Journal of Integer Sequences, Vol. 27 (2024),
Article 24.6.6

Square-Weighted Zero-Sum Constants

Krishnendu Paul and Shameek Paul
School of Mathematical Sciences
Ramakrishna Mission Vivekananda Educational and Research Institute
P.O. Belur Math, Dist. Howrah, West Bengal 711202
India
krishnendu.p.math18@gm.rkmvu.ac.in
shameek.paul@rkmvu.ac.in

Abstract

Let $A \subseteq \mathbb{Z}_n$ be a subset. A sequence $S = (x_1, \ldots, x_k)$ in $\mathbb{Z}_n$ is said to be an A-weighted zero-sum sequence if there exist $a_1, \ldots, a_k \in A$ such that $a_1x_1 + \cdots + a_kx_k = 0$. By a square, we mean a non-zero square in $\mathbb{Z}_n$. We determine the smallest natural number k, such that every sequence in $\mathbb{Z}_n$ whose length is k has a square-weighted zero-sum subsequence. We also determine the smallest natural number k, such that every sequence in $\mathbb{Z}_n$ whose length is k has a square-weighted zero-sum subsequence whose terms are consecutive terms of the given sequence.

1 Introduction

For a finite set A, we let $|A|$ denote the number of elements of A. For $a, b \in \mathbb{Z}$ with $a \leq b$, we let $[a, b]$ denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$.

Let R be a commutative ring with unity, M be an R-module, and $A \subseteq R$. A subsequence T of a sequence $S = (x_1, x_2, \ldots, x_k)$ in M is called an *A-weighted zero-sum subsequence* if the set $J = \{i : x_i \in T\}$ is non-empty, and for every $i \in J$ there exists $a_i \in A$ such that $\sum_{i \in J} a_i x_i = 0$ where 0 is the identity element of M.

For a finite R-module M and $A \subseteq R$, the *A-weighted Davenport constant of M* denoted by D_A is defined to be the least positive integer k, such that every sequence in M whose length is k, has an A-weighted zero-sum subsequence.

Adhikari and Chen [1] introduced this constant for the ring $R = \mathbb{Z}$, i.e., for abelian groups. We define the constant C_A to be the least positive integer k, such that every sequence in

M whose length is k, has an A-weighted zero-sum subsequence whose terms are consecutive terms.

Remark 1. It is easy to observe that $D_A \leq C_A \leq |M|$.

We also denote the ring $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$. Let $U(n)$ denote the group of units in $\mathbb{Z}_n$ and $U(n)^2$ denote the set $\{x^2 : x \in U(n)\}$. For a divisor m of n, the homomorphism $f_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ is given by $f_{n,m}(a + n\mathbb{Z}) = a + m\mathbb{Z}$. Mondal et al. [7, Lem. 7] showed that the image of $U(n)$ under $f_{n,m}$ is $U(m)$.

Let p be a prime divisor of n. We say that $v_p(n) = r$ if $p^r \mid n$ and $p^{r+1} \nmid n$. Suppose $r = v_p(n)$. For every $x \in \mathbb{Z}_n$ we denote the image of x under f_{n,p^r} by $x^{(p)}$. Given a sequence $S = (x_1, \ldots, x_l)$ in $\mathbb{Z}_n$, we get a sequence

$$S^{(p)} = (x_1^{(p)}, \ldots, x_l^{(p)}) \text{ in } \mathbb{Z}_{p^r}.$$

From this point onwards, we will only consider the case when $M = R = \mathbb{Z}_n$.

Adhikari and Rath [3] showed that $D_{U(p)^2} = 3$ when p is an odd prime. Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity. Grynkiewicz and Hennecart [5] showed that $D_{U(n)^2} \geq 2\Omega(n) + \min\{v_3(n), v_5(n)\} + 1$ when n is odd, with equality if either $3 \nmid n$ or $v_3(n) \geq v_5(n)$. This extends a result of Chintamani and Moriya [4], and another of Adhikari, David, and Urroz [2].

These results lead quite naturally to the question of determining the value of $D_{S(n)}$ where

$$S(n) = \{\, x^2 : x \in \mathbb{Z}_n\} \setminus \{0\}.$$

For an odd prime p we observe that $S(p) = U(p)^2$. We determine the value of $D_{S(n)}$ for every n and show that it depends on the parity of n when n is a square, and on the parity of $v_2(n)$ when n is not a square. We also investigate the value of $C_{S(n)}$.

We show that $C_{U(25)^2} = 9$, adding to the results which were obtained by Mondal et al. [8]. Using this fact, we get that $C_{S(n)} \leq 9$ when n is an odd square. The values of $D_{S(n)}$ for all $n \in [2, 37]$ are given in the following table. When p is an odd prime, we see that $S(p) = U(p)^2$ and so the value of $D_{S(p)}$ had been determined in Adhikari and Rath [3].

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$D_{S(n)}$	2	3	4	3	2	3	2	5	2	3	3	3	2	3	4	3	2	3

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
$D_{S(n)}$	3	3	2	3	2	5	2	3	3	3	2	3	2	3	2	3	4	3

Table 1: Values of $D_{S(n)}$ for all $n \in [2, 37]$.

The only values of n in the set $[2, 37]$ for which $C_{S(n)}$ differs from $D_{S(n)}$ are 9 and 25. The smallest n for which we have not been able to determine $C_{S(n)}$ is 81. In this article, we have obtained the following results:

- We determine the size of $S(n)$ for every n.
- When n is a square, we get that $D_{S(n)} = 4$ or 5 when n is even or odd respectively.
- When n is not a square, we get that $D_{S(n)} = 2$ or 3 when $v_2(n)$ is odd or even respectively.
- When n is not a square of an odd number, we get that $C_{S(n)} = D_{S(n)}$.
- When n is a square of an odd, squarefree number, we get that $C_{S(n)} = 9$.
- When n is a square of an odd number m such that m is divisible by p^2 where p is a prime which is at least seven, we get that $C_{S(n)} = D_{S(n)}$.

