Journal of Integer Sequences, Vol. 27 (2024),
Article 24.1.5

Generalized Eulerian Polynomials with a Nonnegative Gamma Vector

Pasquale Petrullo and Domenico Senato
Università degli Studi della Basilicata
Dipartimento di Scienze Umane
Via Nazario Sauro 85
85100 Potenza
Italy
pasquale.petrullo@unibas.it
domenico.senato@unibas.it

Abstract

We define a family of generalized Eulerian polynomials depending on three parameters. We prove that these polynomials have a nonnegative gamma vector, and we provide a combinatorial description of the corresponding gamma coefficients. By assigning suitable integer values to the parameters, we obtain a new expansion of the nth Eulerian polynomial over the symmetric group $\mathfrak{S}_{n-1}$, a new description of the associated gamma vector, and an identity relating the derangements of $\mathfrak{S}_{2n}$ to the alternating permutations of $\mathfrak{S}_{2n+1}$.

1 Introduction

The Eulerian polynomial $A_n(x)$ counts the permutations of the symmetric group $\mathfrak{S}_n$ by number of excedances. More precisely, we have

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{exc}(w)}, \tag{1}$$

where

$$\mathrm{exc}(w) := \big|\{i \,|\, 1 \le i \le n,\, w(i) > i\}\big|.$$

As $A_n(x)$ is a palindromic polynomial of degree $n-1$, namely

$$A_n(x) = x^{n-1}\, A_n(1/x),$$

we may consider its expansion in terms of the so-called gamma-basis:

$$A_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} x^k (x+1)^{n-1-2k}. \tag{2}$$

The array $(\gamma_{n,k})_k$ is known as the gamma vector associated with $A_n(x)$. The coefficients $\gamma_{n,k}$ are known to be nonnegative integers, and that is why the polynomial $A_n(x)$ is said gamma-nonnegative (or gamma-positive).

The idea behind gamma-nonnegativity is contained in the pioneering paper of Foata and Schützenberger [8]. An explicit interpretation of $\gamma_{n,k}$ in terms of permutation enumeration can be given by means of "valley-hopping" [12], which is a technique based on the results of Foata and Strehl [9]. Gamma-nonnegativity implies unimodality, which in turn has been the subject of a considerable amount of research in the recent past [5,6,16]. The literature around this topics is still growing, and provides interesting variants, refinements, and generalizations of Eulerian polynomials. See, for example, the beautiful survey by Athanasiadis [3], and the book of Petersen [12].

More recently, Petrullo [13] provided a characterization and a combinatorial description of gamma-nonnegative Sheffer sequences, obtaining a direct connection with classical orthogonal polynomials. Moreover, Agapito et al. [1,2] introduced a new family of generalized Eulerian polynomials through the action of the Weyl algebra on formal power series. Then, as sometimes happens in the practice of mathematical investigation, in an attempt to prove that some of these polynomials are gamma-nonnegative, we came to define a new family of polynomials with nonnegative gamma vector.

In this paper we show that some of the most remarkable Eulerian-like polynomials can be obtained by assigning suitable integer values to the parameters q, t, y in the polynomials $P_n(q,t;x,y)$ defined by

$$\sum_{n\geq 0} P_n(q,t;x,y)\frac{z^n}{n!} = \left(\frac{x-y}{xe^{yz}-ye^{xz}}\right)^q e^{(x+y)qtz}.$$

As $P_n(q,t;x,y)$ is symmetric and homogeneous of degree n in x, y, by the fundamental theorem of symmetric functions, there exist uniquely determined coefficients $\gamma_{n,k}(q,t)$ such that

$$P_n(q,t;x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k}(q,t)\, x^k y^k\, (x+y)^{n-2k}.$$

