcal{O}\big(n^{131/416}\big).$$

3 The nearest-integer sequence

As was mentioned in the introduction, the number of odd integers in the nearest-integer rounding sequence

$$\left\lfloor \frac{n}{1} \right\rceil, \left\lfloor \frac{n}{2} \right\rceil, \dots, \left\lfloor \frac{n}{n} \right\rceil \tag{7}$$

is $\mathcal{R}_n$. We use the convention that half-integers are rounded up; we consider other conventions in Remark 23. In this subsection, we find an exact formula for $\mathcal{R}_n$. Then, with that formula and the solution to Gauss's circle problem, we obtain an asymptotic formula for $\mathcal{R}_n$.

We first note that $\lfloor n/k \rceil = \lfloor n/k + 1/2 \rfloor$, and thus Seq. (7) is equal to the sequence

$$\left\lfloor \frac{n}{1} + \frac{1}{2} \right\rfloor, \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor, \left\lfloor \frac{n}{3} + \frac{1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{n} + \frac{1}{2} \right\rfloor. \tag{8}$$

Just as we did in obtaining an exact formula for $\mathcal{F}_n$ (Proposition 2), we begin with a lemma.

Lemma 13. *Fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, let $g(k)$ equal the number of terms in Seq. (8) that are greater than or equal to k. Then $g(1) = n$ and, for $k \geq 2$, $g(k) = \lfloor 2n/(2k-1) \rfloor$.*

Proof. Consider an arbitrary term $\lfloor n/d + 1/2 \rfloor$ in the Seq. (8), so that $1 \leq d \leq n$. We first observe that $\lfloor n/d + 1/2 \rfloor \geq \lfloor n/n + 1/2 \rfloor = 1$. Hence, $g(1) = n$.

Next, for $k \geq 2$, we have either $d \leq 2n/(2k-1)$ or $d > 2n/(2k-1)$. We consider these cases separately.

If $d \leq 2n/(2k-1)$, then $k - 1/2 \leq n/d$ which implies $k \leq n/d + 1/2$. Since k is an integer, we take the floor of both sides to find $k \leq \lfloor n/d + 1/2 \rfloor$.

If $d > 2n/(2k-1)$, then $k - 1/2 > n/d$, which implies $k > n/d + 1/2 \geq \lfloor n/d + 1/2 \rfloor$.

Thus, for $k \geq 2$, we have $\lfloor n/d + 1/2 \rfloor \geq k$ for $d \in \{1, 2, \dots, \lfloor 2n/(2k-1) \rfloor\}$. Since $\lfloor 2n/(2k-1) \rfloor \leq n$, we conclude that $g(k) = \lfloor 2n/(2k-1) \rfloor$. □

With this lemma, we can now find an exact formula for $\mathcal{R}_n$.

Proposition 14. *For $n \in \mathbb{N}$,*

$$\mathcal{R}_n = -n + \sum_{d=1}^{n} \left\lfloor \frac{2n}{2d-1} \right\rfloor (-1)^{d+1}.$$

Proof. We wish to count the number of odd terms in Seq. (7). By Lemma 13, there are n terms that are at least 1, and, for $k \geq 2$, there are $\lfloor 2n/(2k-1) \rfloor$ terms that are at least k. We can use this to count the number of terms that are equal to a given value.

For $k = 1$, there are $n - \lfloor 2n/3 \rfloor$ terms that are equal to 1. For $k \geq 2$, there are $\lfloor 2n/(2k-1) \rfloor - \lfloor 2n/(2k+1) \rfloor$ terms that are equal to k. Summing over odd k, we find

$$\begin{aligned} \mathcal{R}_n &= n - \left\lfloor \frac{2n}{3} \right\rfloor + \sum_{\substack{3 \leq k \leq n \\ k \text{ odd}}} \left(\left\lfloor \frac{2n}{2k-1} \right\rfloor - \left\lfloor \frac{2n}{2k+1} \right\rfloor \right) \\ &= -n + \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \left(\left\lfloor \frac{2n}{2k-1} \right\rfloor - \left\lfloor \frac{2n}{2k+1} \right\rfloor \right) \\ &= -n + \sum_{d=1}^{n} \left\lfloor \frac{2n}{2d-1} \right\rfloor (-1)^{d+1}, \end{aligned}$$

which completes the proof. □

We now work toward an asymptotic formula for the summation in $\mathcal{R}_n$ via results of Jacobi and Gauss.

To start, we need some notation. For $n \in \mathbb{N}$, $r \in \mathbb{Z}$, and $m \in \mathbb{N}$, let $d_{r,m}(n)$ denote the number of positive divisors of n that are congruent to r modulo m. We are interested in values of this function for $m = 4$ and r equal to 1 or 3. For example, with $n = 45$, we have $d_{1,4}(45) = 4$ and $d_{3,4}(45) = 2$ because the positive divisors of 45 are 1, 3, 5, 9, 15, and 45.

Lemma 15. *For $n \in \mathbb{N}$,*

$$\mathcal{R}_n = -n + \sum_{k=1}^{2n} \big(d_{1,4}(k) - d_{3,4}(k) \big).$$

Proof. Each integer d divides $\lfloor 2n/d \rfloor$ integers in the interval $[1, 2n]$. If we want to count the number of divisors that are 1 modulo 4 for all of the integers from 1 to $2n$, we see that 1 is a divisor of $\lfloor 2n/1 \rfloor$ terms, 5 is a divisor of $\lfloor 2n/5 \rfloor$ terms, 9 is a divisor of $\lfloor 2n/9 \rfloor$ terms, and so on. In other words, we get

$$\sum_{k=1}^{2n} d_{1,4}(k) = \left\lfloor \frac{2n}{1} \right\rfloor + \left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{9} \right\rfloor + \cdots .$$

Similarly, if we want to add up the number of divisors that are 3 modulo 4 for all of the integers from 1 to $2n$, we get

$$\sum_{k=1}^{2n} d_{3,4}(k) = \left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2n}{7} \right\rfloor + \left\lfloor \frac{2n}{11} \right\rfloor + \cdots .$$

Observe that the terms in each of the two above summations are eventually all 0, and thus these are finite sums.

Hence,

$$\sum_{d=1}^{n} \left\lfloor \frac{2n}{2d-1} \right\rfloor (-1)^{d+1} = \left\lfloor \frac{2n}{1} \right\rfloor - \left\lfloor \frac{2n}{3} \right\rfloor + \left\lfloor \frac{2n}{5} \right\rfloor - \left\lfloor \frac{2n}{7} \right\rfloor + \cdots + (-1)^{n+1} \left\lfloor \frac{2n}{2n-1} \right\rfloor$$
$$= \sum_{k=1}^{2n} \big(d_{1,4}(k) - d_{3,4}(k)\big).$$

The stated result then follows from the formula for $\mathcal{R}_n$ in Proposition 14. □

Next, let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$, the number of representations of n as a sum of two integer squares. For example, still with $n = 45$, we find $r_2(45) = 8$ because

$$45 = (\pm 3)^2 + (\pm 6)^2 = (\pm 6)^2 + (\pm 3)^2,$$

for a total of 8 combinations. Jacobi's two-square theorem relates $r_2(n)$ to the number of divisors of n that are 1 modulo 4 and that are 3 modulo 4.

Theorem 16 (Jacobi's two-square theorem)**.** *For $n \in \mathbb{N}$, the number of representations of n as a sum of integer squares is $r_2(n) = 4(d_{1,4}(n) - d_{3,4}(n))$.*

Jacobi proved this theorem, along with theorems about the number of representations of n using four squares, using six squares, and using eight squares, in 1829 with the use of elliptic theta functions. (See Grosswald [3].) For another approach, one can use the theory of binary quadratic forms. (In Section 5, we use results about binary quadratic forms. See, e.g., Dickson [2].)

One may also prove Theorem 16 in the context of the Gaussian integers, which is the ring

$$\mathbb{Z}[i] = \{a + bi : a, bi \in \mathbb{Z},\ i^2 = -1\}.$$

The norm of the Gaussian integer $a+bi$ is a^2+b^2. It follows that $r_2(n)$ is equal to the number of Gaussian integers with norm equal to a given positive integer n. With an understanding of what the prime elements of $\mathbb{Z}[i]$ are, along with the fact that $\mathbb{Z}[i]$ is a unique factorization domain, one may prove Theorem 16. (For details, see, e.g., Hardy & Wright [4, Theorem 278] or a wonderful video on the 3Blue1Brown channel on YouTube [8].)

Visualizing $\mathbb{Z}[i]$ as a lattice in the complex plane (with $a + bi \in \mathbb{Z}[i]$ corresponding to the point (a, b) in the plane), Jacobi's two-square theorem says that the number of lattice points on a circle of radius $\sqrt{n}$ centered at the origin is $4(d_{1,4}(n) - d_{3,4}(n))$. In general, the point (a, b) is on a circle of radius $\sqrt{a^2 + b^2}$ centered at the origin. Thus, adding the numbers of points on circles of radii $\sqrt{1}, \sqrt{2}, \ldots, \sqrt{2n}$ gives us the total number of lattice points different from the origin in the interior or on the boundary of a circle of radius $\sqrt{2n}$ centered at the origin. Back to our formula for $\mathcal{R}_n$, we now have

$$\mathcal{R}_n = -n + (1/4) \cdot \#\{(a,b) \in \mathbb{Z}^2 : 0 < a^2 + b^2 \le 2n\}. \tag{9}$$

Let $C(x)$ denote the number of lattice points in the interior or on the boundary of a circle of radius $\sqrt{x}$ centered at the origin. Via Eq. (9), we can calculate $C(2n)$ in terms of $\mathcal{R}_n$.

Proposition 17. *For all $n \in \mathbb{N}$, the number of lattice points within a circle of radius $\sqrt{2n}$ centered at the origin is $4\mathcal{R}_n + 4n + 1$. That is, we have $C(2n) = 4\mathcal{R}_n + 4n + 1$.*

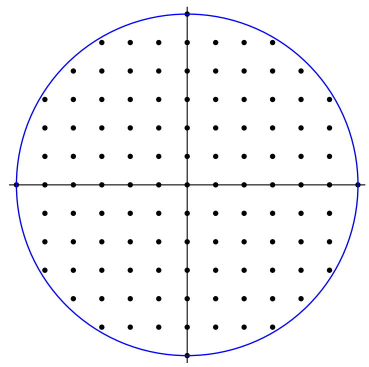

Figure 4: Lattice points in a circle of radius 6.

Here is an example.

Example 18. Suppose we want to count the number of lattice points in a circle of radius 6. (See Figure 4.) This is $C(36)$. By Proposition 17, $C(36) = C(2 \cdot 18) = 4\mathcal{R}_{18} + 4 \cdot 18 + 1$. We can calculate $\mathcal{R}_{18}$ (or look it up in Figure 1) to find $\mathcal{R}_{18} = 10$. Thus, the number of lattice points is $C(36) = 4 \cdot 10 + 4 \cdot 18 + 1 = 113$.