2 The size of $S(n)$

Observation 2. Let $n = p_1^{r_1} \cdots p_s^{r_s}$ where the p_i's are distinct primes. By the Chinese remainder theorem we get an isomorphism

$$\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_s^{r_s}\mathbb{Z}$$

given by $\varphi(a) = (a^{(p_1)}, \ldots, a^{(p_s)})$. As φ is an isomorphism, we have that $a \in S(n)$ if and only if for every prime divisor q of n, we have that $a^{(q)}$ is a square and there exists a prime divisor p of n such that $a^{(p)} \neq 0$.

Hence, it follows that

$$|S(n)| + 1 = \big(|S(p_1^{r_1})| + 1 \big) \cdots \big(|S(p_s^{r_s})| + 1 \big).$$

Thus, it is enough to determine the size of $S(p^r)$ where p is a prime and r is a positive integer.

Observation 3. Let p be a prime, r be a positive integer, and $a \in \mathbb{Z}_{p^r} \setminus \{0\}$. Then there exists a unique $k \in [0, r-1]$ such that $a = p^k u$ where u is a unit.

For a real number x, we let $\lfloor x \rfloor$ denote the greatest integer which is at most equal to x.

Lemma 4. *Let p be a prime, r be a positive integer, and $l = \lfloor (r-1)/2 \rfloor$. Then we have that $S(p^r) = \bigcup_{k \in [0,l]} p^{2k} U(p^r)^2$. Also, this is a disjoint union.*

Proof. Let $a \in S(p^r)$. Then there exists $u \in U(p^r)$ and $k \in [0, r-1]$ such that $a = p^{2k}u^2$. As $a \neq 0$, by Observation 3 we see that $2k \in [0, r-1]$ and so $k \in [0, l]$. □

We omit the proof of the next result.

Lemma 5. *Let p be a prime, r be a natural number, and $l = \lfloor (r-1)/2 \rfloor$. Then for every $k \in [0, l]$ we have that $|\, p^{2k}\, U(p^r)^2 \,| = |\, U(p^{r-2k})^2 \,|$.*

The next result follows from Lemmas 4 and 5.

Theorem 6. *If r is even*

$$|S(p^r)| = |U(p^r)^2| + |U(p^{r-2})^2| + \cdots + |U(p^4)^2| + |U(p^2)^2|$$

and if r is odd

$$|S(p^r)| = |U(p^r)^2| + |U(p^{r-2})^2| + \cdots + |U(p^3)^2| + |U(p)^2|.$$

It remains to determine the size of $U(n)^2$ when n is a prime power. Let $n = p^r$ where p is an odd prime and r is a positive integer. Ireland and Rosen [6, Thm. 2, p. 43] have shown that $U(n)$ is a cyclic group. So there is exactly one element of order two in $U(n)$. Thus, the kernel of the onto map $U(n) \to U(n)^2$ given by $x \mapsto x^2$ has order two. Hence, we see that $U(n)^2$ has index two in $U(n)$. So it follows that

$$|U(n)^2| = |U(n)|/2 = p^{r-1}(p-1)/2.$$

We have that $U(4)^2 = U(2)^2 = \{1\}$. Let $n = 2^r$ where r is at least three. Ireland and Rosen [6, Thm. 2′, p. 43] have shown that $U(n) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{r-2}}$. So there are exactly three elements of order two in $U(n)$. Thus, the kernel of the onto map $U(n) \to U(n)^2$ given by $x \mapsto x^2$ has order four. Hence, we see that $U(n)^2$ has index four in $U(n)$. So it follows that

$$|U(n)^2| = |U(n)|/4 = 2^{r-1}/4 = 2^{r-3}.$$

3 Some general results

Lemma 7. *Let S be a sequence in $\mathbb{Z}_n$ and p be a prime divisor of n such that $v_p(n) = r$. Suppose the sequence $S^{(p)}$ is an $S(p^r)$-weighted zero-sum sequence. Then the sequence S is an $S(n)$-weighted zero-sum sequence.*

Proof. Let $S = (x_1, \ldots, x_l)$. As $S^{(p)} = (x_1^{(p)}, \ldots, x_l^{(p)})$, there exist $b_1, \ldots, b_l \in S(p^r)$ such that

$$b_1 x_1^{(p)} + \cdots + b_l x_l^{(p)} = 0.$$

By Observation 2 we see that for every $i \in [1, l]$ there exists $a_i \in S(n)$ such that $a_i^{(p)} = b_i$ and for each prime divisor q of n/p^r we have $a_i^{(q)} = 0$. Let φ be the isomorphism given by the Chinese remainder theorem as in Observation 2. Since we get $\varphi(a_1x_1 + \cdots + a_lx_l) = 0$, it follows that $a_1x_1 + \cdots + a_lx_l = 0$. Hence, we see that S is an $S(n)$-weighted zero-sum sequence. □

Corollary 8. *Let p be a prime divisor of n and $r = v_p(n)$. Then we have that $C_{S(n)} \leq C_{S(p^r)}$.*

Proof. Let $m = p^r$. Suppose S is a sequence in $\mathbb{Z}_n$ having length $C_{S(m)}$. As $S^{(p)}$ is a sequence in $\mathbb{Z}_m$ having length $C_{S(m)}$, it follows that there exists a subsequence T of S having consecutive terms such that $T^{(p)}$ is an $S(m)$-weighted zero-sum sequence. So from Lemma 7 we see that T is an $S(n)$-weighted zero-sum sequence. Hence, it follows that $C_{S(n)} \leq C_{S(m)}$. $\square$

We will apply the next result later in the case when p is a prime.