So, $P_n(q,t;x) := P_n(q,t;x,1)$ is a palindromic polynomial in x, with gamma coefficients $\gamma_{n,k}(q,t)$. To achieve a combinatorial description of $P_n(q,t;x)$, we define

$$C_n(x) := \sum_{w\in\mathfrak{C}_n} x^{\mathrm{exc}(w)},$$

where $\mathfrak{C}_n$ is the set of all n-cycles in $\mathfrak{S}_n$. Then, from $C_n(x) = x\,A_{n-1}(x)$ we recover the generating function (Proposition 1)

$$\sum_{n\geq 1} C_n(x)\frac{z^n}{n!} := z + \log\left(\frac{x-1}{xe^z - e^{xz}}\right),$$

so that we may write

$$\sum_{n\geq 0} P_n(q,t;x,y)\frac{z^n}{n!} = e^{qtz+q\sum_{n\geq 2} C_n(x)\frac{z^n}{n!}} = e^{qtz+q\sum_{n\geq 2} x\,A_{n-1}(x)\frac{z^n}{n!}}.$$

Given this, the enumerative properties of $P_n(q,t;x)$ (Theorem 2), as well as its gamma-nonnegative expansion (Theorem 4), plainly follow from the classical results on Eulerian polynomials by means of the exponential formula. Moreover, by generalizing some results on derangement polynomials [3, 15], we obtain a combinatorial interpretation of $\gamma_{n,k}(q,t)$. Finally, starting from the curious special case $A_{n+1}(x) = P_n(2,1/2;x,1)$, we establish a new expansion of $A_n(x)$ over the symmetric group $\mathfrak{S}_{n-1}$, a new description of the associated gamma vector, and a new identity relating the derangements of $\mathfrak{S}_{2n}$ to the alternating permutations of $\mathfrak{S}_{2n+1}$ (Example 5.4).

2 The generalized Eulerian polynomials

Let q,t,x,y,z denote indeterminates over the field $\mathbb{Q}$ of rational numbers, set

$$\mathcal{P}(q,t;x,y;z) := \left(\frac{x-y}{xe^{yz} - ye^{xz}}\right)^q e^{(x+y)qtz}, \tag{3}$$

and consider the polynomials $P_n(q,t;x,y)$ that occur in the following formal power series expansion of $\mathcal{P}(q,t;x,y;z)$:

$$\mathcal{P}(q,t;x,y;z) = \sum_{n\geq 0} P_n(q,t;x,y)\frac{z^n}{n!}. \tag{4}$$

Note that, if α is any further independent indeterminate, then we have

$$\mathcal{P}(q,t;\alpha x,\alpha y;z) = \mathcal{P}(q,t;x,y;\alpha z) \text{ and } \mathcal{P}(q,t;x,y;z) = \mathcal{P}(q,t;y,x;z).$$

The identities above say us that $P_n(q,t;x,y)$ is a degree n homogeneous symmetric polynomial in x,y. In particular, this means that $P_n(q,t;x) := P_n(q,t;x,1)$ is palindromic of degree n in x, namely we have

$$x^n\,P_n(q,t;1/x) = P_n(q,t;x).$$

Recall that the Eulerian polynomials (1) have the following exponential generating function:

$$\mathcal{A}(x;z) := 1 + \sum_{n\geq 1} A_n(x)\frac{z^n}{n!} = \frac{1-x}{e^{(x-1)z} - x} = \frac{x-1}{xe^z - e^{xz}}e^z. \tag{5}$$

So, comparing with (3), we have

$$A_n(x) = P_n(1, 1/(x+1); x). \tag{6}$$

Now, for all $S \subseteq \{1, 2, \ldots, n\}$, we let $\mathfrak{C}(S)$ denote the set of all $w \in \mathfrak{S}_n$ that cyclically permute the elements of S, and that fix every $i \notin S$. Then, we define

$$C_S(x) := \sum_{w\in\mathfrak{C}(S)} x^{\mathrm{exc}(w)}. \tag{7}$$

As $C_S(x)$ depends only on $|S|$, we will often write $C_n(x)$ instead of $C_S(x)$ whenever $|S| = n$. The exponential generating function of the polynomials $C_n(x)$ is given below.