Geometrically, we see that $C(n)$ is a non-decreasing function. For any $n \in \mathbb{N}$,

$$0 \le C(2n+2) - C(2n) = 4\mathcal{R}_{n+1} + 4(n+1) + 1 - 4\mathcal{R}_n - 4n - 1 = 4\mathcal{R}_{n+1} + 4 - 4\mathcal{R}_n.$$

Thus, $\mathcal{R}_{n+1} - \mathcal{R}_n \ge -1$. This proves the following corollary.

Corollary 19. *The sequence $(\mathcal{R}_n)_{n\ge1}$ decreases by at most 1 in any step.*

A decrease of 1 from $\mathcal{R}_n$ to $\mathcal{R}_{n+1}$ occurs precisely when $2n+2$ and $2n+1$ each cannot be written as a sum of two integer squares. This occurs when the prime factorizations of $2n+2$ and $2n+1$ each contain some prime which is 3 modulo 4 raised to an odd power. From our data in Figure 1, we see a decrease by 1 for $n = 5$. Observe that $2n + 2 = 12 = 2^2 \cdot 3^1$ and $2n + 1 = 11^1$. Neither 11 nor 12 is a sum of two integer squares.

For comparison, there is no bound for the amount in which the sequences $(\mathcal{F}_n)_{n\ge1}$ and $(\mathcal{C}_n)_{n\ge1}$ can decrease in any step. Indeed, for $k \in \mathbb{N}$ and $n = 2^k$, one finds $\mathcal{F}_n - \mathcal{F}_{n-1} = \mathcal{C}_n - \mathcal{C}_{n-1} = -(k-1)$.

We now want an asymptotic formula for $\mathcal{R}_n$. To get there, we use an asymptotic result for $C(x)$ that is due to Gauss.

Theorem 20. *For $x \ge 1$, the number of lattice points in a closed disc of radius $\sqrt{x}$ is $C(x) = \pi x + \mathcal{O}(\sqrt{x})$.*

This result appears widely in the literature. See, for instance, Rademacher [7, Theorem 41] or Grosswald [3, Chapter 2, Section 7].

We now compute an asymptotic formula for $\mathcal{R}_n$.

Proposition 21. *Let $n \in \mathbb{N}$. Then $\mathcal{R}_n = (\pi/2 - 1)n + \mathcal{O}(\sqrt{n})$ and $\lim_{n\to\infty} \frac{1}{n}\mathcal{R}_n = \pi/2 - 1 \approx 0.570796$.*

Proof. Combining Proposition 17 and Theorem 20, we find

$$\begin{aligned}\mathcal{R}_n &= -n + (1/4) \cdot \big(C(2n) - 1\big) \\ &= -n + (1/4) \cdot \Big(2\pi n - 1 + \mathcal{O}\big(\sqrt{2n}\big)\Big) \\ &= -n + (\pi/2)n + \mathcal{O}\big(\sqrt{n}\big),\end{aligned}$$

which proves the first part. The limit behavior follows. □

Remark 22. The error bound for $C(x)$ in Theorem 20 has been improved since Gauss' original result. The *Gauss circle problem* is to give the best error term. Just as is the case for the Dirichlet divisor problem, the best error bound currently known for $C(x)$ is $\mathcal{O}\big(x^{131/416}\big)$, also due to Huxley [5]. (Note that in that paper, Huxley considers the problem of lattice points in a circle of radius x whereas we consider radius $\sqrt{x}$.) This gives us the following improved error bound for $\mathcal{R}_n$:

$$\mathcal{R}_n = (\pi/2 - 1)n + \mathcal{O}\big(n^{131/416}\big).$$

Remark 23. When we defined $\lfloor x \rceil$ in the introduction, we chose to always round up half-integers. Suppose we choose a different convention for rounding half-integers (e.g., always rounding down, or always rounding to the nearest even integer, or something else). Call the new rounding function $R'(x)$ and consider the sequence $(\mathcal{R}'_n)$ defined by

$$\mathcal{R}'_n = \#\{1 \le k \le n : R'(n/k) \text{ is odd}\}.$$

To compute $|\mathcal{R}'_n - \mathcal{R}_n|$, we need only consider those k for which n/k is a half-integer. But n/k is a half-integer when $n/k = l/2$ for odd l, which means k is a divisor of $2n$. By Lemma 12, the integer $2n$ has at most $2\sqrt{2n}$ divisors. Thus, there are at most $2\sqrt{2n}$ such k, and so

$$|\mathcal{R}'_n - \mathcal{R}_n| \le 2\sqrt{2n},$$

from which we conclude $\mathcal{R}'_n = \mathcal{R}_n + \mathcal{O}\big(\sqrt{n}\big) = n(\pi/2 - 1) + \mathcal{O}\big(\sqrt{n}\big)$. As a result, we have the same asymptotic behavior no matter the convention we use for rounding half-integers.

4 Counting by congruence class with shifted floors

We can now generalize our results from the preceding sections in three ways.

First, recall that we can write $\lfloor x \rceil$ in terms of the floor function by $\lfloor x \rceil = \lfloor x + 1/2 \rfloor$. We may therefore think of $\lfloor x \rceil$ as a $1/2$-*shifted floor function* and the corresponding sequence

$$\left\lfloor \frac{n}{1} \right\rceil, \left\lfloor \frac{n}{2} \right\rceil, \ldots, \left\lfloor \frac{n}{n} \right\rceil = \left\lfloor \frac{n}{1} + \frac{1}{2} \right\rfloor, \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor, \ldots, \left\lfloor \frac{n}{n} + \frac{1}{2} \right\rfloor$$

as a $1/2$-*shifted floor sequence of length* n. We now consider an arbitrary shift $\alpha \in \mathbb{R}$.

Second, we now have sequences of length n where the kth term, for $k \in \{1, 2, \ldots, n\}$, is $\lfloor n/k + \alpha \rfloor$. We can offset the numerator n by some real number ν, resulting in a general term of the form $\lfloor (n - \nu)/k + \alpha \rfloor$.

Third, when we defined $\mathcal{F}_n$, $\mathcal{C}_n$, and $\mathcal{R}_n$ we focused on the number of odd terms in each sequence, which means counting the number of terms in each sequence which are congruent to 1 modulo 2. We now count the number of terms in each sequence that are congruent to r modulo m for integers r and m with $m \geq 1$.

Let's set some notation. For $\alpha, \nu \in \mathbb{R}$, the α-*shifted,* ν-*offset, floor sequence of length* n is

$$\left\lfloor \frac{n - \nu}{1} + \alpha \right\rfloor, \left\lfloor \frac{n - \nu}{2} + \alpha \right\rfloor, \ldots, \left\lfloor \frac{n - \nu}{n} + \alpha \right\rfloor. \tag{10}$$

For $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, let $\mathcal{N}_{n,\alpha,\nu,r,m}$ equal the number of integers in Seq. (10) that are congruent to r modulo m. In other words, let

$$\mathcal{N}_{n,\alpha,\nu,r,m} = \#\left\{ 1 \leq k \leq n : \left\lfloor \frac{n - \nu}{k} + \alpha \right\rfloor \equiv r \pmod{m} \right\}.$$

Connecting to our earlier work, we see $\mathcal{F}_n = \mathcal{N}_{n,0,0,1,2}$ and $\mathcal{R}_n = \mathcal{N}_{n,1/2,0,1,2}$ for all $n \in \mathbb{N}$.

Since every integer is in exactly one congruence class modulo m, we immediately have

$$\sum_{r=1}^{m} \mathcal{N}_{n,\alpha,\nu,r,m} = n \tag{11}$$

for all n, α, ν, and m. If we let $m = 1$, we find $\mathcal{N}_{n,\alpha,\nu,r,1} = n$ for all n, α, ν, and r. In what follows, we assume $m \geq 2$. Furthermore, noting that $\mathcal{N}_{n,\alpha,\nu,r,m} = \mathcal{N}_{n,\alpha-1,\nu,r-1,m}$, we may suppose $\alpha \in [0, 1)$. Next, since the difference $\mathcal{N}_{n,\alpha,\nu,r,m} - \mathcal{N}_{n-1,\alpha,\nu-1,r,m}$ is 1 or 0 depending on whether $\lfloor (n - \nu)/n + \alpha \rfloor$ is congruent to r modulo m or not, we may suppose $\nu \in [0, 1)$. Finally, it turns out to be useful to take $r \in [1, m]$, the least positive integer in a given congruence class modulo m.

4.1 A sum of differences of floors

Our first task is to write $\mathcal{N}_{n,\alpha,\nu,r,m}$ as a sum of differences of floors. We generalize the approaches taken in writing $\mathcal{F}_n$ (Proposition 2) and $\mathcal{R}_n$ (Proposition 14) as summations involving differences of floors.

The following lemma generalizes Lemma 1 and Lemma 13.

Lemma 24. *For $\alpha, \nu \in [0, 1)$, $k \in \mathbb{N}$, and $n \in \mathbb{N}$ with $n\alpha \geq \nu$, let $g(k)$ equal the number of integers in an α-shifted, ν-offset, floor sequence of length n (Seq. (10)) that are greater than or equal to k. Then*

$$g(k) = \begin{cases} n, & \text{if } k = 1; \\ \lfloor (n-\nu)/(k-\alpha) \rfloor, & \text{if } k \geq 2. \end{cases}$$

Proof. To start, we have $\nu < 1 \leq n$ for all n Thus $n - \nu > 0$, which implies $(n-\nu)/1 + \alpha, (n-\nu)/2 + \alpha, \ldots, (n-\nu)/n + \alpha$ is a decreasing sequence. Looking at the final term, since $\alpha \geq \nu/n$, we have

$$\frac{n-\nu}{n} + \alpha \geq \frac{n-\nu}{n} + \frac{\nu}{n} = 1.$$

Thus, every term in this sequence is at least 1, and hence their floors are at least 1. This means $g(1) = n$.

Now, suppose $k \geq 2$ and let $t = (n-\nu)/(k-\alpha)$. Observe that $0 < t < n - \nu \leq n$. We wish to show that the first $\lfloor t \rfloor$ terms of Seq. (10) are at least k and that the remaining terms are less than k. Since Seq. (10) is a non-increasing sequence, it suffices to show that

$$\left\lfloor \frac{n-\nu}{t} + \alpha \right\rfloor \geq k > \left\lfloor \frac{n-\nu}{t+1} + \alpha \right\rfloor. \tag{12}$$

For the first inequality in Eq. (12), observe that

$$\frac{n-\nu}{t} + \alpha = \frac{n-\nu}{(n-\nu)/(k-\alpha)} + \alpha = k.$$

Since k is an integer, we have $\lfloor (n-\nu)/t + \alpha \rfloor = \lfloor k \rfloor = k \geq k$, as desired.