Lemma 9. *Let p be an integer which is at least two, r be an odd number, and T be a sequence in $\mathbb{Z}_{p^r}$. Suppose the image of T under $f_{p^r,p}$ is an $S(p)$-weighted zero-sum sequence. Then T is an $S(p^r)$-weighted zero-sum sequence.*

Proof. Let $T = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_{p^r}$ and $T' = (x'_1, \ldots, x'_k)$ be the image of T under $f_{p^r,p}$. For each $i \in [1, k]$ there exist $a'_i \in S(p)$ such that

$$a'_1 x'_1 + \cdots + a'_k x'_k = 0.$$

Mondal et al. [7, Lem. 7] showed that the image of $U(p^r)^2$ under $f_{p^r,p}$ is $S(p)$. So for each $i \in [1, k]$ there exists $a_i \in U(p^r)^2$ such that $f_{p^r,p}(a_i) = a'_i$. Let $x = a_1x_1 + \cdots + a_kx_k$. As

$$f_{p^r,p}(x) = a'_1 x'_1 + \cdots + a'_k x'_k = 0,$$

it follows that p divides x. We see that $c = p^{r-1} = (p^{(r-1)/2})^2 \in S(p^r)$. As p divides x, we see that $cx = 0$. Thus, it follows that $(c\,a_1)x_1 + \cdots + (c\,a_k)x_k = 0$. For each $i \in [1, k]$ we see that $c\,a_i \in S(p^r)$. Hence, it follows that T is an $S(p^r)$-weighted zero-sum sequence. $\square$

4 $C_{S(n)}$ and $D_{S(n)}$ when n is an even square

The next two results will be used to determine the value of $D_{S(n)}$ when n is an even square. As the image of $U(2^r)^2$ under $f_{2^r,4}$ is $U(4)^2 = \{1\}$ and the sequence $(1, 1, 1)$ in $\mathbb{Z}_4$ does not have any zero-sum subsequence, it follows that S does not have any $U(2^r)^2$-weighted zero-sum subsequence.

Lemma 10. *Let r be a non-zero even number. Let $S = (x_1, x_2, x_3)$ be a sequence in $U(2^r)$ whose image under $f_{2^r,4}$ is the sequence $(1, 1, 1)$. Then S does not have any $S(2^r)$-weighted zero-sum subsequence.*

Proof. Suppose T is an $S(2^r)$-weighted zero-sum subsequence of S. Let

$$I = \{i \in [1, 3] : x_i \text{ is a term of } T\}.$$

For each $i \in I$ there exists $a_i \in S(2^r)$ such that $\sum_{i \in I} a_i x_i = 0$. By Lemma 4 for each $i \in I$ we see that $a_i = 2^{r_i} u_i$ where r_i is an even number which is at most $r - 2$ and $u_i \in U(2^r)^2$. So we get that

$$\sum_{i \in I} 2^{r_i} u_i x_i = 0.$$

Let r' be the minimum of the set $\{r_i : i \in I\}$, let $J = \{i \in I : r_i = r'\}$, and let $f = f_{2^r,4}$. As $r' \leq r-2$ we see that four divides $\sum_{i\in J} u_i\, x_i$ and hence

$$0 = \sum_{i\in J} f(u_i)f(x_i) = \sum_{i\in J} 1.$$

So we get the contradiction that the sequence $(1,1,1)$ in $\mathbb{Z}_4$ has a zero-sum subsequence. Hence, it follows that S does not have any $S(2^r)$-weighted zero-sum subsequence. □

Lemma 11. *Let p be a prime and r be a non-zero even number. Let (z_1, z_2) be a sequence in $U(p^2)$ whose image under $f_{p^2,p}$ is not an $S(p)$-weighted zero-sum sequence. Suppose there exists a sequence $S = (x_1, x_2, y_1)$ in $U(p^r)$ whose image under f_{p^r,p^2} is the sequence (z_1, z_2, p). Then the sequence S does not have any $S(p^r)$-weighted zero-sum subsequence.*

Proof. Suppose the sequence S has an $S(p^r)$-weighted zero-sum subsequence T. Let

$$I = \{i \in [1,2] : x_i \text{ is a term of } T\}.$$

Let $J = \{1\}$ if y_1 is a term of T and let $J = \emptyset$ if y_1 is not a term of T. Then for every $i \in I$ and $j \in J$ there exist $a_i, b_j \in S(p^r)$ such that

$$\sum_{i\in I} a_i x_i + \sum_{j\in J} b_j y_j = 0.$$

As y_1 maps to p under f_{p^r,p^2} there exists $w_1 \in U(p^r)$ such that $y_1 = pw_1$. By Lemma 4, for each $i \in I$ and $j \in J$ we see that $a_i = p^{r_i} u_i$ and $b_j = p^{s_j} v_j$ where $r_i, s_j \in [0, r-2]$ are even and $u_i, v_j \in U(p^r)^2$. So we get that

$$\sum_{i\in I} p^{r_i} u_i\, x_i + \sum_{j\in J} p^{s_j+1} v_j w_j = 0. \tag{1}$$

Consider the set $L = \{r_i : i \in I\} \cup \{s_j + 1 : j \in J\}$. Let r' be the minimum of L. As $r \geq 2$ is even and s_1 is even, it follows that $r' \leq r-1$. Suppose there exists $i \in I$ such that $r_i = r'$.

We claim that $I = \{1,2\}$ and $r_1 = r_2 = r'$. If not, from (1) we get that $p^{r'+1}$ divides $p^{r'} w$ where w is a unit. As $r' \leq r-1$ we get the contradiction that p divides w. By a similar argument, we see that $s_1 + 1 \neq r'$. As $r' \leq r-1$, from (1) we see that p divides $u_1 x_1 + u_2 x_2$. Let $f = f_{p^r,p}$. We get that

$$f(u_1)f(x_1) + f(u_2)f(x_2) = 0.$$

As $u_1, u_2 \in U(p^r)^2$, it follows that $f(u_1), f(u_2) \in S(p)$. So the sequence $\big(f(x_1), f(x_2)\big)$ is an $S(p)$-weighted zero-sum sequence. Thus, we get the contradiction that the image of the sequence (z_1, z_2) under $f_{p^2,p}$ is an $S(p)$-weighted zero-sum sequence. Hence, it follows that S does not have any $S(p^r)$-weighted zero-sum subsequence. □

Theorem 12. *Let n be an even square. Then we have that $D_{S(n)} \geq 4$.*

Proof. Mondal et al. [7, Cor. 2, Lem. 7] have shown that for every odd prime p we can find a sequence (u_p, v_p) in $U(p^2)$ whose image under $f_{p^2,p}$ is not an $S(p)$-weighted zero-sum sequence. Consider the sequence (u_p, v_p, p) in $\mathbb{Z}_{p^2}$.

For each prime divisor p of n, the map $f_{p^{v_p(n)},p^2}$ is onto. So by the Chinese remainder theorem we can find a sequence $S = (x_1, x_2, x_3)$ in $\mathbb{Z}_n$ such that, for every prime divisor p of n, the image of S under f_{n,p^2} is (u_p, v_p, p) when p is odd, and the image of S under $f_{n,4}$ is $(1, 1, 1)$.