Proposition 1. *We have*

$$\mathcal{C}(x;z) := \sum_{n\geq 1} C_n(x)\frac{z^n}{n!} = z + \ln\left(\frac{x-1}{xe^z - e^{xz}}\right). \tag{8}$$

Proof. Recall that every permutation $w \in \mathfrak{S}_n$ induces a partition $\pi = \{S_1, S_2, \ldots, S_k\}$ of the set $\{1, 2, \ldots, n\}$ as follows: set $a, b \in S_i$ if and only if $b = \sigma^j(a)$ for a suitable j. We write $c_1, c_2, \ldots, c_k$ to denote the disjoint cycles (1-cycles included) of w, and we assume that $c_i \in \mathfrak{C}(S_i)$ for $1 \leq i \leq k$. Because $\mathrm{exc}(w) = \mathrm{exc}(c_1) + \mathrm{exc}(c_2) + \cdots + \mathrm{exc}(c_k)$, we may write

$$A_n(x) = \sum_{w\in\mathfrak{S}_n} x^{\mathrm{exc}(w)} = \sum_{\pi\in\Pi_n}\prod_{S\in\pi} C_S(x),$$

where Π_n is the set of all partitions of $\{1, 2, \ldots, n\}$. Now, the exponential formula ensures us that

$$\mathcal{A}(x;z) = e^{\mathcal{C}(x;z)},$$

and (8) comes directly from (5). □

Identity (8) is our starting point to develop a combinatorial description of the polynomial $P_n(q, t; x, y)$. To this aim, for every $w \in \mathfrak{S}_n$, we let $\mathrm{fix}(w)$ denote the number of fixed points of w. Moreover, we let $\mathrm{cyc}(w)$ denote the number of disjoint cycles of w.

Theorem 2. *We have*

$$P_n(q, t; x, y) = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{cyc}(w)} t^{\mathrm{fix}(w)} x^{\mathrm{exc}(w)} y^{\mathrm{exc}(w^{-1})} (x+y)^{\mathrm{fix}(w)}. \tag{9}$$

Proof. From (3), (4), and (8), it follows that

$$e^{q[\mathcal{C}(x;z)-(1-t)z]} = \left(\frac{x-1}{xe^z - e^{xz}}\right)^q e^{qtz} = 1 + \sum_{n\geq 1} \mathcal{P}_n\left(q, \frac{t}{x+1}; x, 1\right)\frac{z^n}{n!}.$$

On the other hand, again by the exponential formula, we recover

$$P_n(q, t/(x+1); x) = \sum_{\pi\in\Pi_n} q^{|\pi|} t^{\mathrm{sing}(\pi)} \prod_{S\in\pi} C_S(x), \tag{10}$$

where $|\pi|$ is the number of blocks of π, and $\mathrm{sing}(\pi)$ is the number of singletons of π. Now, as we have $|\pi| = \mathrm{cyc}(w)$ and $\mathrm{sing}(\pi) = \mathrm{fix}(w)$ for every w that induces π, by virtue of (7) we get

$$\sum_{\pi\in\Pi_n} q^{|\pi|} t^{\mathrm{sing}(\pi)} \prod_{S\in\pi} C_S(x) = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{cyc}(w)} t^{\mathrm{fix}(w)} x^{\mathrm{exc}(w)}. \tag{11}$$

Finally, since $P_n(q, t; x, y)$ is homogeneous of degree n in x, y, from (10) and (11) we conclude

$$\begin{aligned} P_n(q,t;x,y) &= y^n P_n(q,t;x/y) = \\ &= \sum_{w\in\mathfrak{S}_n} q^{\mathrm{cyc}(w)} t^{\mathrm{fix}(w)} x^{\mathrm{exc}(w)} y^{n-\mathrm{exc}(w)-\mathrm{fix}(w)} (x+y)^{\mathrm{fix}(w)}. \end{aligned}$$

At this point, (9) is obtained by observing that $\mathrm{exc}(w) + \mathrm{fix}(w) + \mathrm{exc}(w^{-1}) = n$. □

Remark 3. A slightly different version of the generating function (3) has been studied by Zeng [17] in connection with continued fractions.

