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The Golden Ratio, Factorials, and the Lambert W Function

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Abstract

We study the relationship of the integer sequence A214048 with the Lambert W function and the left factorial numbers A003422.

1 Notation and Introduction

We use the following notation [7]:

- $\mathbb{N}$: the set of positive integers.
- $\mathbb{N}_0$: the set of non-negative integers.
- $\mathbb{Z}_0^-$: the set of negative integers and zero.
- $\mathbb{R}$: the set of real numbers.

- $\mathbb{R}^+$: the set of real positive numbers.
- $\mathbb{C}$: the set of complex numbers.
- L_m: the m'th Lucas number [6, A000032].
- F_m: the m'th Fibonacci number [6, A000045].
- AM_m: the m'th associated Mersenne number [6, A001350].
- φ: the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$.
- $\lfloor \cdot \rfloor$: the floor function of a real number, which is the largest integer not exceeding that real number.
- $W(\cdot)$: the Lambert W function [4] is the solution to the equation $xe^x = z$ for $z \in \mathbb{C}$.
- $\mathrm{Re}(\cdot)$: the real part of the complex number.
- $\Gamma(\cdot)$: the gamma function, defined by [8, p. 1]

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad (\mathrm{Re}(z) > 0).$$

- $\psi^{(m)}(\cdot)$: the polygamma function of order m, [8, p. 22], defined by

$$\psi^{(m)}(z) = \frac{\partial^m}{\partial z^m} \psi(z) = \frac{\partial^{m+1}}{\partial z^{m+1}} \ln\big(\Gamma(z)\big),$$
$$\psi^{(0)}(z) = \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \qquad (m \in \mathbb{N}_0,\ z \in \mathbb{C} \backslash \mathbb{Z}_0^-).$$

- $\arg(\cdot)$: the argument of the complex number.
- $\gamma(\cdot, \cdot)$: the incomplete gamma function, [8, p. 11], defined by

$$\gamma(s, a) = \int_0^a e^{-t} t^{s-1} dt \quad (\mathrm{Re}(s) > 0,\ |\arg(a)| < \pi).$$

Next, let us define the function

$$!z(x) = \int_0^x e^{-t} \frac{t^z - 1}{t - 1} dt \quad (x \in \mathbb{R},\ z \in \mathbb{C}).$$

On the basis of known properties of the Lambert W function and the sequence [6, A214048], we determine $x \in \mathbb{R}$ so that

$$!n(x) < !n < !n(x) + 1 \quad (n \in \mathbb{N}).$$

2 Connection to the Lambert W function

As a multi-valued function, $W(z)$ has infinitely many complex branches and two real branches [3, p. 2]:

$$W_0 : [-1/e, \infty) \to [-1, \infty),$$
$$W_{-1} : [-1/e, 0) \to (-\infty, -1].$$

In what follows, we consider only the real branch $W_{-1}(z)$ of the Lambert W function. The function $W_{-1}(z)$ strictly decreases on $(-1/e, 0)$ and $W_{-1}(-1/e) = -1$. We also have the following well-known inequality [3, p. 3] for the function $W_{-1}(z)$:

$$\frac{e\ln(-x)}{e-1} \le W_{-1}(x) \le \ln(-x) - \ln(-\ln(-x)) \quad (x \in [-1/e, 0)). \tag{1}$$

In proving one of the main results of the paper, the polygamma function of order m plays a key role. The polygamma function satisfies the inequality

$$\frac{(m-1)!}{x^m} + \frac{m!}{2x^{m+1}} \le (-1)^{(m+1)}\psi^{(m)}(x) \le \frac{(m-1)!}{x^m} + \frac{m!}{x^{m+1}} \quad (m \ge 1,\ x > 0). \tag{2}$$

2.1 The real sequence $(a_n)_{n\ge4}$

We have $-\frac{1}{n-1}(4/7)^{\frac{1}{n-1}} \in \left(-\frac{1}{e}, 0\right)$, $(4 \le n \in \mathbb{N})$. Hence the equation $xe^x = -\frac{1}{n-1}(4/7)^{\frac{1}{n-1}}$ has two real solutions, namely $x = W_0\left(-\frac{1}{n-1}(4/7)^{\frac{1}{n-1}}\right)$ and $x = W_{-1}\left(-\frac{1}{n-1}(4/7)^{\frac{1}{n-1}}\right)$. Let us define the sequence of real positive numbers $(a_n)_{n\ge4}$ as follows:

$$a_n = -(n-1)W_{-1}\left(-\frac{1}{n-1}(4/7)^{\frac{1}{n-1}}\right) \quad (a_n \in \mathbb{R}^+). \tag{3}$$

