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# Elementary Proofs of Congruences for POND and PEND Partitions

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**Abstract**

Recently, Ballantine and Welch considered various generalizations and refinements of POD and PED partitions. These are integer partitions wherein the odd parts must be distinct (in the case of POD partitions) or the even parts must be distinct (in the case of PED partitions). In the process, they were led to consider two classes of integer partitions which are, in some sense, the "opposite" of POD and PED partitions. They labeled these POND and PEND partitions, which are integer partitions wherein the odd parts cannot be distinct (in the case of POND partitions) or the even parts cannot be distinct (in the case of PEND partitions). In this work, we study these two types of partitions from an arithmetic perspective. Along the way, we are led to prove two infinite families of Ramanujan–like congruences modulo 3, one satisfied by the function $\mathrm{pond}(n)$, which counts the number of POND partitions of weight $n$, and the other satisfied by the function $\mathrm{pend}(n)$, which counts the number of PEND partitions of weight $n$, where $n$ is a nonnegative integer.

All of the proof techniques used herein are elementary, relying on classical $q$-series identities and generating function manipulations, along with mathematical induction.

## 1 Introduction

In the study of integer partitions, the partitions wherein the parts are distinct have long played a key role, due in large part to Euler's famous identity, which states that the number of partitions of weight $n$ into distinct parts equals the number of partitions of weight $n$ into odd parts. One of the most obvious refinements in this regard is to require distinct parts based on parity; i.e., to require either all of the even parts to be distinct or all of the odd parts to be distinct (while allowing the frequency of the other parts to be unrestricted). This leads to two types of partitions, those that we will call PED partitions (wherein the even parts must be distinct and the odd parts are unrestricted) and POD partitions (wherein the odd parts must be distinct and the even parts are unrestricted). We then define two corresponding enumerating functions, $\mathrm{ped}(n)$ [19, A001935], which counts the number of PED partitions of weight $n$, and $\mathrm{pod}(n)$ [19, A006950], which counts the number of POD partitions of weight $n$. These two functions have been studied from a variety of perspectives; the interested reader may wish to see [1, 2, 3, 4, 6, 7, 8, 9, 10, 13, 14, 15, 18, 20, 22, 23] for examples of work on identities involving, and arithmetic properties satisfied by, $\mathrm{ped}(n)$ and $\mathrm{pod}(n)$.

Recently, Ballantine and Welch [5] generalized and refined these two functions in numerous ways. One of the outcomes of their work was to consider integer partitions, which are, in some sense, the "opposite" of PED partitions and POD partitions. Namely, they considered PEND partitions and POND partitions, wherein the even (respectively, odd) parts are **not allowed** to be distinct. In a vein similar to that shared above, we let $\mathrm{pend}(n)$ denote the number of PEND partitions of weight $n$, and $\mathrm{pond}(n)$ denote the number of POND partitions of weight $n$. The first several values of $\mathrm{pend}(n)$ appear in the OEIS [19, A265254], while the first several values of $\mathrm{pond}(n)$ appear in [19, A265256].

It is worthwhile to share additional historical thoughts to place PEND and POND partitions in context. In his classic *Combinatory Analysis* [16], P. A. MacMahon proved that, for all $n \geq 0$, the number of partitions of weight $n$ wherein no part appears with multiplicity one equals the number of partitions of weight $n$ where all parts must be even or congruent to 3 modulo 6. As an aside, we note that numerous mathematicians have since generalized this theorem of MacMahon and have provided proofs of these results using both generating functions (which was MacMahon's original approach) as well as combinatorial arguments. The first half of the statement of MacMahon's theorem involves the function which counts the number of partitions wherein no part appears with multiplicity one, i.e., no part is allowed to be distinct. It is in this sense that POND and PEND partitions provide a natural, parity–based refinement of the partitions considered by MacMahon.

At the end of their paper, Ballantine and Welch [5] shared the following possibilities for future work:

> In particular, we note two areas of interest. The first is examining the arithmetic properties of these generalizations. Much work has been done in studying arithmetic properties of PED and POD partitions... Hence, this would be a natural topic of further study...