For the second inequality in Eq. (12), we have

$$\left\lfloor \frac{n-\nu}{t+1} + \alpha \right\rfloor \leq \frac{n-\nu}{t+1} + \alpha < \frac{n-\nu}{t} + \alpha = k.$$

This completes the proof. □

Note that in the above proof, we considered the cases $k = 1$ and $k \geq 2$ separately. Our argument for $k \geq 2$ doesn't work for $k = 1$ because, for $n\alpha > \nu$, our value of t would be $t = (n-\nu)/(1-\alpha) > n$, and we cannot have more than n terms in a sequence of n terms. This explains why we had to get rid of "extra" terms in the summation formula for $\mathcal{R}_n$ (Proposition 14), which involves $k = 1$, $\alpha = 1/2$, and $\nu = 0$, whereas we had no such adjustment in the summation formula for $\mathcal{F}_n$ (Proposition 2), which has $\alpha = \nu = 0$.

In what follows, we count the number of terms in Seq. (10) that are congruent to r modulo m. Since we have slightly different results for $k = 1$ and for $k \geq 2$ in Lemma 24, we obtain slightly different results for congruence classes with $r = 1$ and with $2 \leq r \leq m$ in the proposition (and subsequent results) below.

Proposition 25. *Suppose $m \geq 2$ and $\alpha, \nu \in [0, 1)$. Then for all $n \in \mathbb{N}$ with $n\alpha \geq \nu$,*

$$\mathcal{N}_{n,\alpha,\nu,1,m} = n - \left\lfloor \frac{n-\nu}{1-\alpha} \right\rfloor + \sum_{i \geq 0} \left(\left\lfloor \frac{n-\nu}{1+im-\alpha} \right\rfloor - \left\lfloor \frac{n-\nu}{2+im-\alpha} \right\rfloor \right)$$

and, for $2 \leq r \leq m$,

$$\mathcal{N}_{n,\alpha,\nu,r,m} = \sum_{i \geq 0} \left(\left\lfloor \frac{n-\nu}{r+im-\alpha} \right\rfloor - \left\lfloor \frac{n-\nu}{r+1+im-\alpha} \right\rfloor \right).$$

Proof. By Lemma 24, we have a formula for $g(d)$, the number of integers in Seq. (10) that are greater than or equal to a given value d. Now, let $G(d)$ equal the number of integers in Seq. (10) that are equal to a given value d. Then $G(d) = g(d) - g(d+1)$.

We first compute $G(1)$ by $G(1) = g(1) - g(2) = n - \lfloor (n-\nu)/2 - \alpha \rfloor$. Then, for $d \geq 2$,

$$G(d) = g(d) - g(d+1) = \lfloor (n-\nu)/(d-\alpha) \rfloor - \lfloor (n-\nu)/(d+1-\alpha) \rfloor.$$

To compute $\mathcal{N}_{n,\alpha,\nu,r,m}$, we need to count the number of integers in Seq. (10) that are congruent to r modulo m. Since $1 \leq r \leq m$, we have

$$\mathcal{N}_{n,\alpha,\nu,r,m} = G(r) + G(r+m) + G(r+2m) + \cdots .$$

For $2 \leq r \leq m$, we have

$$\mathcal{N}_{n,\alpha,\nu,r,m} = \sum_{i \geq 0} G(r+im) = \sum_{i \geq 0} \left(\left\lfloor \frac{n-\nu}{r+im-\alpha} \right\rfloor - \left\lfloor \frac{n-\nu}{r+1+im-\alpha} \right\rfloor \right).$$

For $r = 1$, we have

$$\mathcal{N}_{n,\alpha,\nu,1,m} = \sum_{i \geq 0} G(1+im) = n - \left\lfloor \frac{n-\nu}{2-\alpha} \right\rfloor + \sum_{i \geq 1} \left(\left\lfloor \frac{n-\nu}{1+im-\alpha} \right\rfloor - \left\lfloor \frac{n-\nu}{2+im-\alpha} \right\rfloor \right).$$

If we include the $i = 0$ term in the summation, to make it more closely resemble the formula for $r \neq 1$, we get the stated result. □

4.2 An asymptotic formula via summation

Our next task is to evaluate the sum

$$\sum_{i \geq 0} \left(\left\lfloor \frac{n-\nu}{r+im-\alpha} \right\rfloor - \left\lfloor \frac{n-\nu}{r+1+im-\alpha} \right\rfloor \right) \tag{13}$$

that appears in Proposition 25.

We start by simplifying $\frac{1}{n-\nu}$ times this sum. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x, that is, $\{x\} = x - \lfloor x \rfloor$. Then

$$\frac{1}{n-\nu}\sum_{i\geq 0}\left(\left\lfloor\frac{n-\nu}{r+im-\alpha}\right\rfloor - \left\lfloor\frac{n-\nu}{r+1+im-\alpha}\right\rfloor\right) = A_{\alpha,r,m} - \frac{1}{n-\nu}B_{n,\alpha,\nu,r,m}$$

for the following quantities:

$$A_{\alpha,r,m} = \sum_{i\geq 0}\left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha}\right),$$

and

$$B_{n,\alpha,\nu,r,m} = \sum_{i\geq 0}\left(\left\{\frac{n-\nu}{r+im-\alpha}\right\} - \left\{\frac{n-\nu}{r+1+im-\alpha}\right\}\right).$$

In general, the series $A_{\alpha,r,m}$ is an alternating series in which the absolute values of the terms decrease to zero. By the alternating series test, this series converges.

The series $B_{n,\alpha,\nu,r,m}$ is also an alternating series. If its terms, in absolute value, were decreasing, then we would have $B_{n,\alpha,\nu,r,m} = \mathcal{O}(1)$ and thus $(1/n)B_{n,\alpha,\nu,r,m} = \mathcal{O}(1/n)$. Unfortunately, this isn't the case.

If we can find an asymptotic formula for $B_{n,\alpha,\nu,r,m}$, then we get an asymptotic formula for $\mathcal{N}_{n,\alpha,\nu,r,m}$. To start, we revisit our results for $\mathcal{F}_n$ and $\mathcal{R}_n$ from earlier in this paper.

Example 26. The floor sequence $(\mathcal{F}_n)_{n\geq 1}$. For $\alpha = \nu = 0$, $r = 1$, and $m = 2$, we have $\mathcal{N}_{n,0,0,1,2} = \mathcal{F}_n$ for all $n \in \mathbb{N}$. Then

$$A_{0,1,2} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

and

$$B_{n,0,0,1,2} = \left\{\frac{n}{1}\right\} - \left\{\frac{n}{2}\right\} + \left\{\frac{n}{3}\right\} - \left\{\frac{n}{4}\right\} + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\left\{\frac{n}{k}\right\}.$$

We see that $A_{0,1,2} = \log 2$. (This is the Maclaurin series for $\ln(1+x)$, which converges for $-1 < x \leq 1$, evaluated at $x = 1$.) Proposition 7 implies that $\mathcal{N}_{n,0,0,1,2} = \mathcal{F}_n = n\log 2 + \mathcal{O}(\sqrt{n})$. With the asymptotic formula for $\mathcal{F}_n$ from Proposition 25, we find $B_{n,0,0,1,2} = \mathcal{O}(\sqrt{n})$.

Example 27. The rounding sequence $(\mathcal{R}_n)_{n\geq 1}$. For $\alpha = 1/2$, $\nu = 0$, $r = 1$, and $m = 2$, we have $\mathcal{N}_{n,1/2,0,1,2} = \mathcal{R}_n$ for all $n \in \mathbb{N}$. Then

$$A_{1/2,1,2} = \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{2}{2k+1}$$

and

$$B_{n,1/2,0,1,2} = \left\{\frac{2n}{1}\right\} - \left\{\frac{2n}{3}\right\} + \left\{\frac{2n}{5}\right\} - \left\{\frac{2n}{7}\right\} + \cdots = \sum_{k=0}^{\infty}(-1)^k\left\{\frac{2n}{2k+1}\right\}.$$

We see that $A_{1/2,1,2} = \pi/2$. (This is the Maclaurin series for $\arctan(x)$, which converges for $-1 \le x \le 1$, evaluated at $x = 1$.) Proposition 21 implies that $\mathcal{N}_{n,1/2,0,1,2} = \mathcal{R}_n = (\pi/2 - 1)n + \mathcal{O}(\sqrt{n})$. With the asymptotic formula for $\mathcal{R}_n$ from Proposition 25, we find $B_{n,1/2,0,1,2} = \mathcal{O}(\sqrt{n})$.

Remark 28. While we will not use any specific properties here, we note that the quantity $A_{\alpha,r,m}$ can be written in terms of the *digamma* function $\psi(z) = \Gamma'(z)/\Gamma(z)$, the logarithmic derivative of the the gamma function, in the following way:

$$A_{\alpha,r,m} = \frac{1}{m}\left(\psi\left(\frac{r+1-\alpha}{m}\right) - \psi\left(\frac{r-\alpha}{m}\right)\right).$$

The examples above suggest that $B_{n,\alpha,\nu,r,m} = \mathcal{O}(\sqrt{n})$ in general. This is in fact the case, as shown in Proposition 32 below. For the proof, we use Dirichlet's hyperbola method. Our method is adapted from the approach given in an answer on Mathematics Stack Exchange [6] which proved that, for an increasing sequence of positive integers $b_1, b_2, b_3, \ldots$,

$$\sum_{k\le n}\left\lfloor\frac{n}{b_k}\right\rfloor(-1)^k = n\sum_{k\le n}\frac{1}{b_k}(-1)^k + \mathcal{O}(\sqrt{n}).$$

The terms in the floors of our series Eq. (13) are not necessarily integers, so we need to prove a slightly more general result. With this in mind, we introduce a generalized notion of "divides."

Definition 29 (Real-ly divides)**.** Let $a \in \mathbb{R}$ and $b \in \mathbb{Z}$. We say *a real-ly divides b* if there is some $d \in \mathbb{Z}$ for which $\lceil da \rceil = b$. We denote this by $a \dagger b$, and we say a is a *real divisor* of b.

To see that this definition generalizes the usual definition of "divides," let's suppose that we have $a, b \in \mathbb{Z}$ with $a \mid b$. Then there is some $d \in \mathbb{Z}$ for which $da = b$. Hence, $\lceil da \rceil = da = b$, and so $a \dagger b$.

Remark 30. We only apply this notion of *real-ly divides* to working with positive numbers. If one wants to use this with negative numbers as well, it may be beneficial to modify the definition so that it has some symmetry with positive and negative numbers. One could say a real number a real-ly divides an integer b if there is some integer d for which one of the following holds: either $b \ge 0$ and $\lceil da \rceil = b$; or $b < 0$ and $\lfloor da \rfloor = b$. With this, one would additionally have $a \dagger b$ if and only if $(-a) \dagger b$.

The key property that we need is the following lemma.