Remark 1. Note that the sequence a_n is strictly increasing. We have

$$a_n^{-(n-1)}e^{a_n} = a_{n+1}^{-n}e^{a_{n+1}} \text{ and } a_{n+1} = a_n + \ln\left(a_{n+1}\left(\frac{a_{n+1}}{a_n}\right)^{n-1}\right).$$

2.2 The golden sequence $(\alpha_n)_{n\ge1}$

In 2012, Kimberling [6] defined the *golden sequence* $(\alpha_n)_{n\ge1}$ [6, A214048] by

$$\alpha_n = m \iff \varphi^{m-1} \le n! \le \varphi^m \quad (m \in \mathbb{N}), \tag{4}$$

or

$$\alpha_1 = 1, \quad \alpha_n = m \iff L_{m-1} \le n! < \text{Ł}_m \quad (1 < n \in \mathbb{N}).$$

Definition (4) produces the following lemma:

Lemma 2. *The following inequalities hold:*

$$\alpha_{n-1} + \left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor \le \alpha_n \le \alpha_{n-1} + \left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor + 1 \quad (1 < n \in \mathbb{N}).$$

Consider the numbers $\lfloor \frac{\ln n}{\ln \varphi} \rfloor$ [6, A371672]. For this purpose, we consider the integer sequence $(d_m)_{m \ge 0}$ [6, A181716] (see also [6, A098600]) which we define as follows:

$$d_m = d_{m-1} + d_{m-2} + (-1)^m, \quad d_0 = 0,\, d_1 = 1.$$

The following equalities hold [6]:

$$d_m = F_{m-2} + F_m + (-1)^m = \mathrm{AM}_{m-1} + 1 \quad (1 < m \in \mathbb{N}).$$

Lemma 3. *The number d_m is the number of natural numbers n for which the equality*

$$\left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor = m \quad (m \in \mathbb{N})$$

holds.

Proof. The proof presented here is due to J. Shallit. If we want to count the number of n for which $\lfloor \frac{\ln n}{\ln \varphi} \rfloor = m$, this is the same as counting the number of integers n such that $m \le \frac{\ln n}{\ln \varphi} < m + 1$, which by rearrangement is the number of integers n such that $\varphi^m \le n < \varphi^{m+1}$. However, we know from the Binet form of the Lucas numbers that

$$\lfloor \varphi^m \rfloor = \begin{cases} L_m - 1, & \text{if } m \text{ is even;} \\ L_m, & \text{if } m \text{ is odd,} \end{cases}$$

so now trivially we get $d_m = L_{m-1} + 1$ if m is even and $d_m = L_{m-1} - 1$ if m is odd, that is,

$$d_m = L_{m-1} + (-1)^m \quad (m \in \mathbb{N}). \tag{5}$$

□

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lfloor \frac{\ln n}{\ln \varphi} \rfloor$	0	1	2	2	3	3	4	4	4	4	4	5	5

Table 1: The numbers $\lfloor \frac{\ln n}{\ln \varphi} \rfloor$ for $n = 1, 2, \ldots, 13$.

Remark 4. Equality (5) was first stated without proof by G. C. Greubel in [6, A181716]. We do not use Lemma 2 and Lemma 3 when proving the new results.