For every prime divisor p of n, we see that the sequence $S^{(p)}$ in $\mathbb{Z}_{p^r}$ has the form as in the statement of Lemma 10 if $p = 2$, or of Lemma 11 if p is odd. So for every prime divisor p of n, if $r = v_p(n)$, it follows that the sequence $S^{(p)}$ does not have any $S(p^r)$-weighted zero-sum subsequence.

Suppose T is an $S(n)$-weighted zero-sum subsequence of S. Let x be a term of T and $a \in S(n)$ be the coefficient of x in an $S(n)$-weighted zero-sum which is obtained from T. As $a \neq 0$, there is a prime divisor p of n such that $a^{(p)} \neq 0$. So we get the contradiction that the sequence $S^{(p)}$ in $\mathbb{Z}_{p^r}$ has an $S(p^r)$-weighted zero-sum subsequence where $r = v_p(n)$.

Thus, it follows that the sequence S does not have any $S(n)$-weighted zero-sum subsequence. Hence, we see that $D_{S(n)} \geq 4$. □

Lemma 13. *Let r be a non-zero even number and p be an integer which is at least two. Suppose T is a sequence in $\mathbb{Z}_{p^r}$ whose image under f_{p^r,p^2} is a $U(p^2)^2$-weighted zero-sum sequence. Then the sequence T is an $S(p^r)$-weighted zero-sum sequence.*

Proof. The proof of this result is similar to the proof of Lemma 9. We need to use the facts that $f_{p^r,p^2}\big(U(p^r)^2\big) = U(p^2)^2$ and that $p^{r-2} = (p^{(r-2)/2})^2 \in S(p^r)$. □

The next result follows immediately from Lemma 13.

Corollary 14. *Let r be a non-zero even number and p be a positive integer. Then we have that $D_{S(p^r)} \leq D_{U(p^2)^2}$ and $C_{S(p^r)} \leq C_{U(p^2)^2}$.*

Theorem 15. *Let r be a non-zero even number. Then we have $C_{S(2^r)} \leq 4$.*

Proof. Mondal et al. [8, Cor. 1] have shown that $C_{\{1\}} = 4$. As $U(4)^2 = \{1\}$, from Corollary 14 it follows that $C_{S(2^r)} \leq 4$. □

Corollary 16. *Let n be an even square. Then we have $D_{S(n)} = C_{S(n)} = 4$.*

Proof. From Theorem 12 we have $D_{S(n)} \geq 4$. By Theorem 15 and Corollary 8 we have $C_{S(n)} \leq 4$. As $D_A(n) \leq C_A(n)$ for every $A \subseteq \mathbb{Z}_n$, it follows that $D_{S(n)} = C_{S(n)} = 4$. □

5 $C_{S(n)}$ and $D_{S(n)}$ when n is not a square

Proposition 17. *Let n be odd. We can find a sequence $S = (u, v)$ in $U(n)$ such that for each prime divisor p of n, the image of S under $f_{n,p}$ does not have any $S(p)$-weighted zero-sum subsequence.*

Proof. Let p be a prime divisor of n and $v_p(n) = r$. By [7, Cor. 2] there exist $u_p, v_p \in U(p)$ such that the sequence (u_p, v_p) does not have any $S(p)$-weighted zero-sum subsequence. As the image of $U(p^r)$ under $f_{p^r,p}$ is $U(p)$, there exist $u'_p, v'_p \in U(p^r)$ such that the image of the sequence (u'_p, v'_p) under $f_{p^r,p}$ is (u_p, v_p).

By the Chinese remainder theorem, there exist $u, v \in U(n)$ such that for each prime divisor p of n if $n_p = p^{v_p(n)}$, then the image of the sequence $S = (u, v)$ under f_{n,n_p} is (u'_p, v'_p). It follows that the image of S under $f_{n,p}$ is (u_p, v_p) which is the same as the image of (u'_p, v'_p) under the map $f_{n_p,p}$. □

Lemma 18. *Let p be an odd prime and r be a positive integer. Suppose $S = (v_1, v_2)$ is a sequence in $U(p^r)$ such that the image of S under $f_{p^r,p}$ is not an $S(p)$-weighted zero-sum sequence. Then the sequence S does not have any $S(p^r)$-weighted zero-sum subsequence.*

Proof. Suppose the sequence S is an $S(p^r)$-weighted zero-sum sequence. Then there exist $a_1, a_2 \in S(p^r)$ such that $a_1v_1 + a_2v_2 = 0$. By Lemma 4 we see that there exist $u_1, u_2 \in U(p^r)^2$ and even $r_1, r_2 \in [0, r-1]$ such that $a_1 = p^{r_1}u_1$ and $a_2 = p^{r_2}u_2$. So we get that

$$p^{r_1}u_1v_1 + p^{r_2}u_2v_2 = 0.$$

By Observation 3 we see that $r_1 = r_2$ and so $p^{r_1}(u_1v_1 + u_2v_2) = 0$. As $r_1 < r$, it follows that p divides $u_1v_1 + u_2v_2$. If f is the map $f_{p^r,p}$, then we see that

$$f(u_1)f(v_1) + f(u_2)f(v_2) = 0.$$

As $u_1, u_2 \in U(p^r)^2$, it follows that $f(u_1), f(u_2) \in S(p)$. Thus, we get the contradiction that the image of S under $f_{p^r,p}$ is an $S(p)$-weighted zero-sum sequence. Hence, it follows that S is not an $S(p^r)$-weighted zero-sum sequence. As $v_1, v_2 \in U(p^r)$, we see that S does not have any $S(p^r)$-weighted zero-sum subsequence of length one. □

Theorem 19. *Let n be an odd number. Then we have that $D_{S(n)} \geq 3$.*

Proof. By Proposition 17 there exists a sequence $S = (u, v)$ in $U(n)$ such that for each prime divisor p of n, the image of S under $f_{n,p}$ does not have any $S(p)$-weighted zero-sum subsequence.