Lemma 31. *Let $n \in \mathbb{N}$, $\nu \in [0,1)$, and $a \in \mathbb{R}$ with $a \geq 1$. Then*

$$\left\lfloor \frac{n-\nu}{a} \right\rfloor = \sum_{\substack{1 \leq d \leq n-\nu \\ a \dagger d}} 1.$$

Proof. To start, we have $n - \nu > 0$ and

$$\lfloor (n-\nu)/a \rfloor = \#\{ka : k \in \mathbb{Z},\ k \geq 1,\ ka \leq n-\nu\}.$$

Since $a \geq 1$, we have $\lceil k_1 a \rceil \neq \lceil k_2 a \rceil$ for all integers $k_1 \neq k_2$. Thus,

$$\lfloor (n-\nu)/a \rfloor = \#\{\lceil ka \rceil : k \in \mathbb{Z},\ k \geq 1,\ ka \leq n-\nu\}.$$

But this is just the cardinality of the set of numbers that a real-ly divides. We obtain

$$\lfloor (n-\nu)/a \rfloor = \#\{d \in \mathbb{Z} : 1 \leq d \leq n-\nu,\ a \dagger d\} = \sum_{\substack{1 \leq d \leq n-\nu \\ a \dagger d}} 1,$$

as desired. □

We can now apply Dirichlet's hyperbola method to show $B_{n,\alpha,\nu,r,m} = \mathcal{O}(\sqrt{n})$.

Proposition 32. *For any increasing sequence $b_0, b_1, b_2, \ldots$ of positive real numbers with the property that $b_k \geq 1$ and $b_k \geq k$ for all k, we have*

$$\sum_{k \leq n-\nu} \left\lfloor \frac{n-\nu}{b_k} \right\rfloor (-1)^k = n \sum_{k \leq n-\nu} \frac{(-1)^k}{b_k} + \mathcal{O}(\sqrt{n}).$$

Proof. Let

$$f(n) = \sum_{k \leq n-\nu} \left\lfloor \frac{n-\nu}{b_k} \right\rfloor (-1)^k.$$

To start, by Lemma 31, we have

$$f(n) = \sum_{k \leq n-\nu} (-1)^k \sum_{\substack{d \leq n-\nu \\ b_k \dagger d}} 1.$$

Changing the order of summation,

$$f(n) = \sum_{d \leq n-\nu} \sum_{\substack{b_k \dagger d \\ k \leq n-\nu}} (-1)^k.$$

We are summing over $d \le n-\nu$ and real divisors b_k of d. Thus, we have $b_k \le n-\nu$. As we have assumed that $b_k \ge k$, this implies $k \le n-\nu$. Thus, we need not explicitly state that $k \le n-\nu$. We have

$$f(n) = \sum_{d \le n-\nu} \sum_{b_k \dagger d} (-1)^k.$$

If $b_k \dagger d$, then $d-1 < b_k d' \le d$. Thus, instead of summing over $d \le n-\nu$, we may sum over d' and k such that $b_k d' \le n-\nu$:

$$f(n) = \sum_{b_k d' \le n-\nu} (-1)^k.$$

We now apply Dirichlet's hyperbola method. Consider the region R in the first quadrant of the xy-plane that is bounded by the hyperbola $xy = n-\nu$, the line $x=1$, and the line $y=1$. Let $A>0$. We split the region R into 3 subregions: R_1, the portion of R which lies above the line $y=(n-\nu)/A$; R_2, the portion of R which lies to the right of the line $x=A$; and R_3, which is the rectangle $[1,A]\times[1,(n-\nu)/A]$. (See Figure 5.)

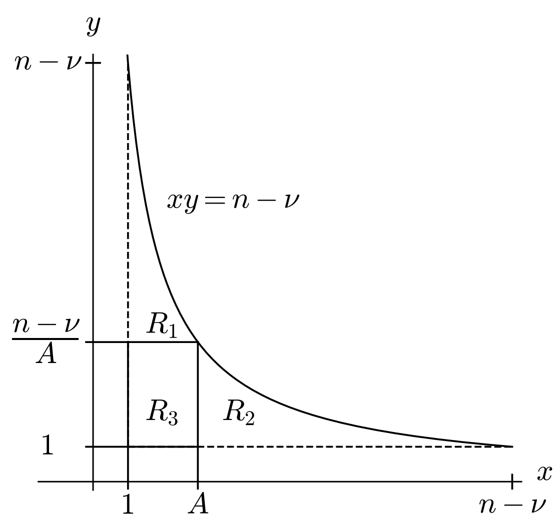

Figure 5: The regions R_1, R_2, R_3.

It follows that for each combination of b_k and d' such that $b_k d' \le n-\nu$, there is a point $(x,y)=(d',b_k)$ in R. Hence, this point is in exactly one of R_1, R_2, R_3.

We can sum over points (d',b_k) in $R_1 \cup R_3$ and $R_2 \cup R_3$, and then subtract a summation over R_3 because we have double counted. We have

$$f(n) = \sum_{d' \le A} \sum_{b_k \le (n-\nu)/d'} (-1)^k + \sum_{b_k \le (n-\nu)/A} \sum_{d' \le (n-\nu)/b_k} (-1)^k - \sum_{d' \le A} \sum_{b_k \le (n-\nu)/A} (-1)^k. \tag{14}$$

Since $b_0, b_1, b_2, \ldots$ is an increasing sequence, the summation $\sum_{b_k \le x} (-1)^k$ is equal to either 0 or 1, which is $\mathcal{O}(1)$. The first and third double sums in Eq. (14) are therefore $\mathcal{O}(A)$. Hence,

$$f(n) = \sum_{b_k \le (n-\nu)/A} \sum_{d' \le (n-\nu)/b_k} (-1)^k + \mathcal{O}(A) = \sum_{b_k \le (n-\nu)/A} (-1)^k \left\lfloor \frac{n-\nu}{b_k} \right\rfloor + \mathcal{O}(A).$$

Then, since $\lfloor (n-\nu)/b_k \rfloor = (n-\nu)/b_k + \mathcal{O}(1)$,

$$f(n) = (n-\nu) \sum_{b_k \le (n-\nu)/A} \frac{(-1)^k}{b_k} + \mathcal{O}\left(A + \frac{n-\nu}{A}\right).$$

Taking $A = \sqrt{n-\nu}$, we have

$$f(n) = (n-\nu) \sum_{b_k \le \sqrt{n-\nu}} \frac{(-1)^k}{b_k} + \mathcal{O}\big(\sqrt{n-\nu}\big)$$

All that remains to do is modify the summation so that we sum over $k \le n-\nu$. Let $K_n = \min\{k : b_k > \sqrt{n-\nu}\}$. (Such a quantity exists because $b_k \ge k$ for all $k \in \mathbb{Z}$.) Then

$$\left| \sum_{K_n \le k \le n} \frac{(-1)^k}{b_k} \right| < \left| \frac{(-1)^{K_n}}{b_{K_n}} \right| = \frac{1}{b_{K_n}} < \frac{1}{\sqrt{n-\nu}} = \mathcal{O}\left(\frac{1}{\sqrt{n-\nu}}\right),$$

where the first inequality holds because we have a finite alternating series. Thus,

$$\begin{aligned}
\sum_{b_k \le \sqrt{n-\nu}} \frac{(-1)^k}{b_k} &= \sum_{k < K_n} \frac{(-1)^k}{b_k} \\
&= \sum_{k \le n-\nu} \frac{(-1)^k}{b_k} - \sum_{K_n \le k \le n-\nu} \frac{(-1)^k}{b_k} \\
&= \sum_{k \le n-\nu} \frac{(-1)^k}{b_k} + \mathcal{O}\big(1/\sqrt{n-\nu}\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(n) &= (n-\nu) \sum_{b_k \le \sqrt{n-\nu}} \frac{(-1)^k}{b_k} + \mathcal{O}\big(\sqrt{n-\nu}\big) \\
&= (n-\nu) \sum_{k \le n-\nu} \frac{(-1)^k}{b_k} + \mathcal{O}\big(\sqrt{n-\nu}\big) \\
&= (n-\nu) \sum_{k \le n-\nu} \frac{(-1)^k}{b_k} + \mathcal{O}\big(\sqrt{n}\big).
\end{aligned}$$

Since ν times the convergent alternating sum is $\mathcal{O}(1)$, we get the stated result. $\square$

Remark 33. As mentioned in the introduction (Section 1.3), the best error bounds known for $\mathcal{F}_n$ and $\mathcal{R}_n$ are better than $\mathcal{O}(\sqrt{n})$. It would be interesting to see if the result of Proposition 25 could be similarly improved.

Combining Proposition 25 and Proposition 32, we obtain an asymptotic formula for $\mathcal{N}_{n,\alpha,\nu,r,m}$. We first give the result for $2 \le r \le m$, followed by a slight modification to get the result for $r = 1$. (In the proof of Proposition 25, we saw that counting with $r = 1$ is slightly different than counting with $r \ne 1$. Fortunately, via Eq. (11), if we can count for $r = 2, \ldots, m$, then we get a count for $r = 1$ for free.)

Corollary 34. *For $2 \le r \le m$, and $\alpha, \nu \in [0, 1)$, and $n \in \mathbb{N}$ with $n\alpha \ge \nu$,*

$$\mathcal{N}_{n,\alpha,\nu,r,m} = n \sum_{i \ge 0} \left(\frac{1}{r + im - \alpha} - \frac{1}{r + 1 + im - \alpha} \right) + \mathcal{O}(\sqrt{n}).$$

Proof. For integers r, m with $2 \le r \le m$ and for $\alpha \in [0, 1)$, define the sequence $b_0, b_1, b_2, \ldots$ as follows. For $i \ge 0$, let $b_{2i} = (r + im) - \alpha$ and $b_{2i+1} = (r + im) - \alpha + 1$. Then, for $\nu \in [0, 1)$,

$$\sum_{k \le n-\nu} \left\lfloor \frac{n-\nu}{b_k} \right\rfloor (-1)^k = \sum_{i \ge 0} \left(\left\lfloor \frac{n-\nu}{r + im - \alpha} \right\rfloor - \left\lfloor \frac{n-\nu}{r + 1 + im - \alpha} \right\rfloor \right). \tag{15}$$

(While one series is finite and the other is infinite, the terms in the infinite series are zero for all $i > (n - \nu - r + \alpha)/m$ and hence there are no convergence issues.)

We wish to show the sequence $b_0, b_1, b_2, \ldots$ satisfies the conditions of Proposition 32. We will show $b_k \ge 1$, $b_k \ge k$, and $b_{k+1} > b_k$ for all $k \ge 0$.

If k is even, then $k = 2i$ for some i and we have $b_k = (r + km/2) - \alpha$. Since $m \ge 2$, we have $b_k \ge k + r - \alpha \ge k$. If k is odd, then $k = 2i + 1$ for some i and we have $b_k = r + (k-1)m/2 - \alpha + 1$. Since $m \ge 2$, we have $b_k \ge (k-1) + r - \alpha + 1 \ge k$. Thus, $b_k \ge k$ for all $k \ge 0$. Additionally, since $b_0 = r - \alpha \ge 2 - \alpha > 1$ and $b_k \ge k \ge 1$ for all $k \ge 1$, we have $b_k \ge 1$ for all $k \ge 0$. Finally, for k even, $b_{k+1} - b_k = 1$, and for k odd, $b_{k+1} - b_k = m - 1 \ge 1$. Hence, $b_{k+1} > b_k$ for all $k \ge 0$.