Example 5. We have

$$\begin{aligned}
\left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor &= 1 \text{ for } n = 2 && \Rightarrow d_1 = 1, \\
\left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor &= 2 \text{ for } n = 3, 4 && \Rightarrow d_2 = 2, \\
\left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor &= 3 \text{ for } n = 5, 6 && \Rightarrow d_3 = 2, \\
\left\lfloor \frac{\ln n}{\ln \varphi} \right\rfloor &= 4 \text{ for } n = 7, 8, 9, 10, 11 && \Rightarrow d_4 = 5.
\end{aligned}$$

2.3 The main result

We give our main result in the following theorem:

Theorem 6. *The following inequality holds:*

$$\alpha_n > a_n \quad (4 < n \in \mathbb{N}).$$

Proof. The inequality is easily checked for $4 \leq n \leq 78$. We prove that it holds for $n > 78$. Since $\varphi^{\alpha_n} > n!$, we have $\alpha_n > \frac{\ln n!}{\ln \varphi} > 2\ln(n!)$. Inequality (1) produces

$$-(n-1)\frac{e}{e-1}\ln\left(\frac{1}{n-1}(4/7)^{\frac{1}{n-1}}\right) \geq a_n.$$

Therefore we have to prove $2\ln(n!) > a_n$, or, equivalently

$$2\ln(n!) > -(n-1)\frac{e}{e-1}\ln\left(\frac{1}{n-1}(4/7)^{\frac{1}{n-1}}\right). \tag{6}$$

The last inequality holds for $n = 79$. Let us define the function

$$\omega(x) = 2\ln\big(\Gamma(x+1)\big) + (x-1)\frac{e}{e-1}\ln\left(\frac{1}{x-1}(4/7)^{\frac{1}{x-1}}\right).$$

Therefore we need to prove that the first derivative of the function $\omega(x)$ is higher than 0 for $79 \leq x \in \mathbb{R}$. It is easy to check that

$$\omega'(x) = 2\psi(x+1) + \frac{e}{e-1}\left(\ln\left(\frac{1}{x-1}(4/7)^{\frac{1}{x-1}}\right) - 1 + \frac{\ln 7/4}{x-1}\right) > 0 \quad (x = 38).$$

Further, it is necessary to show that $\omega'(x)$ is an increasing function for $79 \leq x \in \mathbb{R}$. We need to prove the following inequality:

$$\frac{\partial^2}{\partial x^2}\omega(x) = 2\psi^{(1)}(x+1) - \frac{e}{(e-1)(x-1)} > 0 \quad (79 \leq x \in \mathbb{R}).$$

Hence using Inequality (2) it is sufficient to show that

$$2\left(\frac{1}{x+1}+\frac{1}{2(x+1)^2}\right) > \frac{e}{(e-1)(x-1)}.$$

Equivalently, it suffices to show that

$$2x^2+x-3 > (x+1)^2\frac{16}{10} > (x+1)^2\frac{e}{(e-1)}.$$

It follows straightforwardly that

$$2x^2+x-3 > (x+1)^2\frac{16}{10}$$

holds for $x \geq 8$, which completes our proof. □

3 The left factorial

In 1971, Kurepa [1] defined the function *left factorial* [6, A003422] for natural numbers $!n$ by

$$!0 = 0, \quad !n = \sum_{k=0}^{n-1} k! \quad (n \in \mathbb{N}).$$

In the aforementioned paper [1, p. 151] Kurepa extended left factorial function to the complex half-plane $\mathrm{Re}(z) > 0$ as

$$!z = \int_0^{+\infty} e^{-t}\frac{t^z-1}{t-1}dt.$$

We can also extend such function analytically to the whole complex plane [2] by

$$!z = !(z+1) - \Gamma(z+1).$$

Remark 7. In [1, p. 149], Kurepa proposed the conjecture for the left factorial as follows:

$$\text{If } 1 < n \in \mathbb{N}, \text{ then } \gcd(!n, n!) = 2.$$

Over the past fifty years there have been many attempts to find a solution to Kurepa's conjecture. The problem remains open. For more details, see [5].

3.1 The left factorial and the Lambert W function

Let us define the sequence of real positive numbers $(b_n)_{n\geq 4}$ as follows:

$$b_n = \int_0^{a_n} e^{-x}\frac{x^n-1}{x-1}dx = \sum_{k=0}^{n-1}\gamma(k+1, a_n) \quad (b_n \in \mathbb{R}^+).$$

The sequence a_n is given by (3).

n	4	5	6	7	8	9
a_n	5.87	9.61	13.62	17.85	22.29	26.90
b_n	8.86	33.01	153.08	873.12	5913.15	46233.17
$!n$	10	34	154	874	5914	46234

Table 2: The numbers a_n, b_n and $!n$ for $n = 4, 5, 6, 7, 8, 9$.