3 Gamma-nonnegativity

As we have already pointed out, $P_n(q, t; x, y)$ is symmetric and homogeneous polynomial of degree n in x, y. So, by the fundamental theorem of symmetric functions, there exist uniquely determined coefficients $\gamma_{n,k}(q, t)$ such that

$$P_n(q,t;x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k}(q,t)\, x^k y^k (x+y)^{n-2k}. \tag{12}$$

We are going to prove that $P_n(q, t; x)$ is gamma-nonnegative as a polynomial in x. Namely, we will show that each $\gamma_{n,k}(q, t)$ occurring in (12) is itself a polynomial in q, t with nonnegative integer coefficients. To this goal, we first recall that $A_n(x)$ is gamma-nonnegative, which means that the coefficients $\gamma_{n,k}$ in (2) are nonnegative integers. In addition, we observe that from (5) and (8) it follows that

$$1 + \frac{1}{x}\frac{d}{dz}\mathcal{C}(x,z) = \frac{1}{x} + \frac{x-1}{xe^z - e^{xz}} e^z = \frac{1}{x} + \mathcal{A}(x,z),$$

whence

$$C_n(x) = x\,A_{n-1}(x) = \sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor} \gamma_{n-1,k}\, x^{k+1}\,(x+1)^{n-2(k+1)}, \text{ for } n \geq 2. \tag{13}$$

Moreover, note that every product $A_{n_1}(x)A_{n_2}(x)\cdots A_{n_i}(x)$ of Eulerian polynomials is a gamma-nonnegative polynomial of degree $n_1+n_2+\cdots+n_i-i$. More precisely, we have

$$A_{n_1}(x)A_{n_2}(x)\cdots A_{n_i}(x) = \sum_{k=0}^{\lfloor\frac{n-i}{2}\rfloor} \gamma_{n_1,n_2,\ldots,n_i;k}\, x^k\,(x+1)^{n-i-2k},$$

where $n := n_1+n_2+\cdots+n_i$ and

$$\gamma_{n_1,n_2,\ldots,n_i;k} = \sum_{j_1+j_2+\cdots+j_i=k} \gamma_{n_1,j_1}\gamma_{n_2,j_2}\cdots\gamma_{n_i,j_i}. \tag{14}$$

Theorem 4. *Each polynomial $\gamma_{n,k}(q,t)$ in (12) has nonnegative integer coefficients.*

Proof. From (10) and (13) it follows that

$$P_n(q,t;x) = \sum_{\pi\in\Pi_n} q^{|\pi|}t^{\mathrm{sing}(\pi)}x^{|\pi|-\mathrm{sing}(\pi)}(x+1)^{\mathrm{sing}(\pi)}\prod_{S\in\pi}A_{|S|-1}(x).$$

As $\prod_{S\in\pi}A_{|S|-1}(x)$ is a gamma-nonnegative polynomial of degree

$$n_\pi := n-2|\pi|+\mathrm{sing}(\pi),$$

there exist nonnegative integers $\gamma_{\pi,i}$ satisfying

$$\prod_{S\in\pi}A_{|S|-1}(x) = \sum_{i=0}^{\lfloor\frac{n_\pi}{2}\rfloor}\gamma_{\pi,i}\,x^i\,(x+1)^{n_\pi-2i}.$$

Hence, we have

$$P_n(q,t;x) = \sum_{\pi\in\Pi_n} q^{|\pi|}t^{\mathrm{sing}(\pi)}\sum_{i=0}^{\lfloor\frac{n_\pi}{2}\rfloor}\gamma_{\pi,i}\,x^{i+|\pi|-\mathrm{sing}(\pi)}\,(x+1)^{n-2(i+|\pi|-\mathrm{sing}(\pi))},$$

which ensures us that $\gamma_{n,k}(q,t)$ is a polynomial in q,t with nonnegative integer coefficients. □

Since gamma-nonnegativity implies unimodality, the following result directly follows from Theorem 4.