Suppose T is an $S(n)$-weighted zero-sum subsequence of S. As the terms of S are in $U(n)$, we see that T must be S. Thus, there exist $a, b \in S(n)$ such that $au + bv = 0$. As $a \neq 0$, there exists a prime divisor p of n such that $a^{(p)} \neq 0$. Let $k = v_p(n)$ and (u_p, v_p) be the image of S under f_{n,p^k}.

It follows that the sequence (u_p, v_p) in $\mathbb{Z}_{p^k}$ has an $S(p^k)$-weighted zero-sum subsequence. As the image of $S = (u, v)$ under $f_{n,p}$ is the same as the image of (u_p, v_p) under $f_{p^k,p}$, it follows that (u_p, v_p) is a sequence in $\mathbb{Z}_{p^k}$ whose image under $f_{p^k,p}$ does not have any $S(p)$-weighted zero-sum subsequence.

So by Lemma 18 we get the contradiction that the sequence (u_p, v_p) does not have any $S(p^k)$-weighted zero-sum subsequence. Thus, we see that S does not have any $S(n)$-weighted zero-sum subsequence. Hence, it follows that $D_{S(n)} \geq 3$. □

Theorem 20. *We have that $D_{S(n)} \geq 3$ when $v_2(n)$ is even and at least two.*

Proof. By the results by Mondal et al. [7, Cor. 2, Lem. 7] and by the Chinese remainder theorem, we can find a sequence $S = (v_1, v_2)$ in $U(n)$ (by a similar method as in Proposition 17) such that for every odd prime divisor p of n the image of S under $f_{n,p}$ is not an $S(p)$-weighted zero-sum sequence and the image of S under $f_{n,4}$ is $(1, 1)$.

Suppose T is an $S(n)$-weighted zero-sum subsequence of S. As the terms of S are in $U(n)$, we see that T must be S. Thus, there exists $a, b \in S(n)$ such that $au + bv = 0$. As $a \neq 0$, there is a prime divisor q of n such that $a^{(q)} \neq 0$. We now use a similar argument as in the proof of Theorem 19, where we use Lemma 10 in addition to Lemma 18. □

Theorem 21. *Let p be an odd prime and r be odd. Then we have $C_{S(p^r)} \leq 3$.*

Proof. Let $S = (x, y, z)$ be a sequence in $\mathbb{Z}_{p^r}$ and let S' be the image of S under $f_{p^r,p}$. Mondal et al. [8, Thm. 4] showed that for an odd prime p we have $C_{S(p)} = 3$. Thus, we can find a subsequence T whose terms are consecutive terms of S such that the image of T under $f_{p^r,p}$ is an $S(p)$-weighted zero-sum subsequence of S'. So by Lemma 9 we see that T is an $S(p^r)$-weighted zero-sum sequence. Hence, it follows that $C_{S(p^r)} \leq 3$. □

Corollary 22. *Suppose n is not a square and $v_2(n)$ is a non-negative, even integer. Then we have that $D_{S(n)} = C_{S(n)} = 3$.*

Proof. By Theorem 20 we have $D_{S(n)} \geq 3$. From the assumptions on n, we see that there is an odd prime divisor p of n such that $v_p(n)$ is odd. Thus, by Corollary 8 and Theorem 21 we have that $C_{S(n)} \leq 3$. As $D_A(n) \leq C_A(n)$ for every $A \subseteq \mathbb{Z}_n$, it follows that $D_{S(n)} = C_{S(n)} = 3$. □

Theorem 23. *We have that $C_{S(2^r)} \leq 2$ where r is an odd number.*

Proof. Let $S = (x, y)$ be a sequence in $\mathbb{Z}_{2^r}$ and $S' = (x', y')$ be the image of S under $f_{2^r,2}$. We can find a subsequence T of S such that the image of T under $f_{2^r,2}$ is a zero-sum sequence. So by Lemma 9 we see that T is an $S(2^r)$-weighted zero-sum sequence. Hence, it follows that $C_{S(2^r)} \leq 2$. □

Corollary 24. *Suppose n is an even positive integer such that $v_2(n)$ is odd. Then we have that $D_{S(n)} = C_{S(n)} = 2$.*

Proof. It is easy to see that $D_{S(n)} \geq 2$. From Corollary 8 and Theorem 23, we get that $C_{S(n)} \leq 2$. For every $A \subseteq \mathbb{Z}_n$ as $D_A(n) \leq C_A(n)$, it follows that $D_{S(n)} = C_{S(n)} = 2$. □

6 $D_{S(n)}$ when n is an odd square

Lemma 25. *Let p be a prime and r be a non-zero even number. Suppose (w_1, w_2) is a sequence in $U(p^r)$ whose image under $f_{p^r,p}$ is not an $S(p)$-weighted zero-sum sequence. Let $u \in U(p^r)$ and*

$$S = (u\,w_1, u\,w_2, p\,w_1, p\,w_2).$$

Then the sequence S in $\mathbb{Z}_{p^r}$ does not have any $S(p^r)$-weighted zero-sum subsequence.

Proof. Suppose T is an $S(p^r)$-weighted zero-sum subsequence of S. Let

$$I = \{i \in [1,2] : u\,w_i \text{ is a term of } T\} \text{ and } J = \{j \in [1,2] : p\,w_j \text{ is a term of } T\}.$$

As T is an $S(p^r)$-weighted zero-sum sequence, for each $i \in I$ there exists $a_i \in S(p^r)$ and for each $j \in J$ there exists $b_j \in S(p^r)$ such that $\sum_{i\in I} a_i u\,w_i + \sum_{j\in J} b_j\,p\,w_j = 0$. From Lemma 4, for each $i \in I$ we have $a_i = p^{r_i}u_i$ for some even $r_i < r$ and $u_i \in U(p^r)^2$ and for each $j \in J$ we have $b_j = p^{s_j}v_j$ for some even $s_j < r$ and $v_j \in U(p^r)^2$. So we have

$$u\sum_{i\in I} p^{r_i}u_i\,w_i + \sum_{j\in J} p^{s_j+1}v_j w_j = 0. \tag{2}$$

Consider the set $L = \{r_i : i \in I\} \cup \{s_j + 1 : j \in J\}$. Let r' be the minimum of L. As r is even and s_j is even for each $j \in J$, it follows that $r' \leq r - 1$. Observe that $\{r_i : i \in I\} \cap \{s_j + 1 : j \in J\} = \emptyset$ as the r_i's and s_j's are even. Suppose there exists $i \in I$ such that $r_i = r'$. We claim that $I = \{1, 2\}$ and $r_1 = r_2 = r'$. If not, from (2) we get that $p^{r'+1}$ divides $p^{r'}w$ where w is a unit. As $r' \leq r - 1$ we get the contradiction that p divides w. By a similar argument if there exists $j \in J$ such that $s_j + 1 = r'$, then $J = \{1, 2\}$ and $s_1 + 1 = s_2 + 1 = r'$.