Lemma 8. *We have*

$$2n - 4 < \frac{3}{4} a_n \quad (4 \leq n \in \mathbb{N}).$$

Proof. Applying Remark 1, we have

$$\begin{aligned} a_{n+1} &> a_n + \ln a_{n+1} && (4 \leq n \in \mathbb{N}) \\ &> a_n + 2.8 > a_n + \frac{8}{3} && (\ln a_{6+1} > 2.88). \end{aligned}$$

Hence by induction on n we obtain our inequality. □

Lemma 9. *The following inequality holds:*

$$\sum_{t=0}^{k} \frac{a_n^t}{t!} < \frac{a_n^{k+1}}{(k+1)!} \quad (5 \leq n \in \mathbb{N},\ n - 2 > k \in \mathbb{N}_0).$$

Proof. We prove the lemma by induction on k. For $k = 0$ we have $1 < a_n$, which is valid. Assume that the statement holds for $k - 1 < n - 3$, i.e.,

$$\sum_{t=0}^{k-1} \frac{a_n^t}{t!} < \frac{a_n^k}{k!}.$$

Then

$$\sum_{t=0}^{k} \frac{a_n^t}{t!} = \frac{a_n^k}{k!} + \sum_{t=0}^{k-1} \frac{a_n^t}{t!} < 2\frac{a_n^k}{k!}.$$

Hence Lemma 8 produces

$$2\frac{a_n^k}{k!} < \frac{a_n^{k+1}}{(k+1)!} \quad (k < n - 2),$$

which completes the proof. □

Theorem 10. *The following inequalities hold:*

$$b_n < !n < b_n + 1 \quad (5 \leq n \in \mathbb{N}).$$

Proof. The claim holds for $5 \leq n \leq 9$. Since

$$e^{-x}\frac{x^n-1}{x-1} > 0,$$

the left side of the inequality holds. Furthermore, we have

$$\int_0^{a_n} e^{-x}\frac{x^n-1}{x-1}dx = !n - e^{-a_n}\sum_{t=0}^{n-1}\frac{!n-!t}{t!}a_n^t. \tag{7}$$

To prove the right side of the inequality, we use Equation (7). It suffices to show

$$\sum_{t=0}^{n-1}\frac{!n-!t}{t!}a_n^t < e^{a_n}.$$

Firstly, by induction on n we have the proof of the following inequality:

$$2n+1 < \frac{3}{4}\,a_n \qquad (9 \leq n \in \mathbb{N}). \tag{8}$$

Next, we have

$$\begin{aligned}\sum_{t=0}^{n-1}\frac{!n-!t}{t!}a_n^t &= a_n^{n-1} + na_n^{n-2} + \sum_{t=0}^{n-3}\frac{!n-!t}{t!}a_n^t < a_n^{n-1} + na_n^{n-2} + \,!n\sum_{t=0}^{n-3}\frac{a_n^t}{t!} \\ &< a_n^{n-1} + na_n^{n-2} + \,!n\frac{a_n^{n-2}}{(n-2)!} \qquad \text{(Lemma 9 for } k = n-3) \\ &< a_n^{n-1} + na_n^{n-2} + (n+1)a_n^{n-2} \qquad \left(\frac{!(n-2)}{(n-2)!} < 1\right) \\ &< \frac{7}{4}a_n^{n-1}. \qquad \text{(Equation (8))}.\end{aligned}$$

Since one real solution to the equation

$$e^x - \frac{7}{4}x^{n-1} = 0$$

is a_n, we have

$$\sum_{t=0}^{n-1}\frac{!n-!t}{t!}a_n^t < \frac{7}{4}a_n^{n-1} = e^{a_n},$$

which proves our theorem. □

Remark 11. If we replace the constant $\frac{4}{7}$ with numbers $\frac{4+\frac{1}{31}}{7}$ or $\frac{4-\frac{1}{31}}{7}$ in Definition (3), then Lemma 9 and Inequality (8) hold for the sequence $(a_n)_{n\geq 4}$ defined in this way. We conclude that our constant $\frac{4}{7}$ is not optimal. We do not deal with this issue in this paper.