Since the sequence $b_0, b_1, b_2, \ldots$ satisfies the conditions of Proposition 32, we conclude that

$$\sum_{k \le n-\nu} \left\lfloor \frac{n-\nu}{b_k} \right\rfloor (-1)^k = n \sum_{k \le n-\nu} \frac{(-1)^k}{b_k} + \mathcal{O}(\sqrt{n}).$$

Next, since we have an alternating series and $b_k \ge k$ for all k, we have

$$\left| \sum_{k > n-\nu} \frac{(-1)^k}{b_k} \right| < \left| \frac{(-1)^n}{b_n} \right| = \frac{1}{b_n} < \frac{1}{n} = \mathcal{O}\Big(\frac{1}{n}\Big).$$

Thus,

$$\sum_{k\le n-\nu}\left\lfloor\frac{n-\nu}{b_k}\right\rfloor(-1)^k = n\sum_{k\ge 0}\frac{(-1)^k}{b_k} - n\sum_{k>n-\nu}\frac{(-1)^k}{b_k} + \mathcal{O}(\sqrt{n})$$
$$= n\sum_{k\ge 0}\frac{(-1)^k}{b_k} - n\mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}(\sqrt{n})$$
$$= n\sum_{k\ge 0}\frac{(-1)^k}{b_k} + \mathcal{O}(1) + \mathcal{O}(\sqrt{n}).$$

Combined with Eq. (15), we conclude

$$\sum_{i\ge 0}\left(\left\lfloor\frac{n-\nu}{r+im-\alpha}\right\rfloor - \left\lfloor\frac{n-\nu}{r+1+im-\alpha}\right\rfloor\right)$$

is equal to

$$n\sum_{i\ge 0}\left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha}\right) + \mathcal{O}(\sqrt{n}).$$

Together with Proposition 25, this proves the result for $2 \le r \le m$. □

We now use Eq. (11) and Corollary 34 to compute $\mathcal{N}_{n,\alpha,\nu,r,m}$ for $r = 1$.

Corollary 35. *For any $m \ge 2$, $\alpha, \nu \in [0, 1)$, and $n \in \mathbb{N}$ with $n\alpha \ge \nu$,*

$$\mathcal{N}_{n,\alpha,\nu,1,m} = \frac{-\alpha n}{1-\alpha} + n\sum_{i\ge 0}\left(\frac{1}{1+im-\alpha} - \frac{1}{2+im-\alpha}\right) + \mathcal{O}(\sqrt{n}).$$

Proof. By Eq. (11),

$$\mathcal{N}_{n,\alpha,\nu,1,m} = n - \sum_{r=2}^{m}\mathcal{N}_{n,\alpha,\nu,r,m}.$$

Then, by Corollary 34

$$\mathcal{N}_{n,\alpha,\nu,1,m} = n - \sum_{r=2}^{m}\left(n\sum_{i\ge 0}\left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha}\right) + \mathcal{O}(\sqrt{n})\right)$$
$$= n - \sum_{r=2}^{m} n\sum_{i\ge 0}\left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha}\right) + \mathcal{O}(\sqrt{n})$$
$$= n - n\sum_{i\ge 0}\sum_{r=2}^{m}\left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha}\right) + \mathcal{O}(\sqrt{n}),$$

where we have changed the order of summation in the last step because we have a finite number of convergent alternating series. We have a telescoping series for each $i \geq 0$ which we can manipulate by

$$\begin{aligned}
\mathcal{N}_{n,\alpha,\nu,1,m} &= n - n \sum_{i\geq 0} \left(\frac{1}{2+im-\alpha} - \frac{1}{m+1+im-\alpha} \right) + \mathcal{O}(\sqrt{n}) \\
&= n - \frac{n}{1-\alpha} + \frac{n}{1-\alpha} - n \sum_{i\geq 0} \left(\frac{1}{2+im-\alpha} - \frac{1}{m+1+im-\alpha} \right) + \mathcal{O}(\sqrt{n}) \\
&= n - \frac{n}{1-\alpha} + n \sum_{i\geq 0} \left(\frac{1}{1+im-\alpha} - \frac{1}{2+im-\alpha} \right) + \mathcal{O}(\sqrt{n}).
\end{aligned}$$

Note that we have merely inserted $-n/(1-\alpha) + n/(1-\alpha)$ into our expression, and that we have not changed the order of summation in doing so. Hence,

$$\mathcal{N}_{n,\alpha,\nu,1,m} = \frac{-\alpha n}{1-\alpha} + n \sum_{i\geq 0} \left(\frac{1}{1+im-\alpha} - \frac{1}{2+im-\alpha} \right) + \mathcal{O}(\sqrt{n}),$$

as desired. □

4.3 An asymptotic formula via integration

From integral calculus, we know how to evaluate sums like $1 - 1/2 + 1/3 - 1/4 + \cdots$ and $1 - 1/3 + 1/5 - 1/7 + \cdots$ via integration of Maclaurin series as mentioned in Example 26 and Example 27. We take the same approach to evaluate the summation that appears in Corollary 34 and Corollary 35. The following proposition shows us how. As with previous work in this section, we first obtain a result for $2 \leq r \leq m$ and then use Eq. (11) to obtain a result for $r = 1$.

Proposition 36. *For integers r and m with $2 \leq r \leq m$, and for any $\alpha \in [0, 1)$,*

$$\sum_{i\geq 0} \left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha} \right) = \int_0^1 \frac{(1-x)x^{r-1-\alpha}}{1-x^m} \mathrm{d}x.$$

Proof. To start, for $\beta = r - 1 - \alpha$, a positive real number, and for any non-negative integer k, let

$$f_k(x) = x^\beta \sum_{i=0}^{k} \left(x^{im} - x^{im+1} \right).$$

Observe that $f_k(0) = f_k(1) = 0$. For all $x \in (0,1)$ and all $k \geq 0$,

$$\begin{aligned}
|f_k(x)| &= \left| x^\beta (1-x) \sum_{i=0}^{k} x^{im} \right| \\
&= \left| x^\beta (1-x) \frac{1 - x^{(k+1)m}}{1 - x^m} \right| \\
&= \left| \frac{x^\beta}{1 + x + x^2 + \cdots + x^{m-1}} \right| \left| \left(1 - x^{(k+1)m}\right) \right|,
\end{aligned}$$

a product of absolute values of two functions. Each absolute value is at most 1 for all $x \in (0,1)$. (We're using the fact that $\beta > 0$ here.) It follows that $|f_k(x)| \leq 1 \cdot 1 = 1$ for all $x \in (0,1)$. Earlier, we saw $f_k(0) = f_k(1) = 0$, so we can extend this interval to $[0,1]$. Since 1 is integrable on $[0,1]$, by the Dominated Convergence Theorem we have

$$\lim_{k\to\infty} \int_0^1 f_k(x)\,\mathrm{d}x = \int_0^1 \lim_{k\to\infty} f_k(x)\,\mathrm{d}x.$$

Starting with the integral in the statement of this proposition, and applying this limit and integration interchange, we find

$$\begin{aligned}
\int_0^1 \frac{(1-x)x^{r-1-\alpha}}{1-x^m}\mathrm{d}x &= \int_0^1 \lim_{k\to\infty} f_k(x)\mathrm{d}x = \lim_{k\to\infty} \int_0^1 f_k(x)\mathrm{d}x \\
&= \lim_{k\to\infty} \int_0^1 x^\beta \sum_{i=0}^{k} \left(x^{im} - x^{im+1}\right)\mathrm{d}x \\
&= \lim_{k\to\infty} \sum_{i=0}^{k} \left(\frac{x^{\beta+1+im}}{\beta+1+im} - \frac{x^{\beta+2+im}}{\beta+2+im} \right) \Bigg|_{x=0}^{x=1} \\
&= \lim_{k\to\infty} \sum_{i=0}^{k} \left(\frac{1}{\beta+1+im} - \frac{1}{\beta+2+im} \right) \\
&= \sum_{i=0}^{\infty} \left(\frac{1}{r+im-\alpha} - \frac{1}{r+1+im-\alpha} \right),
\end{aligned}$$

as desired. □

Combining Corollary 34 and Proposition 36, we obtain the following asymptotic formula for $\mathcal{N}_{n,\alpha,\nu,r,m}$ with $2 \leq r \leq m$, which we may then extend to $r = 1$ via Eq. (11). This is our main result.

Theorem 37. *Suppose $m \geq 1$, $\alpha, \nu \in [0,1)$, and $n \in \mathbb{N}$ with $n\alpha \geq \nu$. Then*

$$\mathcal{N}_{n,\alpha,\nu,1,m} = \frac{-\alpha n}{1-\alpha} + n \int_0^1 \frac{(1-x)x^{-\alpha}}{1-x^m} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big),$$

and, for $2 \leq r \leq m$,

$$\mathcal{N}_{n,\alpha,\nu,r,m} = n \int_0^1 \frac{(1-x)x^{r-1-\alpha}}{1-x^m} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big).$$

Proof. For $2 \leq r \leq m$, the result follows from Corollary 34 and Proposition 36. We need to prove the result for $r = 1$.

By Eq. (11),

$$\mathcal{N}_{n,\alpha,\nu,1,m} = n - \sum_{r=2}^{m} \mathcal{N}_{n,\alpha,\nu,r,m}.$$

Thus,

$$\begin{aligned}\mathcal{N}_{n,\alpha,\nu,1,m} = n - \sum_{r=2}^{m} \mathcal{N}_{n,\alpha,\nu,r,m} &= n - \sum_{r=2}^{m} \left(n \int_0^1 \frac{(1-x)x^{r-1-\alpha}}{1-x^m} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big) \right) \\ &= n - n \int_0^1 \frac{x^{-\alpha}(1-x)}{1-x^m} \sum_{r=2}^{m} x^{r-1} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big).\end{aligned}$$

The finite series inside the integral will cancel with the denominator nicely if we include one more term. We do so as follows:

$$\begin{aligned}\mathcal{N}_{n,\alpha,\nu,1,m} - n \int_0^1 \frac{(1-x)x^{-\alpha}}{1-x^m} \mathrm{d}x &= n - n \int_0^1 \frac{x^{-\alpha}(1-x)}{1-x^m} \sum_{r=1}^{m} x^{r-1} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big) \\ &= n - n \int_0^1 x^{-\alpha} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big).\end{aligned}$$

Since $0 \leq \alpha < 1$, this improper integral converges to $1/(1-\alpha)$. Solving for $\mathcal{N}_{n,\alpha,\nu,1,m}$, we find

$$\mathcal{N}_{n,\alpha,\nu,1,m} = n - \frac{n}{1-\alpha} + n \int_0^1 \frac{(1-x)x^{-\alpha}}{1-x^m} \mathrm{d}x + \mathcal{O}\big(\sqrt{n}\big),$$

which simplifies to the stated result. □

Observe that $\mathcal{N}_{n,\alpha,\nu,r,m}$ is asymptotically linear and our formula is independent of ν. We record the corresponding slope below in the following corollary.