Corollary 5. *If q,t are nonnegative integers, then $P_n(q,t;x)$ is a unimodal polynomial in x.*

Remark 6. Thanks to the "transformation fondamentale" [8], the Eulerian polynomial $A_n(x)$ can be defined in the equivalent way

$$A_n(x) = \sum_{w\in\mathfrak{S}_n} x^{\mathrm{asc}(w)},$$

where $\mathrm{asc}(w) := \big|\{i \,|\, w_i < w_{i+1},\, 1 \le i \le n-1\}\big|$ is the number of *ascents* in $w = w_1w_2\cdots w_n$. Now, let $\mathfrak{C}_n := \mathfrak{C}(\{1,2,\ldots,n\})$. For all $w = (a_1, a_2, \ldots, a_{n-1}, n) \in \mathfrak{C}_n$, set $w' := a_1a_2\cdots a_{n-1} \in \mathfrak{S}_{n-1}$. As $w \mapsto w'$ is bijective, we have

$$C_n(x) = \sum_{w\in\mathfrak{C}_n} x^{\mathrm{exc}(w)} = \sum_{w'\in\mathfrak{S}_{n-1}} x^{1+\mathrm{asc}(w)} = xA_{n-1}(x),$$

and a bijective proof of (13) is obtained.

4 A combinatorial description of gamma coefficients

Given $w \in \mathfrak{S}_n$ and $i \in \{1,2,\ldots,n\}$, we say that i is a *double excedance* for w if and only if we have $w^{-1}(i) < i < w(i)$. Let $\mathrm{dexc}(w)$ denote the number of double excedances of w. Moreover, let $\mathfrak{D}_n \subset \mathfrak{S}_n$ denote the set of *derangements* (permutations with no fixed points) in $\mathfrak{S}_n$, and consider the expansion of the *derangement polynomial* $d_n(x)$ in terms of the gamma basis [3]:

$$d_n(x) := \sum_{w\in\mathfrak{D}_n} x^{\mathrm{exc}(w)} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \xi_{n,k}x^k(x+1)^{n-2k}.$$

By means of "valley-hopping", Athanasiadis and Savvidou [4] obtained the following neat combinatorial interpretation of $\xi_{n,k}$:

$$\xi_{n,k} = \big|\{w \,|\, w \in \mathfrak{D}_n,\, \mathrm{exc}(w) = k,\, \mathrm{dexc}(w) = 0\}\big|. \tag{15}$$

This last result plainly extends to the *q-derangement polynomial* $d_n(q;x)$, for which we have [3, 15]

$$d_n(q;x) := \sum_{w\in\mathfrak{D}_n} q^{\mathrm{cyc}(w)}x^{\mathrm{exc}(w)} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \xi_{n,k}(q)x^k(x+1)^{n-2k},$$

where

$$\xi_{n,k}(q) = \sum_{\substack{w\in\mathfrak{D}_n \\ \mathrm{exc}(w)=k,\,\mathrm{dexc}(w)=0}} q^{\mathrm{cyc}(w)} = \sum_{i=1}^{n} \xi_{n,k,i}q^i,$$

and

$$\xi_{n,k,i} = \big|\{w \,|\, w \in \mathfrak{D}_n,\, \mathrm{exc}(w) = k,\, \mathrm{dexc}(w) = 0,\, \mathrm{cyc}(w) = i\}\big|. \tag{16}$$

With this established, a combinatorial description of $\gamma_{n,k}(q)$ can be easily carried out.