In light of this suggestion from Ballantine and Welch, our overarching goal in this work is to study $\mathrm{pond}(n)$ and $\mathrm{pend}(n)$ from an arithmetic perspective. With this in mind, we will first prove the following Ramanujan–like congruences satisfied by $\mathrm{pond}(n)$ and $\mathrm{pend}(n)$:

**Theorem 1.** *For all integers $n \geq 0$, we have*

$$\begin{aligned} \mathrm{pond}(3n+2) &\equiv 0 \pmod{2}, && (1)\\ \mathrm{pond}(27n+26) &\equiv 0 \pmod{3}, \quad \textit{and} && (2)\\ \mathrm{pond}(3n+1) &\equiv 0 \pmod{4}. && (3) \end{aligned}$$

**Theorem 2.** *For all integers $n \geq 0$, we have*

$$\mathrm{pend}(27n+19) \equiv 0 \pmod{3}.$$

We will then prove that each of these two functions satisfies an internal congruence modulo 3.

**Theorem 3.** *For all integers $n \geq 0$, we have* $\mathrm{pond}(27n+17) \equiv \mathrm{pond}(3n+2) \pmod{3}$.

**Theorem 4.** *For all integers $n \geq 0$, we have* $\mathrm{pend}(27n+10) \equiv \mathrm{pend}(3n+1) \pmod{3}$.

Finally, with the above results in hand, we will prove the following infinite families of non–nested Ramanujan–like congruences modulo 3 by induction.

**Theorem 5.** *For all integers $\alpha \geq 1$ and all $n \geq 0$, we have*

$$\mathrm{pond}\left(3^{2\alpha+1}n + \frac{23 \cdot 3^{2\alpha}+1}{8}\right) \equiv 0 \pmod{3}.$$

**Theorem 6.** *For all integers $\alpha \geq 1$ and all $n \geq 0$, we have*

$$\mathrm{pend}\left(3^{2\alpha+1}n + \frac{17 \cdot 3^{2\alpha}-1}{8}\right) \equiv 0 \pmod{3}.$$

Section 2 is devoted to providing the tools necessary for the remainder of the paper. In Section 3, we prove Theorems 1, 3, and 5. In Section 4, we prove Theorems 2, 4, and 6. All of the proof techniques used herein are elementary, relying on classical $q$-series identities and generating function manipulations, along with mathematical induction.

# 2 Preliminaries

Throughout this work, we will use the following shorthand notation for $q$-Pochhammer symbols:

$$f_r := (q^r;q^r)_\infty = (1-q^r)\cdot(1-q^{2r})\cdot(1-q^{3r})\dots$$

In order to prove the congruences mentioned above, several important 3–dissections of various $q$-series will be needed. These results will allow us to write the necessary generating functions in an appropriate fashion. We now catalog these results here.

**Lemma 7.** *We have*

$$\frac{f_2}{f_1f_4} = \frac{f_{18}^9}{f_3^2f_9^3f_{12}^2f_{36}^3} + q\frac{f_6^2f_{18}^3}{f_3^3f_{12}^3} + q^2\frac{f_6^4f_9^3f_{36}^3}{f_3^4f_{12}^4f_{18}^3}.$$

*Proof.* A proof of this identity appears in [21, Lemma 2.1]. □

**Lemma 8.** *We have*

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}.$$

*Proof.* A proof of this identity can be found in [14]. □

**Lemma 9.** *We have*

$$\frac{1}{f_1f_2} = \frac{f_9^9}{f_3^6f_6^2f_{18}^3} + q\frac{f_9^6}{f_3^5f_6^3} + 3q^2\frac{f_9^3f_{18}^3}{f_3^4f_6^4} - 2q^3\frac{f_{18}^6}{f_3^3f_6^5} + 4q^4\frac{f_{18}^9}{f_3^2f_6^6f_9^3}.$$

*Proof.* This lemma is equivalent to [17, Equation (39)]. □

**Lemma 10.** *We have*

$$\frac{f_2^2}{f_1} = \frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}.$$

*Proof.* For a proof of this result, see [11, (14.3.3)]. □

**Lemma 11.** *We have*

$$\frac{f_2}{f_1^2} = \frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}.$$

*Remark* 12. Note that

$$\frac{f_2}{f_1^2} = \sum_{n=0}^{\infty} \overline{p}(n)q^n$$

where $\overline{p}(n)$ is the number of overpartitions of $n$.