Corollary 38. *For $\alpha, \nu \in [0,1)$ and $1 \le r \le m$,*

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,\alpha,\nu,r,m} = \begin{cases} \dfrac{-\alpha}{1-\alpha} + \displaystyle\int_0^1 \frac{(1-x)x^{-\alpha}}{1-x^m}\mathrm{d}x, & \text{if } r = 1; \\ \displaystyle\int_0^1 \frac{(1-x)x^{r-1-\alpha}}{1-x^m}\mathrm{d}x, & \text{if } 2 \le r \le m. \end{cases}$$

Via integration, we can compute specific values of $\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,\alpha,\nu,r,m}$ for $1 \le r \le m \le 4$. For $\alpha = \nu = 0$, exact and rounded values are in Figure 6 and Figure 7. For $\alpha = 1/2$ and $\nu = 0$, exact and rounded values are in Figure 8 and Figure 9.

r \ m	1	2	3	4
1	1	$\log 2$	$\frac{1}{9}\sqrt{3}\pi$	$\frac{1}{8}\pi + \frac{1}{4}\log 2$
2		$-\log 2 + 1$	$-\frac{1}{18}\sqrt{3}\pi + \frac{1}{2}\log 3$	$\frac{1}{8}\pi - \frac{1}{4}\log 2$
3			$-\frac{1}{18}\sqrt{3}\pi - \frac{1}{2}\log 3 + 1$	$-\frac{1}{8}\pi + \frac{3}{4}\log 2$
4				$-\frac{1}{8}\pi - \frac{3}{4}\log 2 + 1$

Figure 6: Values of $\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,0,0,r,m}$ for $1 \le m \le 4$.

r \ m	1	2	3	4
1	1.000000	0.693147	0.604600	0.565986
2		0.306853	0.247006	0.219412
3			0.148394	0.127161
4				0.087441

Figure 7: $\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,0,0,r,m}$ for $1 \le m \le 4$, rounded to 6 decimal places.

5 Applications to finding parity and counting lattice points

Now that we have a formula for $\mathcal{N}_{n,\alpha,\nu,r,m}$, we focus on a few applications: computing a floor shift which results in an asymptotic 50/50 split of even and odd terms; and counting lattice points in a few families of ellipses.

$r \backslash m$	1	2	3	4
1	1	$\frac{1}{2}\pi - 1$	$\frac{1}{6}\sqrt{3}\pi + \frac{1}{2}\log 3 - 1$	$\frac{1}{4}\pi + \frac{1}{4}\sqrt{2}\log\left(3+2\sqrt{2}\right) - 1$
2		$-\frac{1}{2}\pi + 2$	$\frac{1}{6}\sqrt{3}\left(\pi - \sqrt{3}\log 3\right)$	$\frac{1}{4}\pi\left(\sqrt{2}-1\right)$
3			$-\frac{1}{3}\sqrt{3}\pi + 2$	$\frac{1}{4}\pi - \frac{1}{4}\sqrt{2}\log\left(3+2\sqrt{2}\right)$
4				$-\frac{1}{4}\pi\left(\sqrt{2}+1\right) + 2$

Figure 8: Values of $\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,1/2,0,r,m}$ for $1 \le m \le 4$.

$r \backslash m$	1	2	3	4
1	1.000000	0.570796	0.456206	0.408623
2		0.429204	0.357594	0.325323
3			0.186201	0.162173
4				0.103881

Figure 9: $\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,1/2,0,r,m}$ for $1 \le m \le 4$, rounded to 6 decimal places.

5.1 Shifting for parity

We return to the case where $m = 2$ and consider the problem of determining a shift α so that half of the terms are odd and half are even. It amounts to computing α for which

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,\alpha,\nu,1,2} = \lim_{n\to\infty} \frac{1}{n}\mathcal{N}_{n,\alpha,\nu,2,2} = 1/2.$$

By Corollary 38 (with the formula for $r = 2$ to avoid the extra term out front), we need α such that

$$\int_0^1 \frac{(1-x)x^{1-\alpha}}{1-x^2}\mathrm{d}x = \int_0^1 \frac{x^{1-\alpha}}{1+x}\mathrm{d}x = 1/2.$$

(Note that this is independent of ν.) Since we're using $r = 2$, for the remainder of this subsection, we count even integers instead of odd integers in an α-shifted floor sequence.

Let

$$p_e(\alpha) = \int_0^1 \frac{x^{1-\alpha}}{1+x}\mathrm{d}x.$$

Then $p_e(\alpha)$ is the (asymptotic) proportion of terms in an α-shifted floor sequence of length n that are even. We immediately see that $p_e(\alpha)$ is continuous, increasing, and concave up for $\alpha \in [0, 1]$. Furthermore, $p_e(0) = 1 - \log 2 < 1/2$ and $p_e(1) = \log 2 > 1/2$. Thus, there is a unique shift $\alpha_0 \in (0, 1)$ for which $p_e(\alpha_0) = 1/2$. A plot of $p_e(\alpha)$ appears in Figure 10. We see that the value of α for which $p_e(\alpha) = 1/2$ is between 0.6 and 0.7.

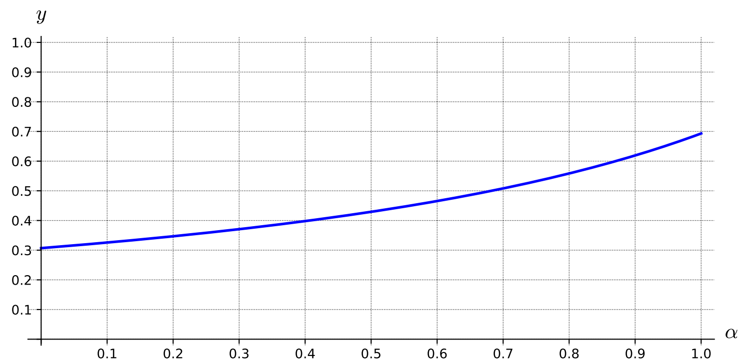

Figure 10: Plot of $y = p_e(\alpha) = \int_0^1 \frac{x^{1-\alpha}}{1+x} \mathrm{d}x$ for $\alpha \in [0, 1]$.

Computing with Sage [11], we can shrink the interval down. We compute

$$p_e(0.682379227335) < 1/2 \text{ and } p_e(0.682379227345) > 1/2.$$

Hence, we have $p_e(\alpha_0) = 1/2$ for

$$\alpha_0 \approx 0.68237922734.$$

We have generated some data with this approximate value. A plot of $y = \mathcal{N}_{n,0.68237922734,0,2,2}$ appears in Figure 11 along with the graph of $y = n/2$. In Figure 12 gives values of $\mathcal{N}_{n,0.68237922734,0,2,2}$ for random n with $10^5 < n < 10^6$.

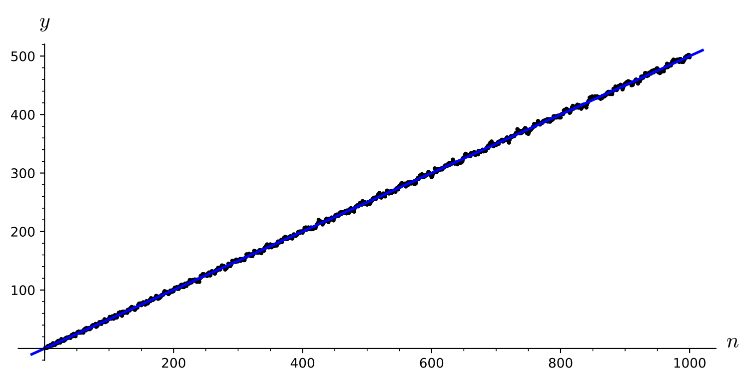

Figure 11: Plot of $y = \mathcal{N}_{n,\alpha,0,2,2}$ for $\alpha = 0.68237933734$ and $1 \leq n \leq 1000$ (black) along with the graph of $y = n/2$ (blue).

n	number of even terms	proportion of even terms
7015	3503	49.9359%
179220	89632	50.0123%
213788	106901	50.0033%
267093	133562	50.0058%
439675	219839	50.0003%
491213	245600	49.9987%
503521	251741	49.9961%
689325	344631	49.9954%
775294	387629	49.9977%
978029	489010	49.9995%

Figure 12: The number and proportion of even terms in an α-shifted, ν-offset, floor sequence of length n for $\alpha = 0.68237933634$ and various random n with $10^5 < n < 10^6$.

5.2 Lattice points in selected ellipses, number rings, and plane tilings

Since $\mathcal{N}_{n,1/2,0,1,2} = \mathcal{R}_n$, Proposition 17 says that the number of lattice points in a disc of radius $\sqrt{2n}$ centered at the origin is $4\mathcal{N}_{n,1/2,0,1,2} + 4n + 1$. In this subsection, we present a formula for the number of lattice points in a disc of radius $\sqrt{n}$ in terms of $\mathcal{N}_{n,\alpha,\nu,r,m}$. We then find similar formulas for the number of lattice points contained in the ellipses

$$x^2 + xy + y^2 = n \quad \text{and } x^2 + 2y^2 = n.$$

In Proposition 25, we wrote $\mathcal{N}_{n,\alpha,\nu,1,m}$ with a summation involving a difference of floors. In what follows, it is useful to have the formula for $\mathcal{N}_{n,\alpha,\nu,1,m}$ in the case where α is rational. If we suppose $\alpha = p/q$, then

$$\mathcal{N}_{n,p/q,\nu,1,m} = n - \left\lfloor \frac{(n-\nu)q}{q-p} \right\rfloor + \sum_{i\geq 0} \left(\left\lfloor \frac{(n-\nu)q}{q+qim-p} \right\rfloor - \left\lfloor \frac{(n-\nu)q}{2q+qim-p} \right\rfloor \right) \tag{16}$$

for all $n \in \mathbb{N}$ with $n\alpha \geq \nu$.

Also, for $n \in \mathbb{N}$, recall the function $d_{r,m}(n)$ which counts the number of positive divisors of n that are congruent to r modulo m.

5.2.1 Lattice points in the region $x^2 + y^2 \leq n$

In Proposition 17, we found a formula for the number of lattice points $(x, y) \in \mathbb{Z}^2$ contained in the disc $x^2 + y^2 \leq 2n$ in terms of $\mathcal{R}_n$. This is a result for discs with radius equal to the square root of an even number. We can extend the result to square roots of odd numbers as well, eventually obtaining a formula for the number of lattice points contained in the disc $x^2 + y^2 \leq n$.

For $n \geq 0$, again let $C(n)$ be the number of lattice points contained in the circular region $x^2 + y^2 \leq n$. Then

$$(C(n))_{n\geq 0} = (1, 5, 9, 9, 13, 21, 21, 21, 25, 29, 37, \dots),$$

which is sequence A057655 in OEIS. By the above proposition, we can compute $C(n)$ as follows with the help of $\mathcal{N}_{\lceil n/2\rceil,1/2,\{n/2\},1,2}$, where $\{n/2\}$ denotes the fractional part of $n/2$.