Theorem 7. *We have*

$$\gamma_{n,k}(q,t) = \sum_{\substack{w\in\mathfrak{S}_n\\ \mathrm{exc}(w)=k,\,\mathrm{dexc}(w)=0}} q^{\mathrm{cyc}(w)}t^{\mathrm{fix}(w)} = \sum_{i,j=0}^{n} \gamma_{n,k,i,j}q^i t^j, \tag{17}$$

where

$$\gamma_{n,k,i,j} = \big|\{w \mid w\in\mathfrak{S}_n,\, \mathrm{exc}(w)=k,\, \mathrm{dexc}(w)=0,\, \mathrm{cyc}(w)=i,\, \mathrm{fix}(w)=j\}\big|. \tag{18}$$

Proof. From (4) we obtain

$$P_n(q,t;x) = \sum_{j=0}^{n}\binom{n}{j}t^j q^j (x+1)^j d_{n-j}(q;x),$$

then

$$\gamma_{n,k}(q,t) = \sum_{j=0}^{n}\binom{n}{j}t^j q^j \xi_{n-j,k}(q). \tag{19}$$

On the other hand, it is easily seen that

$$\gamma_{n,k,i,j} = \binom{n}{j}\xi_{n-j,k,i-j}.$$

Indeed, for every positive integer i, and for every $F\subseteq\{1,2,\ldots,n\}$ such that $|F|=j$, there are exactly $\xi_{n-j,k,i-j}$ permutations $w\in\mathfrak{S}_n$ with i cycles, k excedances, no double excedances, and with the set F of fixed points. Hence, we may write

$$\sum_{i,j=0}^{n}\gamma_{n,k,i,j}q^i t^j = \sum_{j=0}^{n}\binom{n}{j}q^j t^j \xi_{n-j,k}(q). \tag{20}$$

Finally, comparing (19) and (20), we obtain (17). □

Corollary 8. *We have $P_{2n+1}(q,t;-1)=0$ and*

$$(-1)^n P_{2n}(q,t;-1) = \sum_{w\in\mathfrak{D}_{2n}} q^{\mathrm{cyc}(w)}(-1)^{n-\mathrm{exc}(w)} \tag{21}$$

$$= \sum_{\substack{w\in\mathfrak{D}_{2n}\\ \mathrm{exc}(w)=n,\,\mathrm{dexc}(w)=0}} q^{\mathrm{cyc}(w)}. \tag{22}$$

Proof. Setting $x=-1$ and $y=1$ in (9), we get (21). Analogously, setting $x=-1$ and $y=1$ in (12), we obtain $P_{2n+1}(q,t;-1)=0$ and

$$P_{2n}(q,t;-1) = (-1)^n\gamma_{2n,n}(q,t).$$

Besides, from (17) we have

$$P_{2n}(q,t;-1) = (-1)^n \sum_{\substack{w \in \mathfrak{S}_{2n} \\ \mathrm{exc}(w)=n,\, \mathrm{dexc}(w)=0}} q^{\mathrm{cyc}(w)} t^{\mathrm{fix}(w)}. \tag{23}$$

Now, given $w \in \mathfrak{S}_{2n}$ such that $\mathrm{exc}(w) = n$ and $\mathrm{dexc}(w) = 0$, we have that i is an excedance for w if and only if $i < w(i) > w^2(i)$. So, $w^2(i)$ is an excedance for w^{-1}, and $\mathrm{exc}(w^{-1}) = n$. Finally, as $\mathrm{exc}(w) + \mathrm{exc}(w^{-1}) + \mathrm{fix}(w) = 2n$, we deduce that $\mathrm{fix}(w) = 0$, and we may replace $\mathfrak{S}_{2n}$ with $\mathfrak{D}_{2n}$ in (23) to obtain (22). □

Note that the permutations $w \in \mathfrak{S}_{2n}$ satisfying $\mathrm{exc}(w) = n$ and $\mathrm{dexc}(w) = 0$ bijectively correspond to the alternating permutations via the "transformation fondamentale" [8]. For instance, starting from the standard representation of $w = (5,1,4,2)(8,3,7,6) \in \mathfrak{S}_8$ (with $\mathrm{exc}(w) = 4$ and $\mathrm{dexc}(w) = 0$), via the "transformation fondamentale"we obtain $\hat{w} = 51428376$, which in fact is an alternating permutation. So, (22) can be seen as a q-analogue of the classical identity $A_{2n+1}(-1) = (-1)^n E_{2n+1}$ relating Eulerian polynomials to the Euler numbers E_n.