*Proof.* For a proof of Lemma 11, see [12, Theorem 1]. □

**Lemma 13.** *We have*

$$\frac{f_4}{f_1} = \frac{f_{12}f_{18}^4}{f_3^3f_{36}^2} + q\frac{f_6^2f_9^3f_{36}}{f_3^4f_{18}^2} + 2q^2\frac{f_6f_{18}f_{36}}{f_3^3}.$$

*Remark* 14. Note that

$$\frac{f_4}{f_1} = \sum_{n=0}^{\infty} \mathrm{ped}(n)q^n$$

where $\mathrm{ped}(n)$ is the number of partitions of $n$ wherein even parts are distinct (as mentioned in the introductory comments above).

*Proof.* Lemma 13 follows from [2, Theorem 3.1] and [11, (33.2.6)]. □

One additional $q$-series identity will be beneficial in the proof of Theorem 3.

**Lemma 15.** *We have*

$$\frac{f_3^3}{f_1} - q\frac{f_{12}^3}{f_4} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}}.$$

*Proof.* This identity appears in [11, (22.7.5)]. □

Lastly, we will utilize the following result which, at its core, relies on the binomial theorem and the divisibility properties of various binomial coefficients.

**Lemma 16.** *For all primes $p$ and all $j, k, m \geq 1$, $f_m^{p^j k} \equiv f_{pm}^{p^{j-1}k} \pmod{p^j}$.*

With all of these tools in hand, we are now in a position to prove the theorems listed above.

# 3 Congruences for pond($n$)

We begin by considering the function pond($n$). Although one can derive the generating function for pond($n$) from the work of Ballantine and Welch [5], we provide a proof of the result here for the sake of completeness.

**Theorem 17.** *We have*

$$\sum_{n=0}^{\infty} \mathrm{pond}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}.$$

*Proof.* By definition,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{pond}(n)q^n &= \frac{1}{f_2}\prod_{i=1}^{\infty}\left(\frac{1}{1-q^{2i-1}} - q^{2i-1}\right) \\
&= \frac{1}{f_2}\prod_{i=1}^{\infty}\left(\frac{1-q^{2i-1}+q^{4i-2}}{1-q^{2i-1}}\right) \\
&= \frac{1}{f_2}\prod_{i=1}^{\infty}\left(\frac{1+q^{6i-3}}{(1+q^{2i-1})(1-q^{2i-1})}\right) \\
&= \frac{1}{f_2}\cdot\frac{(-q^3;q^6)_\infty}{(q^2;q^4)_\infty} \\
&= \frac{1}{f_2}\cdot\frac{f_4}{f_2}\cdot\frac{(q^6;q^{12})_\infty}{(q^3;q^6)_\infty} \\
&= \frac{f_4}{f_2^2}\cdot\frac{f_6}{f_{12}}\cdot\frac{f_6}{f_3}
\end{aligned}$$

$$= \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}.$$

□

We can now move to a proof of Theorem 1.

*Proof of Theorem 1.* Our first goal is to 3–dissect the generating function for $\mathrm{pond}(n)$. Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{pond}(n) q^n &= \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}} \\
&= \frac{f_4}{f_2^2} \cdot \frac{f_6^2}{f_3 f_{12}} \\
&= \left( \frac{f_{12}^4 f_{18}^6}{f_6^8 f_{36}^3} + 2q^2 \frac{f_{12}^3 f_{18}^3}{f_6^7} + 4q^4 \frac{f_{12}^2 f_{36}^3}{f_6^6} \right) \cdot \frac{f_6^2}{f_3 f_{12}}
\end{aligned}$$

thanks to Lemma 11. This means we know the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{pond}(3n) q^{3n} &= \frac{f_6^2}{f_3 f_{12}} \cdot \frac{f_{12}^4 f_{18}^6}{f_6^8 f_{36}^3}, \\
\sum_{n=0}^{\infty} \mathrm{pond}(3n+1) q^{3n+1} &= \frac{f_6^2}{f_3 f_{12}} \cdot 4q^4 \frac{f_{12}^2 f_{36}^3}{f_6^6}, \quad \text{and} \\
\sum_{n=0}^{\infty} \mathrm{pond}(3n+2) q^{3n+2} &= \frac{f_6^2}{f_3 f_{12}} \cdot 2q^2 \frac{f_{12}^3 f_{18}^3}{f_6^7}.
\end{aligned}$$