Suppose $I = \{1, 2\}$ and $r_1 = r_2 = r'$. As $r' \leq r - 1$, from (2) we see that p divides $u\,(u_1w_1 + u_2w_2)$. As $u \in U(p^r)$, it follows that $f(u) \in U(p)$ where $f = f_{p^r,p}$. So we get that

$$f(u_1)f(w_1) + f(u_2)f(w_2) = 0.$$

As $u_1, u_2 \in U(p^r)^2$, it follows that $f(u_1), f(u_2) \in S(p)$. Thus, we get the contradiction that the image of the sequence (w_1, w_2) under $f_{p^r,p}$ is an $S(p)$-weighted zero-sum sequence. We will get the same contradiction if $J = \{1, 2\}$ and $s_1 + 1 = s_2 + 1 = r'$. Thus, it follows that S does not have any $S(p^r)$-weighted zero-sum subsequence. □

Theorem 26. *Let n be an odd square. Then we have that $D_{S(n)} \geq 5$.*

Proof. Let m be the radical of n, i.e., the largest squarefree divisor of n. By Proposition 17 there exists a sequence (u, v) in $U(n)$ such that for each prime divisor p of n, the image of the sequence (u, v) under $f_{n,p}$ does not have any $S(p)$-weighted zero-sum subsequence. Let

$$S = (u, v, mu, mv).$$

We claim that this sequence S does not have any $S(n)$-weighted zero-sum subsequence. From this it follows that $D_{S(n)} \geq 5$.

Suppose T is an $S(n)$-weighted zero-sum subsequence of S. Let x be a term of T and $a \in S(n)$ be the coefficient of x in an $S(n)$-weighted zero-sum which we obtain from T. As $a \neq 0$, there exists a prime divisor p of n such that $a^{(p)} \neq 0$. Let $r = v_p(n)$. It follows that the sequence $S^{(p)}$ has an $S(p^r)$-weighted zero-sum subsequence.

The image of the sequence $(u^{(p)}, v^{(p)})$ under $f_{p^r,p}$ does not have any $S(p)$-weighted zero-sum subsequence. As m is the largest squarefree divisor of n, it follows that $m^{(p)} = p\,w$ where $w \in U(p^r)$. It follows that $S^{(p)}$ is a sequence in $\mathbb{Z}_{p^r}$ which has the form as in the statement of Lemma 25. As n is a square, we see that r is a non-zero even number. So by Lemma 25 we arrive at the contradiction that the sequence $S^{(p)}$ does not have any $S(p^r)$-weighted zero-sum subsequence. Hence, our claim must be true. □

We get the next result from the proof of [2, Thm. 7].

Lemma 27. *Let p be an odd prime, r be a positive integer, and $A = U(p^r)^2$. Suppose we are given $y_1, y_2, y_3 \in U(p^r)$. Then we have that*

$$Ay_1 + (Ay_2 \cup \{0\}\,) + (Ay_3 \cup \{0\}\,) = \mathbb{Z}_{p^r}.$$

Theorem 28. *Let r be a non-zero even number and p be an odd prime. Then we have that $D_{S(p^r)} \leq 5$.*

Proof. Suppose $S = (x_1, \ldots, x_5)$ is a sequence in $\mathbb{Z}_{p^r}$. If p^2 divides some term x_i of S, then $p^{r-2}x_i = 0$. As r is even, we see that $p^{r-2} = (p^{(r-2)/2})^2 \in S(n)$. So we see that $T = (x_i)$ is an $S(p^r)$-weighted zero-sum subsequence of S of length one. Thus, we may assume that p^2 does not divide any term of S.

It follows that each term of S is either a unit or a unit multiple of p. If at least three terms of S are units or at least three terms of S are unit multiples of p, by using Lemma 27 we get an $S(p^r)$-weighted zero-sum subsequence of S. Thus, we see that $D_{S(p^r)} \leq 5$. □

Corollary 29. *Let n be an odd square. Then we have that $D_{S(n)} = 5$.*

Proof. From Theorem 26 we see that $D_{S(n)} \geq 5$ when n is an odd square. Since every prime divisor p of n is odd and $v_p(n)$ is even, by using Corollary 8 and Theorem 28 we see that $D_{S(n)} \leq 5$. Thus, it follows that $D_{S(n)} = 5$. □

7 $C_{S(n)}$ when n is an odd square

We will use the following notation.

If T is a subsequence of S, then $S - T$ denotes the subsequence which is obtained by removing the terms of T from S. The concatenation of the sequences $S - T$ and T gives us a sequence whose terms are a permutation of the terms of the sequence S.

If S is a sequence in $\mathbb{Z}_n$ and $d \in \mathbb{Z}_n$ such that all the terms of S are divisible by d, then S/d denotes the sequence in $\mathbb{Z}_n$ whose terms are obtained by dividing the corresponding terms of S by d.

Theorem 30. *We have that* $C_{U(25)^2} = 9$.

Proof. Let $S = (x_1, \dots, x_9)$ be a sequence in $\mathbb{Z}_{25}$. We may assume that all the terms of S are non-zero.

Suppose at least four terms of S are units. From [4, Lem. 2] it follows that S is a $U(25)^2$-weighted zero-sum sequence. Let

$$S_1 = (x_1, x_2, x_3),\ S_2 = (x_4, x_5, x_6),\ \text{and } S_3 = (x_7, x_8, x_9).$$

Suppose at most two terms of S are units. Then we see that there exists $i \in [1, 3]$ such that all the terms of S_i are divisible by 5. Let S'_i denote the sequence in $\mathbb{Z}_5$ which is the image of $S_i/5$ under $f_{25,5}$. From [8, Thm. 4] we have that $C_{Q_5} = 3$. Thus, the sequence S'_i has a Q_5-weighted zero-sum subsequence having consecutive terms. By [7, Lem. 5] it follows that the sequence S_i (and hence the sequence S) has a $U(25)^2$-weighted zero-sum subsequence having consecutive terms.