5 Some special cases and new results on Eulerian polynomials

The polynomial $P_n(q,t;x,y)$ reduces to well-known gamma-nonnegative polynomials once that the parameters q,t,y are suitably specialized. For each of these special cases, (9) provides the corresponding interpretation in terms of enumeration of permutations, while (17) and (18) lead to the combinatorial description of the associated gamma vector.

5.1 Powers of the binomial

From (4) we have

$$\lim_{q\to 0} P_n\left(q, \frac{t}{q}; x, y\right) = t^n (x+y)^n.$$

In this case, the identity permutation e is the unique $w \in \mathfrak{S}_n$ providing a nonzero contribution in (9). In particular, we have $\mathrm{cyc}(e) = \mathrm{fix}(e) = n$ and $\mathrm{dexc}(e) = \mathrm{exc}(e) = 0$.

5.2 Derangement polynomials and their q-analogues

This case was essentially discussed in the previous section. We recall that we have

$$d_n(q;x) = P_n(q,0;x,1). \tag{24}$$

By virtue of (4), the corresponding generating function is

$$\sum_{n\geq 0} d_n(q;x)\frac{z^n}{n!} = \left(\frac{x-1}{xe^z - e^{xz}}\right)^q. \tag{25}$$

The combinatorial expansion obtained from (9) is

$$d_n(q;x) = \sum_{w\in\mathfrak{D}_n} q^{\mathrm{cyc}(w)} x^{\mathrm{exc}(w)}.$$

Of course, for the gamma vector we recover the known expression [3, 4, 15]

$$\gamma_{n,k}(q,0) = \xi_{n,k}(q).$$

The derangement polynomial arises as $d_n(x) = d_n(1;x)$, for which we have

$$\gamma_{n,k}(1,0) = \xi_{n,k}.$$

5.3 Binomial Eulerian polynomials

Set

$$\tilde{A}_n(x) := P_n(1,1;x) = \sum_{w\in\mathfrak{S}_n} x^{\mathrm{exc}(w)}(x+1)^{\mathrm{fix}(w)}. \tag{26}$$

From (3), (4), and (5) we recover

$$\sum_{n\geq 0} \tilde{A}_n(x)\frac{z^n}{n!} = \frac{x-1}{xe^z - e^{xz}}\, e^{(x+1)z} = \mathcal{A}(x;z)\, e^{xz} = e^z + x\big(\mathcal{A}(x;z) - 1\big)\, e^z,$$

then

$$\tilde{A}_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(x) = 1 + x\sum_{k=1}^{n}\binom{n}{k} A_k(x), \tag{27}$$

where $A_0(x) := 1$. This says that $\tilde{A}_n(x)$ is a *binomial Eulerian polynomials* [3]. For these polynomials, (17) and (18) lead to the known interpretation of the gamma vector $\tilde{\gamma}_{n,k}$:

$$\tilde{\gamma}_{n,k} = \gamma_{n,k}(1,1) = \big|\{w \in \mathfrak{S}_n \,|\, \mathrm{exc}(w) = k, \mathrm{dexc}(w) = 0\}\big|.$$