This is equivalent to the following 3–dissection for the generating function for $\mathrm{pond}(n)$:

$$\sum_{n=0}^{\infty} \mathrm{pond}(3n) q^n = \frac{f_2^2}{f_1 f_4} \cdot \frac{f_4^4 f_6^6}{f_2^8 f_{12}^3} = \frac{f_4^3 f_6^6}{f_1 f_2^6 f_{12}^3},$$

$$\sum_{n=0}^{\infty} \mathrm{pond}(3n+1) q^n = 4q \frac{f_2^2}{f_1 f_4} \cdot \frac{f_4^2 f_{12}^3}{f_2^6} = 4q \frac{f_4 f_{12}^3}{f_1 f_2^4}, \quad \text{and} \tag{4}$$

$$\sum_{n=0}^{\infty} \mathrm{pond}(3n+2) q^n = 2 \frac{f_2^2}{f_1 f_4} \cdot \frac{f_4^3 f_6^3}{f_2^7} = 2 \frac{f_4^2 f_6^3}{f_1 f_2^5}. \tag{5}$$

We pause here to note that (5) implies (1) while (4) implies (3). Thus, in order to complete the proof of Theorem 1, we simply need to prove (2), and this requires us to 3–dissect the generating function for $\mathrm{pond}(3n+2)$, which appears in (5):

$$\sum_{n=0}^{\infty} \mathrm{pond}(3n+2) q^n$$

$$
\begin{aligned}
&= 2\frac{f_4^2 f_6^3}{f_1 f_2^5} \\
&\equiv 2\frac{f_4^2 f_2^9}{f_1 f_2^5} \pmod 3 \quad \text{(thanks to Lemma 16)} \\
&= 2\frac{f_4^2 f_2^4}{f_1} \\
&= 2(f_2 f_4)^3 \cdot \frac{f_2^2}{f_1} \cdot \frac{1}{f_2 f_4} \\
&\equiv 2 f_6 f_{12} \cdot \frac{f_2^2}{f_1} \cdot \frac{1}{f_2 f_4} \pmod 3 \\
&\equiv 2 f_6 f_{12} \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9} \right) \\
&\qquad \times \left( \frac{f_{18}^9}{f_6^6 f_{12}^2 f_{36}^3} + q^2 \frac{f_{18}^6}{f_6^5 f_{12}^3} + q^6 \frac{f_{36}^6}{f_6^3 f_{12}^5} + q^8 \frac{f_{36}^9}{f_6^2 f_{12}^6 f_{18}^3} \right) \pmod 3
\end{aligned}
$$

thanks to Lemmas 9 and 10. Thus, we know

$$\sum_{n=0}^{\infty} \text{pond}(9n+8) q^{3n+2} \equiv 2 f_6 f_{12} \cdot \frac{f_6 f_9^2}{f_3 f_{18}} \left( q^2 \frac{f_{18}^6}{f_6^5 f_{12}^3} + q^8 \frac{f_{36}^9}{f_6^2 f_{12}^6 f_{18}^3} \right) \pmod 3.$$

Therefore,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \text{pond}(9n+8) q^{n} &\equiv 2\frac{f_2^2 f_3^2 f_4}{f_1 f_6} \left( \frac{f_6^6}{f_2^5 f_4^3} + q^2 \frac{f_{12}^9}{f_2^2 f_4^6 f_6^3} \right) \pmod 3 \\
&= 2\frac{f_3^2}{f_1 f_4^2} \left( \frac{f_6^5}{f_2^3} + q^2 \frac{f_{12}^9}{f_4^3 f_6^4} \right) \\
&\equiv 2\frac{f_3^2}{f_1 f_4^2} \left( \frac{f_6^5}{f_6} + q^2 \frac{f_{12}^9}{f_{12} f_6^4} \right) \pmod 3 \quad \text{(using Lemma 16)} \\
&= 2\frac{f_3^2 f_4}{f_1 f_4^3} \left( f_6^4 + q^2 \frac{f_{12}^8}{f_6^4} \right) \\
&\equiv 2\frac{f_4}{f_1} \cdot \frac{f_3^2}{f_{12}} \left( f_6^4 + q^2 \frac{f_{12}^8}{f_6^4} \right) \pmod 3.
\end{aligned}
$$