So we may assume that exactly three terms of S are units. If at least three consecutive terms of S are non-units, by a similar argument as in the previous paragraph we get a $U(25)^2$-weighted zero-sum subsequence of S having consecutive terms. So it follows that for each $i \in [1, 3]$ there is exactly one term y_i in the sequence S_i which is a unit.

As $C_{Q_5} = 3$ we see that the sequence (y_1, y_2, y_3) has a subsequence S_4 having consecutive terms whose image S'_4 under $f_{25,5}$ is a Q_5-weighted zero-sum sequence. As $f_{25,5}$ is onto, it follows that there exists $k \in \mathbb{Z}_{25}$ such that a $U(25)^2$-weighted sum of the terms of S_4 is $-5k$. We will use this observation a bit later in this proof. Let

$$J = \{i \in [1, 3] : y_i \text{ is a term of } S_4\}.$$

Let T be the concatenation of the sequences S_i where $i \in J$. It follows that T is a subsequence of S having consecutive terms. We claim that T is a $U(25)^2$-weighted zero-sum sequence. Let $T_1 = T - S_4$. As all the terms of S are non-zero, all the terms of T_1 are of the form $5u$ where $u \in U(25)$.

Let T'_1 denote the image of $T_1/5$ under $f_{25,5}$. Chintamani and Moriya [4, Lem. 2] showed that $f_{25,5}(k) \in \mathbb{Z}_5$ is a Q_5-weighted sum of the terms of T'_1. Mondal et al. [7, Lem. 5] showed that $5k$ is a $U(25)^2$-weighted sum of the terms of T_1. As we have seen that $-5k$ is a $U(25)^2$-weighted sum of the terms of S_4, it follows that T is a $U(25)^2$-weighted zero-sum sequence.

Thus, every sequence in $\mathbb{Z}_{25}$ having length nine has a $U(25)^2$-weighted zero-sum subsequence whose terms are consecutive terms of the given sequence. So it follows that $C_{U(25)^2} \leq 9$. Mondal et al. [8, Cor. 5] showed that $C_{U(25)^2} \geq 9$. Hence, it follows that $C_{U(25)^2} = 9$. □

Theorem 31. *Let n be an odd square. Then we have that $C_{S(n)} \leq 9$.*

Proof. Mondal et al. [8, Cor. 6] showed that $C_{U(p^2)^2} = 9$ when p is a prime which is at least seven. From Remark 1 we see that $C_{U(9)^2} \leq 9$ and from Theorem 30 we see that $C_{U(25)^2} \leq 9$. Thus, from Corollaries 8 and 14 it follows that $C_{S(n)} \leq 9$. □

Chintamani and Moriya [4, Lem. 1] showed the next result.

Lemma 32. *Let p be a prime which is at least seven and $A = U(p^r)^2$. Then for every $x_1, x_2, x_3 \in U(p^r)$ we have that $Ax_1 + Ax_2 + Ax_3 = \mathbb{Z}_{p^r}$.*

Mondal et al. [8, Lem. 7] showed the next result, which follows easily from Lemma 32.

Lemma 33. *Let p be a prime which is at least seven and $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_{p^r}$. Suppose at least three terms of S are units. Then S is a $U(p^r)^2$-weighted zero-sum sequence.*

Theorem 34. *Let p be a prime which is at least seven and r be an even number which is at least four. Then we have that $C_{S(p^r)} \leq 5$.*

Proof. Let $S = (x_1, \ldots, x_5)$ be a sequence in $\mathbb{Z}_{p^r}$. As r is even, we see that $p^{r-2} = (p^{(r-2)/2})^2$ and hence $p^{r-2} \in S(p^r)$. If p^2 divides some term x of S, then it follows that $p^{r-2}x = 0$ and so S has an $S(p^r)$-weighted zero-sum subsequence of length one. Thus, we may assume that p^2 does not divide any term of S. So every term of S is either a unit or of the form $p\,u$ where u is a unit.

If at least three terms of S are units, by Lemma 33 we see that S is an $S(p^r)$-weighted zero-sum sequence. Thus, we may assume that at most two terms of S are units. Then at least three terms of S are of the form $p\,u$ where u is a unit. We may assume that $x_1 = pu_1,\ x_2 = pu_2,\ x_3 = pu_3$ where $u_1, u_2, u_3 \in U(p^r)$. Consider the sequence

$$S' = (u_1, u_2, u_3, px_4, px_5).$$

By Lemma 33 we see that S' is a $U(p^r)^2$-weighted zero-sum sequence. So there exist a_i's in $U(p^r)^2$ such that

$$a_1u_1 + a_2u_2 + a_3u_3 + a_4px_4 + a_5px_5 = 0.$$

Thus, it follows that

$$a_1x_1 + a_2x_2 + a_3x_3 + p^2a_4x_4 + p^2a_5x_5 = 0.$$

As $r \geq 4$, we see that $p^2 \neq 0$. Hence, it follows that S is an $S(p^r)$-weighted zero-sum sequence. □

Corollary 35. *Let n be an odd square which is divisible by p^4 where p is a prime which is at least seven. Then we have that $C_{S(n)} = 5$.*

Proof. As n is a square, it follows that $v_p(n)$ is even. So from Corollary 8 and Theorem 34 we have $C_{S(n)} \leq 5$. As we have that $D_A(n) \leq C_A(n)$ for every $A \subseteq \mathbb{Z}_n$, from Theorem 26 it follows that $C_{S(n)} = 5$. □

The next result follows easily from a result by Chintamani and Moriya [4, Lem. 2].