Proposition 39. *For $n \in \mathbb{N}$, the number of lattice points in a disc of radius $\sqrt{n}$ centered at the origin is*

$$C(n) = 4\mathcal{N}_{\lceil n/2\rceil,1/2,\{n/2\},1,2} + 4\lfloor n/2\rfloor + 1.$$

Proof. Let $C(n) = \#\big\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 \leq n\big\}$. We consider the cases of even n and odd n separately.

If n is even, then $n = 2k$ for some $k \in \mathbb{N}$. By Proposition 17

$$C(n) = C(2k) = 4\mathcal{R}_k + 4k + 1 = 4\mathcal{N}_{k,1/2,0,1,2} + 4k + 1 = 4\mathcal{N}_{n/2,1/2,0,1,2} + 2n + 1.$$

Note that $\lceil n/2\rceil = k = n/2$, $\{n/2\} = \{k\} = 0$, and $4\lfloor n/2\rfloor = 4\lfloor k\rfloor = 4k = 2n$. This proves the formula for $C(n)$ with n even.

If n is odd, then $n = 2k - 1$ for some $k \in \mathbb{N}$. By Jacobi's two-square theorem (Theorem 16),

$$C(n) = 1 + 4\sum_{j=1}^{n}\big(d_{1,4}(j) - d_{3,4}(j)\big) = 1 + 4\left(\left\lfloor\frac{n}{1}\right\rfloor - \left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{n}{5}\right\rfloor - \left\lfloor\frac{n}{7}\right\rfloor + \cdots\right).$$

In order to get this alternating floor sum, we use $\alpha = \nu = 1/2$, $r = 1$, and $m = 2$ with Eq. (16). We have

$$\mathcal{N}_{k,1/2,1/2,1,2} = 1 - k + \sum_{i\geq 0}\left(\left\lfloor\frac{2k-1}{4i+1}\right\rfloor - \left\lfloor\frac{2k-1}{4i+3}\right\rfloor\right) = \frac{1-n}{2} + \sum_{i\geq 0}\left(\left\lfloor\frac{n}{4i+1}\right\rfloor - \left\lfloor\frac{n}{4i+3}\right\rfloor\right)$$

for all $k \geq \nu/\alpha = 1$. Then,

$$C(n) = 1 + 4\left(\mathcal{N}_{k,1/2,1/2,1,2} + \frac{n-1}{2}\right) = 1 + 4\mathcal{N}_{(n+1)/2,1/2,1/2,1,2} + 2n - 2.$$

Note that $\lceil n/2\rceil = k = (n+1)/2$, $\{n/2\} = \{k + 1/2\} = 1/2$, and $4\lfloor n/2\rfloor = 4\lfloor k - 1/2\rfloor = 4(k-1) = 2n - 2$. This proves the formula for $C(n)$ with n odd. □

Example 40. To illustrate Proposition 39, we can compute the number of lattice points in a disc of radius $\sqrt{13}$. Note that $\lceil 13/2 \rceil = 7$. The number of lattice points in a disc of radius $\sqrt{13}$ involves the quantity $4\mathcal{N}_{7,1/2,1/2,1,2}$. To compute this, we examine the length-7 sequence

$$\left\lfloor \frac{7-1/2}{1} + \frac{1}{2} \right\rfloor, \left\lfloor \frac{7-1/2}{2} + \frac{1}{2} \right\rfloor, \ldots, \left\lfloor \frac{7-1/2}{7} + \frac{1}{2} \right\rfloor = 7, 3, 2, 2, 1, 1, 1,$$

which contains 5 odd terms. Thus, $\mathcal{N}_{7,1/2,1/2,1,2} = 5$. By Proposition 39, the number of lattice points in a disc of radius $\sqrt{13}$ is therefore

$$C(13) = 4\mathcal{N}_{7,1/2,1/2,1,2} + 4\lfloor 13/2 \rfloor + 1 = 4 \cdot 5 + 4 \cdot 6 + 1 = 45.$$

A disc of radius $\sqrt{13}$ appears in Figure 13, and one confirms there are 45 lattice points.

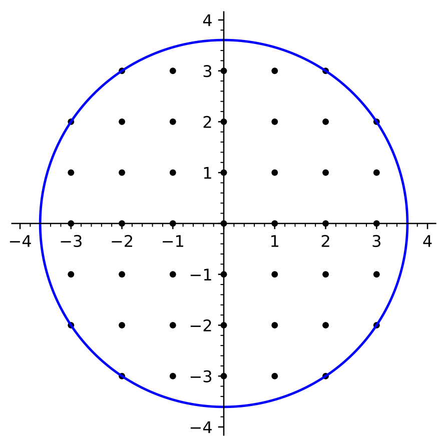

Figure 13: The 45 lattice points in a disc of radius $\sqrt{13}$, which are also the 45 Gaussian integers with norm at most 13.

Remark 41. We mentioned the ring of Gaussian integers, $\mathbb{Z}[i]$, earlier. For any $z = a + bi \in \mathbb{Z}[i]$, the norm of z is $N(z) = N(a + bi) = a^2 + b^2$. Thus, Proposition 39 gives a formula for the number of Gaussian integers with norm at most n. Following from Example 40, we know there are 45 Gaussian integers with norm at most 13. We visualize these Gaussian integers in Figure 13.

5.2.2 Lattice points in the region $x^2 + xy + y^2 \leq n$

Next, we consider the ellipse $x^2 + xy + y^2 = n$. In general, the quantity $ax^2 + bxy + cy^2$, for constants a, b, c, is a *binary quadratic form*. We take results about the number of representations of an integer n by a binary quadratic form from Dickson's book [2].

Proposition 42 ([2, Exercise XXII.2]). *Let $n \in \mathbb{N}$. Then the number of representations of $n = x^2 + xy + y^2$, for integers x, y, is $6\big(d_{1,3}(n) - d_{2,3}(n)\big)$.*

For $n \geq 0$, let $E(n)$ be the number of lattice points contained in the elliptical region $x^2 + xy + y^2 \leq n$. Then

$$(E(n))_{n\geq 0} = (1, 7, 7, 13, 19, 19, 19, 31, 31, 37, 37, 37, \dots),$$

which is sequence A038589 in OEIS. By the above proposition, we can compute $E(n)$ by counting the number of terms in a floor sequence of length n that are 1 modulo 3.

Corollary 43. *For $n \in \mathbb{N}$, the number of lattice points contained in the elliptical region $x^2 + xy + y^2 \leq n$ is*

$$E(n) = 6\mathcal{N}_{n,0,0,1,3} + 1.$$

Proof. Let $f(n) = \#\{(x, y) \in \mathbb{Z}^2 : x^2 + xy + y^2 = n\}$. Observe that $f(0) = 1$. Thus,

$$E(n) = \sum_{k=0}^{n} f(k) = 1 + \sum_{k=1}^{n} f(k). \tag{17}$$

Next, by Proposition 42,

$$\sum_{k=1}^{n} f(k) = 6\left(\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor - \cdots\right),$$

which is a finite sum since the floors are eventually zero. Using $\alpha = \nu = 0$, $r = 1$, and $m = 3$ with Eq. (16), we get

$$\mathcal{N}_{n,0,0,1,3} = \left(\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor - \cdots\right)$$

for all $n \in \mathbb{N}$. We conclude that $E(n) = 1 + 6\mathcal{N}_{n,0,0,1,3}$, as desired. □

Example 44. We compute the number of lattice points contained in the ellipse defined by the equation $x^2 + xy + y^2 = 30$. To do so, we consider the sequence

$$\left\lfloor \frac{30}{1} \right\rfloor, \left\lfloor \frac{30}{2} \right\rfloor, \dots, \left\lfloor \frac{30}{30} \right\rfloor = 30, 15, 10, 7, 6, 5, 4, 3(3), 2(5), 1(15),$$

where the number in parentheses indicates the number of terms with that value. There are 18 terms in this sequence that are congruent to 1 modulo 3. Hence, the ellipse defined by the equation $x^2 + xy + y^3 = 30$ contains $6\mathcal{N}_{30,0,0,1,3} + 1 = 6 \cdot 18 + 1 = 109$ lattice points. See Figure 14.

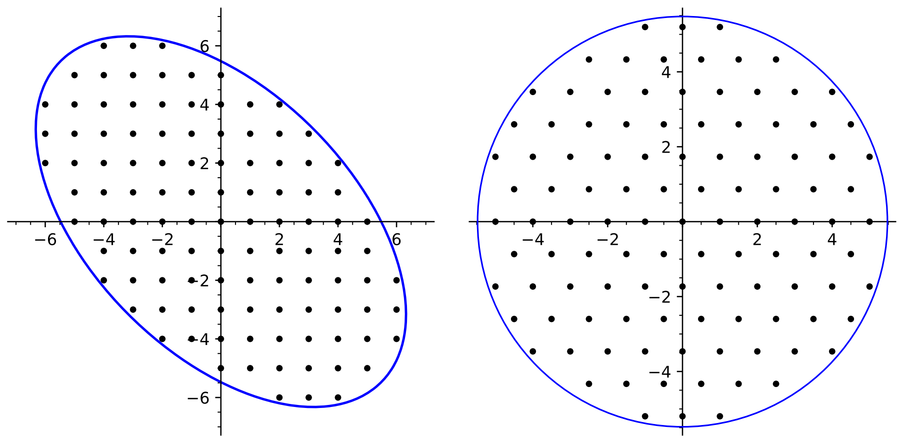

Figure 14: The 109 lattice points within the ellipse $x^2+xy+y^2=30$, and the 109 Eisenstein integers with norm at most 30.

From Theorem 37 and Figure 6, we see that $\mathcal{N}_{n,0,0,1,3}=\frac{1}{9}\sqrt{3}\pi n+\mathcal{O}\big(\sqrt{n}\big)$. We immediately obtain the following corollary.

Corollary 45. *For $n\in\mathbb{N}$, the number of lattice points in the elliptical region $x^2+xy+y^2\leq n$ is $\frac{2}{3}\sqrt{3}\pi n+\mathcal{O}\big(\sqrt{n}\big)$.*

Remark 46. Consider the ring of Eisenstein integers, $\mathbb{Z}[\omega]$, where $\omega=\frac{-1+\sqrt{-3}}{2}$, a primitive 3rd root of unity. For $z=a-b\omega\in\mathbb{Z}[\omega]$, the norm of z is $N(z)=N(a-b\omega)=a^2+ab+b^2$. Thus, Corollary 43 gives a formula for the number of Eisenstein integers with norm at most n. Following from Example 44, we know there are 109 Eisenstein integers with norm at most 30. We visualize these Eisenstein integers in Figure 14.

In general, the Eisenstein integers form a hexagonal (or triangular) lattice in the complex plane where neighboring Eisenstein integers are 1 unit apart. We can now count the number of such lattice points contained in a disc.