We now use Lemma 13 to see that

$$\sum_{n=0}^{\infty} \text{pond}(27n+26) q^{3n+2} \equiv 2\frac{f_3^2}{f_{12}} \left( q^2 \frac{f_{12}^8}{f_6^4} \cdot \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + 2q^2 \frac{f_6^5 f_{18} f_{36}}{f_3^3} \right) \pmod 3$$

so that

$$\sum_{n=0}^{\infty} \text{pond}(27n+26) q^{n} \equiv 2\frac{f_1^2}{f_4} \left( \frac{f_4^9 f_6^4}{f_2^4 f_1^3 f_{12}^2} + 2\frac{f_2^5 f_6 f_{12}}{f_1^3} \right) \pmod 3$$

$$
\begin{aligned}
&= \frac{f_2^5}{f_1 f_4}\left(2\frac{f_4^9 f_6^4}{f_2^9 f_{12}^2} + 4f_6 f_{12}\right) \\
&\equiv \frac{f_2^5}{f_1 f_4}\left(2\frac{f_{12}^3 f_6^4}{f_6^3 f_{12}^2} + 4f_6 f_{12}\right) \pmod 3 \\
&\equiv \frac{f_2^5}{f_1 f_4}(6f_6 f_{12}) \pmod 3 \\
&\equiv 0 \pmod 3.
\end{aligned}
$$

This completes the proof of (2) and, therefore, Theorem 1. □

Equation (2) will serve as the base case for the proof by induction of Theorem 5. However, before we turn to the proof of Theorem 5, we first prove Theorem 3, which will be the "engine" for that proof by induction.

*Proof of Theorem 3.* Our goal is to prove that, for all $n \geq 0$,

$$\text{pond}(27n+17) \equiv \text{pond}(3n+2) \pmod 3.$$

From our work above, we know

$$\sum_{n=0}^{\infty} \text{pond}(3n+2)q^n \equiv 2\frac{f_4^2}{f_1}(f_2^4) \pmod 3. \tag{6}$$

Next, we need to determine a corresponding congruence for the generating function for pond$(27n+17)$. In our earlier work, we showed that

$$\sum_{n=0}^{\infty} \text{pond}(9n+8)q^n \equiv 2\frac{f_4}{f_1} \cdot \frac{f_3^2}{f_{12}}\left(f_6^4 + q^2\frac{f_{12}^8}{f_6^4}\right) \pmod 3.$$

We can then use Lemma 13 to see that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \text{pond}(9(3n+1)+8)q^{3n+1} \\
&\equiv 2\frac{f_3^2}{f_{12}}\left(f_6^4 \cdot q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + q^2\frac{f_{12}^8}{f_6^4} \cdot 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}\right) \pmod 3
\end{aligned}
$$

or

$$
\begin{aligned}
\sum_{n=0}^{\infty} \text{pond}(27n+17)q^n &\equiv 2\frac{f_1^2}{f_4}\left(\frac{f_2^6 f_3^3 f_{12}}{f_1^4 f_6^2} + 2q\frac{f_4^8 f_6 f_{12}}{f_1^3 f_2^3}\right) \pmod 3 \\
&\equiv 2\frac{f_2^6 f_3^3 f_{12}}{f_1^2 f_4 f_6^2} + 4q\frac{f_4^7 f_6 f_{12}}{f_1 f_2^3} \pmod 3
\end{aligned}
$$

$$\equiv 2\frac{f_2^6 f_1^9 f_4^3}{f_1^2 f_4 f_2^6} + 4q\frac{f_4^7 f_2^3 f_4^3}{f_1 f_2^3} \pmod 3$$
$$\equiv 2f_1^7 f_4^2 + 4q\frac{f_4^{10}}{f_1} \pmod 3$$
$$= 2\frac{f_4^2}{f_1}\left(f_1^8 + 2qf_4^8\right). \tag{7}$$