5.4 Eulerian polynomials

Classical Eulerian polynomials arise by means of a less obvious specialization. In fact, (4) and (5) give

$$\frac{d}{dz}\mathcal{A}(x,z) = \left(\frac{x-1}{xe^z - e^{xz}}\right)^2 e^{(x+1)z} = \mathcal{P}(2, 1/2; x, 1; z),$$

from which it follows that

$$A_{n+1}(x) = P_n(2, 1/2; x, 1), \text{ for } n \geq 1.$$

Now, (9) leads to the following alternative combinatorial interpretation of the Eulerian polynomial $A_n(x)$:

$$A_n(x) = \sum_{w \in \mathfrak{S}_{n-1}} 2^{\mathrm{cyc}(w)-\mathrm{fix}(w)} x^{\mathrm{exc}(w)} (x+1)^{\mathrm{fix}(w)}. \tag{28}$$

Moreover, from (17) and (18) we obtain a new description of the gamma vector of $A_n(x)$:

$$\gamma_{n,k} = \gamma_{n,k}(2, 1/2) = \sum_{\substack{w \in \mathfrak{S}_{n-1} \\ \mathrm{exc}(w)=k,\, \mathrm{dexc}(w)=0}} 2^{\mathrm{cyc}(w)-\mathrm{fix}(w)}. \tag{29}$$

Finally, from (28) we get the following identity relating the derangements of $\mathfrak{S}_{2n}$ to the alternating permutations of $\mathfrak{S}_{2n+1}$:

$$\sum_{w \in \mathfrak{D}_{2n}} 2^{\mathrm{cyc}(w)} (-1)^{\mathrm{exc}(w)} = (-1)^n E_{2n+1}. \tag{30}$$

An analogous relation involving the derangements and the alternating permutations of $\mathfrak{S}_{2n}$ was stated by Roselle [14]:

$$\sum_{w \in \mathfrak{D}_{2n}} (-1)^{\mathrm{exc}(w)} = (-1)^n E_{2n}.$$

It would be interesting to achieve neat bijective proofs of (28), (29), and (30).

5.5 A q,t-analogue of Eulerian polynomials

A very natural generalization of the Eulerian polynomial $A_n(x)$ is given by

$$Q_n(q, t; x) := \sum_{w \in \mathfrak{S}_n} q^{\mathrm{cyc}(w)} t^{\mathrm{fix}(w)} x^{\mathrm{exc}(w)} = P_n(q, t/(x+1); x). \tag{31}$$

This polynomial reduces to $A_n(x)$ when $q = t = 1$. Also, $Q_n(q, t; x)$ gives the q-derangement polynomial $d_n(q; x)$ when $t = 0$, and the derangement polynomial $d_n(x)$ when $q = 1$ and $t = 0$. Combinatorial properties of $Q_n(q, t; x)$ have been studied by G. Kasavrelof and J. Zeng [10], and by S.-M. Ma [11]. Moreover, by setting $t = 1$ in (31) we obtain the polynomial $A_n(x, q) = x\, Q_n(q, 1; x)$ introduced by Brenti [7]. It is easily seen that $Q_n(q, t; x)$ is not palindromic in x, hence there is no gamma vector associated with $Q_n(q, t; x)$. Nevertheless, by comparing (31) with (3) and (4) we recover

$$\sum_{n \geq 0} P_n(q, t; x) \frac{z^n}{n!} = e^{xqtz} \sum_{n \geq 0} Q_n(q, t; x) \frac{z^n}{n!}.$$

So, we obtain

$$P_n(q,t;x) = \sum_{k=0}^{n} \binom{n}{k} (qtx)^{n-k} Q_k(q,t;x), \tag{32}$$

which is a generalization of the first identity in (27). In fact, the first part of (27) comes from (32) by setting $q = t = 1$.

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2020 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A10, 05A19, 11B65, 11B83.

Keywords: Eulerian polynomial, excedance, derangement, alternating permutation.

(Concerned with sequences A000166, A001250, and A008292.)

Received August 5 2023; revised version received August 6 2023; November 18 2023; November 20 2023. Published in *Journal of Integer Sequences*, January 13 2024.

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