Corollary 47. *In a hexagonal lattice where neighboring lattice points are one unit apart, the number of lattice points in a disc of radius $\sqrt{n}$ is $1+6\mathcal{N}_{n,0,0,1,3}$, where $\mathcal{N}_{n,0,0,1,3}$ is the number of integers in the sequence*

$$\left\lfloor\frac{n}{1}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor,\ldots,\left\lfloor\frac{n}{n}\right\rfloor$$

that are 1 modulo 3.

The sequence $(\mathcal{N}_{n,0,0,1,3})_{n\geq 0}=(0,1,1,2,3,3,3,5,5,\ldots)$ is A014202 in OEIS.

5.2.3 Lattice points in the region $x^2 + 2y^2 \le n$

We now consider the ellipse $x^2 + 2y^2 = n$. The result below gives the number of representations of a natural number n in terms of the binary quadratic form $x^2 + 2y^2$.

Proposition 48 ([2, Exercise XXII.1]). *Let $n \in \mathbb{N}$. Then the number of representations of $n = x^2 + 2y^2$, for integers x, y, is $2\big(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)\big)$.*

Corollary 49. *Let $F(n)$ be the number of lattice points contained in the elliptical region $x^2 + 2y^2 \le n$. Then $F(0) = 1$, $F(1) = 3$, $F(2) = 5$, $F(5) = 11$, and for all $n \in \mathbb{N}$ with $n \ne 1, 2, 5$,*

$$F(n) = 1 + 2\mathcal{N}_{\lceil n/4\rceil,3/4,\{-n/4\},1,2} + 2\mathcal{N}_{\lceil n/4\rceil,1/4,\{-n/4\},1,2} + 2n + 2\lfloor n/3\rfloor - 4\lceil n/4\rceil.$$

Proof. Let $f(n) = \#\big\{(x, y) \in \mathbb{Z}^2 : x^2 + 2y^2 = n\big\}$. Observe that $f(0) = 1$. To start, we have

$$F(n) = \sum_{k=0}^{n} f(k) = 1 + \sum_{k=1}^{n} f(k). \tag{18}$$

Next, by Proposition 48,

$$\sum_{k=1}^{n} f(k) = 2\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{3}\right\rfloor - \left\lfloor\frac{n}{5}\right\rfloor - \left\lfloor\frac{n}{7}\right\rfloor + \left\lfloor\frac{n}{9}\right\rfloor + \cdots\right),$$

which is a finite sum since the floors are eventually zero. We need two alternating floor sums here: one for $\lfloor n/1\rfloor - \lfloor n/5\rfloor + \lfloor n/9\rfloor - \cdots$; and one for $\lfloor n/3\rfloor - \lfloor n/7\rfloor + \lfloor n/11\rfloor - \cdots$. Each requires $q = 4$ and $m = 2$. As was the case with computing the number of lattice points in a disc of radius $\sqrt{n}$ (Proposition 39), we have $\nu \ne 0$ here as well.

For $n \in \mathbb{N}$, let $a = \lceil n/4\rceil$ and $b = 4a - n$. Then $0 \le b < 4$. Using $\alpha = p/q = 3/4$, $\nu = b/4$, $r = 1$, and $m = 2$ with Eq. (16), we get

$$\mathcal{N}_{a,3/4,b/4,1,2} = a - \left\lfloor\frac{n}{1}\right\rfloor + \sum_{i\ge 0}\left(\left\lfloor\frac{n}{1+4i}\right\rfloor - \left\lfloor\frac{n}{5+4i}\right\rfloor\right)$$

for all $a \ge \nu/\alpha = b/3$. Using $\alpha = p/q = 1/4$, $\nu = b/4$, $r = 1$, and $m = 2$ with Eq. (16), we get

$$\mathcal{N}_{a,1/4,b/4,1,2} = a - \left\lfloor\frac{n}{3}\right\rfloor + \sum_{i\ge 0}\left(\left\lfloor\frac{n}{3+4i}\right\rfloor - \left\lfloor\frac{n}{7+4i}\right\rfloor\right)$$

for all $a \ge \nu/\alpha = b$.

Both equations hold for $a \ge b$. Since $a \ge 1$ and $0 \le b < 4$, this inequality is satisfied for all a, b except for $(a, b) = (1, 2), (1, 3), (2, 3)$, which correspond, respectively, to $n = 2$, $n = 1$, and $n = 5$.

Thus, for $n \in \mathbb{N}$ with $n \neq 1, 2, 5$,

$$\begin{aligned} F(n) &= 1 + 2\big(\mathcal{N}_{a,3/4,b/4,1,2} + \mathcal{N}_{a,1/4,b/4,1,2} - a + \lfloor n/3 \rfloor - a + n\big) \\ &= 1 + 2\mathcal{N}_{\lceil n/4 \rceil,3/4,\{-n/4\},1,2} + 2\mathcal{N}_{\lceil n/4 \rceil,1/4,\{-n/4\},1,2} + 2n + 2\lfloor n/3 \rfloor - 4\lceil n/4 \rceil. \end{aligned}$$

We can fill in the missing values by hand. We compute $f(1) = 2$, $f(2) = 2$, and $f(5) = 0$. And by the formula above with $n = 4$,

$$F(4) = 1 + 2\mathcal{N}_{1,3/4,0,1,2} + 2\mathcal{N}_{1,1/4,0,1,2} + 4 + 2\lfloor 4/3 \rfloor - 0 = 1 + 2 \cdot 1 + 2 \cdot 1 + 4 + 2 = 11.$$

Thus, $F(0) = 1$, $F(1) = F(0) + f(1) = 1 + 2 = 3$, $F(2) = F(1) + f(2) = 3 + 2 = 5$, and $F(5) = F(4) + f(5) = 11 + 0 = 11$. □

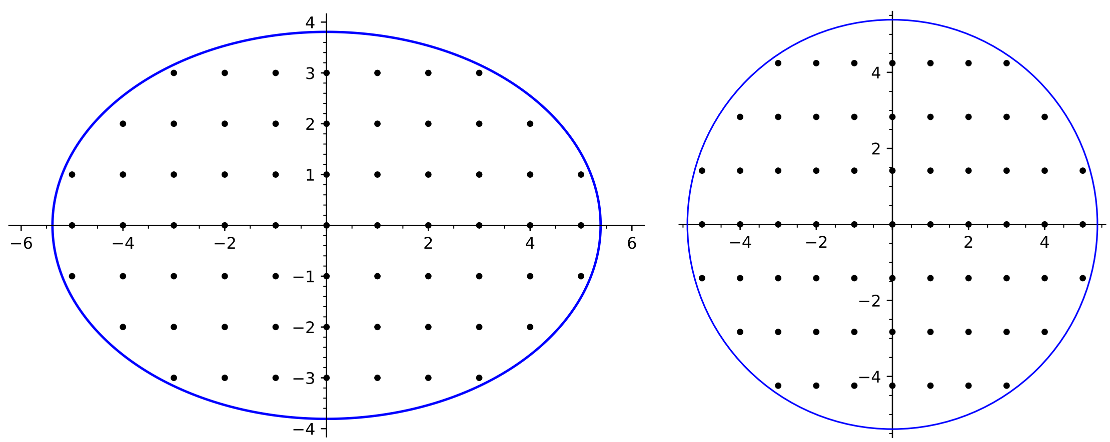

Figure 15: The 65 lattice points within the ellipse $x^2 + 2y^2 = 29$, and the 65 elements of $\mathbb{Z}[\sqrt{-2}]$ with norm at most 29.

Example 50. We compute the number of lattice points contained in the ellipse defined by $x^2 + 2y^2 = 29$. We have $n = 29$, $\lceil 29/4 \rceil = 8$, and $\{-29/4\} = 3/4$.

To compute $\mathcal{N}_{8,3/4,3/4,1,2}$, we consider the sequence

$$\left\lfloor \frac{8 - 3/4}{1} + 3/4 \right\rfloor, \left\lfloor \frac{8 - 3/4}{2} + 3/4 \right\rfloor, \ldots, \left\lfloor \frac{8 - 3/4}{8} + 3/4 \right\rfloor = 8, 4, 3, 2, 2, 1, 1, 1.$$

There are 4 odd numbers, so $\mathcal{N}_{8,3/4,3/4,1,2} = 4$.

To compute $\mathcal{N}_{8,1/4,3/4,1,2}$, we consider the sequence

$$\left\lfloor \frac{8 - 3/4}{1} + 1/4 \right\rfloor, \left\lfloor \frac{8 - 3/4}{2} + 1/4 \right\rfloor, \ldots, \left\lfloor \frac{8 - 3/4}{8} + 1/4 \right\rfloor = 7, 3, 2, 2, 1, 1, 1, 1.$$

There are 6 odd numbers, so $\mathcal{N}_{8,1/4,3/4,1,2} = 6$.

By Corollary 49, the number of lattice points in this ellipse is

$$\begin{aligned} F(29) &= 1 + 2\mathcal{N}_{8,3/4,3/4,1,2} + 2\mathcal{N}_{8,1/4,3/4,1,2} + 29 + 2\lfloor 29/3 \rfloor - 3 \\ &= 1 + 2 \cdot 4 + 2 \cdot 6 + 29 + 2 \cdot 9 - 3 \\ &= 65. \end{aligned}$$

See Figure 15.

Remark 51. Consider the ring $\mathbb{Z}[\sqrt{-2}]$. For $z = a + b\sqrt{-2} \in \mathbb{Z}[\sqrt{-2}]$, the norm of z is $N(z) = a^2 + 2b^2$. Thus, Corollary 49 gives a formula for the number of elements of $\mathbb{Z}[\sqrt{-2}]$ with norm at most n. Following from Example 50, we know there are 65 elements of $\mathbb{Z}[\sqrt{-2}]$ with norm at most 29. We visualize these elements in Figure 15.

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References

[1] T. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, 1976.

[2] L. E. Dickson, *Introduction to the Theory of Numbers*, Dover, 1957.

[3] E. Grosswald, *Representations of Integers as Sums of Squares*, Springer-Verlag, 1985.

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, The Clarendon Press, Oxford University Press, fifth edition, 1979.

[5] M. N. Huxley, Integer points, exponential sums and the Riemann zeta function, in *Number Theory for the Millennium II*, A K Peters/CRC Press, 2002, pp. 275–290.

[6] E. Naslund, Rounding is asymptotically useless? Mathematics Stack Exchange, 2012. Available at https://math.stackexchange.com/q/116687.

[7] H. Rademacher, *Lectures on Elementary Number Theory*, Robert E. Krieger Publishing Co., 1977.

[8] G. Sanderson, Pi hiding in prime regularities, 3Blue1Brown, YouTube, 2017. Available at https://youtu.be/NaL_Cb42WyY.

[9] C. Shor, Reflective numerical semigroups, *Albanian J. Math.* **17** (2023), 55–82.

[10] N. J. A. Sloane et al., The On-line Encyclopedia of Integer Sequences, 2020. Available at https://oeis.org/.

[11] W. A. Stein et al., *Sage Mathematics Software (Version 9.5)*, The Sage Development Team, 2022. Available at http://www.sagemath.org/.

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