Therefore, in order to prove this theorem, we know from (6) and (7) that we must show the following:

$$2\frac{f_4^2}{f_1}\left(f_1^8 + 2qf_4^8\right) \equiv 2\frac{f_4^2}{f_1}(f_2^4) \pmod 3$$

or

$$f_1^8 + 2qf_4^8 \equiv f_2^4 \pmod 3.$$

To complete this proof, we are reminded of Lemma 15:

$$\frac{f_3^3}{f_1} - q\frac{f_{12}^3}{f_4} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}}.$$

Note that this implies that

$$\frac{f_1^9}{f_1} + 2q\frac{f_4^9}{f_4} \equiv \frac{f_{12} f_2^6}{f_2^2 f_{12}} \pmod 3$$

or

$$f_1^8 + 2qf_4^8 \equiv f_2^4 \pmod 3,$$

which is the desired result. □

With Theorems 1 and 3 in hand, we can now turn to proving the infinite family of Ramanujan–like congruences modulo 3 satisfied by pond$(n)$.

*Proof of Theorem 5.* We prove this theorem by induction on $\alpha$. Note that the base case, $\alpha = 1$, which corresponds to the arithmetic progression

$$3^3 n + \frac{23 \cdot 3^2 + 1}{8} = 27n + 26,$$

has already been proved in Theorem 1 above. Thus, we assume that, for some $\alpha \geq 1$ and all $n \geq 0$,

$$\text{pond}\left(3^{2\alpha+1}n + \frac{23 \cdot 3^{2\alpha} + 1}{8}\right) \equiv 0 \pmod 3.$$

We then want to prove that

$$\text{pond}\left(3^{2\alpha+3}n + \frac{23 \cdot 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod 3.$$

Note that

$$
\begin{aligned}
3^{2\alpha+1}n + \frac{23\cdot 3^{2\alpha}+1}{8} &= 3\left(3^{2\alpha}n\right) + \frac{23\cdot 3^{2\alpha}-15+16}{8}\\
&= 3\left(3^{2\alpha}n\right) + 3\left(\frac{23\cdot 3^{2\alpha-1}-5}{8}\right)+2\\
&= 3\left(3^{2\alpha}n + \frac{23\cdot 3^{2\alpha-1}-5}{8}\right)+2
\end{aligned}
$$

and it is easy to argue that

$$3^{2\alpha}n + \frac{23\cdot 3^{2\alpha-1}-5}{8}$$

is an integer for any $\alpha \geq 1$. Therefore, we have the following:

$$
\begin{aligned}
&\text{pond}\left(3^{2\alpha+1}n + \frac{23\cdot 3^{2\alpha}+1}{8}\right)\\
&= \text{pond}\left(3\left(3^{2\alpha}n + \frac{23\cdot 3^{2\alpha-1}-5}{8}\right)+2\right)\\
&\equiv \text{pond}\left(27\left(3^{2\alpha}n + \frac{23\cdot 3^{2\alpha-1}-5}{8}\right)+17\right) \pmod 3 \quad \text{(thanks to Theorem 3)}\\
&= \text{pond}\left(3^{2\alpha+3}n + \frac{23\cdot 3^{2\alpha+2}-27\cdot 5+17\cdot 8}{8}\right)\\
&= \text{pond}\left(3^{2\alpha+3}n + \frac{23\cdot 3^{2\alpha+2}+1}{8}\right)\\
&\equiv 0 \pmod 3
\end{aligned}
$$

thanks to the induction hypothesis. This completes the proof. □

# 4 Congruences for pend($n$)

We now turn our attention to proving Theorems 2, 4, and 6. We begin by finding the generating function for pend($n$).