3.2 The left factorial and the golden sequence

Let us define the sequence of real numbers $(\beta_n)_{n\geq 1}$ as follows:

$$\beta_n = \int_0^{\alpha_n} e^{-x}\frac{x^n-1}{x-1}dx \quad (\beta_n \in \mathbb{R}).$$

The sequence α_n is given by (4).

n	1	2	3	4	5	6	7	8
α_n	1	2	4	7	10	14	18	23
β_n	0.63	1.46	3.13	9.44	33.23	153.29	873.21	5913.49
$!n$	1	2	4	10	34	154	874	5914

Table 3: The numbers $!n$, α_n and β_n for $n = 1, 2, \ldots, 8$.

Theorem 12. *The following inequalities hold:*

$$\beta_n < !n < \beta_n + 1 \quad (n \in \mathbb{N}).$$

Proof. Let us define the family of functions $f_n(x)$ as follows:

$$f_n(x) = x - (n-1)\ln x - \ln\frac{7}{4} \quad (0 < x \in \mathbb{R},\ 3 < n \in \mathbb{N}).$$

Then

$$\frac{\partial}{\partial x}f_n(x) = f_n'(x) = 1 - \frac{n-1}{x}, \quad \frac{\partial^2}{\partial x^2}f_n(x) = \frac{n-1}{x^2}, \quad f_n'(n-1) = 0, \quad f_n(n-1) < 0.$$

For the equation $f_n(x) = 0$, we have the real solution $x = a_n$. On the interval $(0, +\infty)$ the function $f_n(x)$ has the minimum for $x = n-1$. The function $f_n(x)$ increases on the interval $(0, +\infty)$. For the sequence $(\alpha_n)_{n\geq 5}$ the analogs of Lemma 9 and Inequality (8) are valid. Hence Theorem 6 produces

$$\beta_n < !n < \beta_n + 1.$$

□

Remark 13. If Lemma 9 and Inequality (8) hold for the positive real sequence $(c_n)_{n\geq n_0}$, and if $c_n \geq a_n$, $(n \geq n_0 \geq 5)$, then

$$\int_0^{c_n} e^{-x}\frac{x^n-1}{x-1}dx < !n < \int_0^{c_n} e^{-x}\frac{x^n-1}{x-1}dx + 1 \quad (n \geq n_0).$$

3.3 Generalization

Let us define two-dimensional real sequences $(A_{n,m})_{n\geq 5, m\geq 1}$ and $(B_{n,m})_{n\geq 5, m\geq 1}$ as follows:

$$\begin{aligned} A_{n,m} &= -(n-1)W_{-1}\left(-\frac{1}{n-1}\left(\frac{4}{7m}\right)^{\frac{1}{n-1}}\right), \\ B_{n,m} &= \int_0^{A_{n,m}} e^{-x}\frac{x^n-1}{x-1}dx. \end{aligned}$$

n	5	6	7	8	9
$B_{n,1}$	33.01	153.08	873.12	5913.15	46233.17
$B_{n,2}$	33.53	153.56	873.57	5913.58	46233.60
$B_{n,3}$	33.70	153.71	873.72	5913.73	46233.70
$B_{n,4}$	33.78	153.79	873.79	5913.80	46233.80
$B_{n,5}$	33.82	153.83	873.83	5913.84	46233.84
$!n$	34	154	874	5914	46234

Table 4: The numbers $B_{n,m}$ and $!n$ for $n = 5, 6, 7, 8, 9$ and $m = 1, 2, 3, 4, 5$.

Theorem 14. *The following inequalities hold:*

$$B_{n,m} < \, !n < B_{n,m} + \frac{1}{m} \quad (5 \leq n \in \mathbb{N}, \; m \in \mathbb{N}).$$

Proof. The Theorem 14 follows from the application of the procedures used in the proof of Theorem 10 and Remark 13. □

Corollary 15. *The following equalities hold:*

$$\begin{aligned} !n &= \lfloor \beta_n \rfloor + 1 & (n \in \mathbb{N}) \\ &= \lfloor b_n \rfloor + 1 = \left\lfloor B_{n,m} + \frac{1}{m} \right\rfloor & (4 < n \in \mathbb{N}). \end{aligned}$$

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