**Theorem 18.** *We have*

$$\sum_{n=0}^{\infty} \text{pend}(n)q^n = \frac{f_2 f_{12}}{f_1 f_4 f_6}.$$

*Proof.* Using the definition of the partitions counted by pend($n$), we know

$$\sum_{n=0}^{\infty} \text{pend}(n)q^n = \frac{1}{(q;q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{1}{1-q^{2i}} - q^{2i}\right)$$

$$= \frac{f_2}{f_1} \prod_{i=1}^{\infty} \left( \frac{1 - q^{2i} + q^{4i}}{1 - q^{2i}} \right)$$

$$= \frac{f_2}{f_1} \prod_{i=1}^{\infty} \left( \frac{1 + q^{6i}}{(1 + q^{2i})(1 - q^{2i})} \right)$$

$$= \frac{f_2}{f_1} \frac{(-q^6; q^6)_\infty}{f_4}$$

$$= \frac{f_2}{f_1} \cdot \frac{f_{12}}{f_4 f_6}$$

$$= \frac{f_2 f_{12}}{f_1 f_4 f_6}.$$

□

We now turn our attention to proving Theorem 2. This will require that we 3–dissect the generating function for pend$(n)$ in a particular way.

*Proof of Theorem 2.* Thanks to Theorem 18, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \text{pend}(n) q^n &= \frac{f_2 f_{12}}{f_1 f_4 f_6} \\
&\equiv \frac{f_4^2}{f_1 f_2^2} \pmod 3 \quad \text{(from Lemma 16)} \\
&= \frac{f_4^3}{f_2^3} \cdot \frac{f_2}{f_1 f_4} \\
&\equiv \frac{f_{12}}{f_6} \cdot \frac{f_2}{f_1 f_4} \pmod 3.
\end{aligned}$$

From Lemma 7, we then know that

$$\sum_{n=0}^{\infty} \text{pend}(3n+1) q^{3n+1} \equiv \frac{f_{12}}{f_6} \left( q \frac{f_6^2 f_{18}^3}{f_3^3 f_{12}^3} \right) \pmod 3,$$

which means

$$\begin{aligned}
\sum_{n=0}^{\infty} \text{pend}(3n+1) q^n &\equiv \frac{f_4}{f_2} \cdot \frac{f_2^2 f_6^3}{f_1^3 f_4^3} \pmod 3 \\
&= \frac{f_2 f_6^3}{f_1^3 f_4^2} \\
&\equiv (f_2 f_4) \frac{f_6^3}{f_3 f_{12}} \pmod 3. \qquad (8)
\end{aligned}$$

Thanks to Lemma 8, we see that

$$\sum_{n=0}^{\infty} \mathrm{pend}(3n+1)q^n \equiv \frac{f_6^3}{f_3 f_{12}} \left( \frac{f_{12} f_{18}^4}{f_6 f_{36}^2} + 2q^2 f_{18} f_{36} + q^4 \frac{f_6 f_{36}^4}{f_{12} f_{18}^2} \right) \pmod{3}.$$

This now allows us to perform an additional 3–dissection to obtain

$$\sum_{n=0}^{\infty} \mathrm{pend}(9n+1)q^{3n} \equiv \frac{f_6^3}{f_3 f_{12}} \cdot \frac{f_{12} f_{18}^4}{f_6 f_{36}^2} \pmod{3},$$

which yields

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{pend}(9n+1)q^n &\equiv \frac{f_2^2 f_6^4}{f_1 f_{12}^2} \pmod{3} \\ &= \frac{f_2^2}{f_1} \cdot \frac{f_6^4}{f_{12}^2}. \end{aligned}$$

From Lemma 10, we can rewrite this result as

$$\sum_{n=0}^{\infty} \mathrm{pend}(9n+1)q^n \equiv \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right) \frac{f_6^4}{f_{12}^2} \pmod{3}. \tag{9}$$

Note that the power series representation of the right–hand side of the above congruence contains no terms of the form $q^{3n+2}$. Thus,

$$\sum_{n=0}^{\infty} \mathrm{pend}(9(3n+2)+1)q^{3n+2} \equiv 0 \pmod{3},$$

which means that, for all $n \geq 0$,

$$\mathrm{pend}(9(3n+2)+1) = \mathrm{pend}(27n+19) \equiv 0 \pmod{3}.$$

□

We next consider the proof of Theorem 4.

*Proof of Theorem 4.* Our goal here is to prove that, for all $n \geq 0$,

$$\mathrm{pend}(27n+10) \equiv \mathrm{pend}(3n+1) \pmod{3}.$$

Thanks to (9), we see that

$$\sum_{n=0}^{\infty} \mathrm{pend}(27n+10)q^{3n+1} \equiv q \frac{f_6^4 f_{18}^2}{f_9 f_{12}^2} \pmod{3},$$

which means

$$\sum_{n=0}^{\infty} \operatorname{pend}(27n+10)q^n \equiv \frac{f_2^4 f_6^2}{f_3 f_4^2} \pmod{3}. \tag{10}$$

From (8), we know

$$\begin{aligned}
\sum_{n=0}^{\infty} \operatorname{pend}(3n+1)q^n &\equiv \frac{f_2 f_4 f_6^3}{f_3 f_{12}} \pmod{3} \\
&\equiv \frac{f_2 f_4 f_6^3}{f_3 f_4^3} \pmod{3} \\
&= \frac{f_2 f_6^3}{f_3 f_4^2} \\
&\equiv \frac{f_2 f_2^3 f_6^2}{f_3 f_4^2} \pmod{3} \\
&= \frac{f_2^4 f_6^2}{f_3 f_4^2} \\
&\equiv \sum_{n=0}^{\infty} \operatorname{pend}(27n+10)q^n \pmod{3}
\end{aligned}$$

thanks to (10). □

We are now in a position to prove the infinite family of congruences in Theorem 6.

*Proof of Theorem 6.* We prove this theorem by induction on $\alpha$. Note that the base case, $\alpha = 1$, which corresponds to the arithmetic progression

$$3^3 n + \frac{17 \cdot 3^2 - 1}{8} = 27n + 19,$$

has already been proved in Theorem 2. Thus, we assume that, for some $\alpha \geq 1$ and all $n \geq 0$,

$$\operatorname{pend}\left(3^{2\alpha+1} n + \frac{17 \cdot 3^{2\alpha} - 1}{8}\right) \equiv 0 \pmod{3}.$$

We then want to prove that

$$\operatorname{pend}\left(3^{2\alpha+3} n + \frac{17 \cdot 3^{2\alpha+2} - 1}{8}\right) \equiv 0 \pmod{3}.$$

Note that

$$3^{2\alpha+1} n + \frac{17 \cdot 3^{2\alpha} - 1}{8} = 3\left(3^{2\alpha} n\right) + \frac{17 \cdot 3^{2\alpha} - 9 + 8}{8}$$

$$= 3\left(3^{2\alpha}n\right) + 3\left(\frac{17\cdot 3^{2\alpha-1}-3}{8}\right) + 1$$
$$= 3\left(3^{2\alpha}n + \frac{17\cdot 3^{2\alpha-1}-3}{8}\right) + 1$$

and it is easy to argue that

$$3^{2\alpha}n + \frac{17\cdot 3^{2\alpha-1}-3}{8}$$

is an integer for any $\alpha \geq 1$. Therefore, we have the following:

$$\begin{aligned}
&\text{pend}\left(3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{8}\right)\\
&= \text{pend}\left(3\left(3^{2\alpha}n + \frac{17\cdot 3^{2\alpha-1}-3}{8}\right) + 1\right)\\
&\equiv \text{pend}\left(27\left(3^{2\alpha}n + \frac{17\cdot 3^{2\alpha-1}-3}{8}\right) + 10\right) \pmod 3 \quad \text{(thanks to Theorem 4)}\\
&= \text{pend}\left(3^{2\alpha+3}n + \frac{17\cdot 3^{2\alpha+2}-27\cdot 3+10\cdot 8}{8}\right)\\
&= \text{pend}\left(3^{2\alpha+3}n + \frac{17\cdot 3^{2\alpha+2}-1}{8}\right)\\
&\equiv 0 \pmod 3
\end{aligned}$$

thanks to the induction hypothesis. This completes the proof.

□

# 5 Closing thoughts

While it is very satisfying to see the proofs provided above, it would be interesting to see combinatorial proofs of these divisibility properties. We leave it to the interested reader to obtain such proofs.

It may also be fruitful to consider further refinements of the functions pend($n$) and pond($n$). For example, rather than requiring that even parts must be repeated, one could restrict this requirement to only those parts which are divisible by 4 (with no such requirements on the other parts). It is certainly straightforward to find the generating functions for such refinements, which means that an analysis such as that above should be possible. Ballantine and Welch [5] share comments about such partitions (and their enumerating functions) near the end of their manuscript. The interested reader may wish to study such functions from an arithmetic perspective.

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