Lemma 36. *Let r be a positive integer and S be a sequence in $\mathbb{Z}_{5^r}$. Suppose at least four terms of S are units. Then S is a $U(5^r)^2$-weighted zero-sum sequence.*

Theorem 37. *We have that $C_{S(5^r)} \leq 7$ when r is an even number which is at least four.*

Proof. We use a similar argument as in the proof of Theorem 34. The only change is that we replace Lemma 33 with Lemma 36. □

Corollary 38. *Let n be a square which is divisible by 5^4. Then we have that $C_{S(n)} \leq 7$.*

Proof. As n is a square, it follows that $v_5(n)$ is even. Also, we have that $v_5(n) \geq 4$. So from Corollary 8 and Theorem 37 we get that $C_{S(n)} \leq 7$. □

Theorem 39. *Let n be a square of an odd squarefree number. Then we have that $C_{S(n)} = 9$.*

Proof. By Theorem 31 we get that $C_{S(n)} \leq 9$. We will construct a sequence S of length eight in $\mathbb{Z}_n$ which has no $S(n)$-weighted zero-sum subsequence having consecutive terms. Hence, it will follow that $C_{S(n)} = 9$.

By Proposition 17 there exists a sequence $S' = (u, v)$ in $U(n)$ such that for every prime divisor p of n, the image (u_p, v_p) of S' under $f_{n,p}$ does not have any $S(p)$-weighted zero-sum subsequence. By the Chinese remainder theorem there exist $x, y \in \mathbb{Z}_n$ such that for each prime divisor p of n we have that $x^{(p)} = p\,u^{(p)}$ and $y^{(p)} = p\,v^{(p)}$. In this proof, for every $c \in \mathbb{Z}_n$ we will denote $f_{n,p}(c)$ by c_p. So it follows that $x_p = y_p = 0$. Consider the sequence S in $\mathbb{Z}_n$ defined as follows:

$$S = (x, y, u, x, y, v, x, y).$$

Suppose there exists a subsequence T of S having consecutive terms which is an $S(n)$-weighted zero-sum sequence.

Case 1: Either u or v is a term of T.

Without loss of generality, we may assume that u is a term of T.

Let $a \in S(n)$ be the coefficient of u in the $S(n)$-weighted zero-sum which is obtained from T. As $a \neq 0$, there exists a prime divisor p of n such that $a^{(p)} \neq 0$. As n is the square of a squarefree number, it follows that $v_p(n) = 2$. So we see that $a^{(p)} \in S(p^2)$. As every non-zero term of $\mathbb{Z}_{p^2}$ is either a unit or a unit multiple of p, we see that $S(p^2) = U(p^2)^2$. As $a^{(p)} \in U(p^2)^2$, it follows that $f_{p^2,p}\big(a^{(p)}\big) \in S(p)$ and so $a_p \in S(p)$.

We claim that the sequence (u_p, v_p) has an $S(p)$-weighted zero-sum subsequence.

Suppose v is a term of T. Let $b \in S(n)$ be the coefficient of v in the $S(n)$-weighted zero-sum which is obtained from T. As $b \in S(n)$, it follows that $b_p \in S(p) \cup \{0\}$. So we get that $a_p u_p + b_p v_p = 0$. If v is not a term of T then $a_p u_p = 0$. This proves our claim which contradicts our choice of the sequence (u_p, v_p).

Case 2: Neither u nor v is a term of T.

As T is a subsequence of consecutive terms, it follows that T is a subsequence of the sequence (x, y). Suppose x is a term of T. Let $a \in S(n)$ be the coefficient of x in the $S(n)$-weighted zero-sum which is obtained from T. By a similar argument as in the previous case, we see that there is a prime divisor p of n such that $a_p \in S(p)$.

We claim that the sequence (u_p, v_p) has an $S(p)$-weighted zero-sum subsequence.

Suppose y is a term of T. Then there exists $b \in S(n)$ such that $ax + by = 0$. As $b \in S(n)$, it follows that $b_p \in S(p) \cup \{0\}$. As $ax + by = 0$, it follows that $a^{(p)}x^{(p)} + b^{(p)}y^{(p)} = 0$ in $\mathbb{Z}_{p^2}$. Thus, we get that

$$p\left(a^{(p)}u^{(p)} + b^{(p)}v^{(p)}\right) = 0$$

and so $a^{(p)}u^{(p)} + b^{(p)}v^{(p)}$ is divisible by p. Hence, it follows that $a_p u_p + b_p v_p = 0$. By a similar argument, we see that if y is not a term of T then $a_p u_p = 0$. This proves our claim which contradicts our choice of the sequence (u_p, v_p).

Hence, it follows that the sequence S does not have any $S(n)$-weighted zero-sum subsequence of consecutive terms. □

8 Concluding remarks

We have been unable to determine the constants $C_{S(3^r)}$ and $C_{S(5^r)}$ where r is an even number which is at least four. For every such r we have shown that $C_{S(3^r)} \in [5, 9]$ and $C_{S(5^r)} \in [5, 7]$. If the values of these constants are known, we can determine the value of $C_{S(n)}$ for every n.

We can try to characterize the sequences in $\mathbb{Z}_n$ of length $C_{S(n)} - 1$ which do not have any $S(n)$-weighted zero-sum subsequence having consecutive terms. We can also try to characterize sequences in $\mathbb{Z}_n$ of length $D_{S(n)} - 1$ which do not have any $S(n)$-weighted zero-sum subsequence.

9 Acknowledgments

We thank the anonymous referee for the helpful suggestions.

References

[1] S. D. Adhikari and Y. G. Chen, Davenport constant with weights and some related questions II, *J. Combin. Theory Ser. A* **115** (2008), 178–184.

[2] S. D. Adhikari, C. David, and J. J. Urroz, Generalizations of some zero-sum theorems, *Integers* **8** (2008), #A52.

[3] S. D. Adhikari and P. Rath, Davenport constant with weights and some related questions, *Integers* **6** (2006), #A30.

[4] M. N. Chintamani and B. K. Moriya, Generalizations of some zero-sum theorems, *Proc. Indian Acad. Sci. Math. Sci.* **122** (2012), 15–21.

[5] D. J. Grynkiewicz and F. Hennecart, A weighted zero-sum problem with quadratic residues, *Unif. Distrib. Theory* **10** (2015), 69–105.

[6] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer, New York, 1990.

[7] S. Mondal, K. Paul, and S. Paul, Extremal sequences for a weighted zero-sum constant, *Integers* **22** (2022), #A93.

[8] S. Mondal, K. Paul, and S. Paul, On a different weighted zero-sum constant, *Discrete Math.* **346** (2023), 113350.

2020 *Mathematics Subject Classification*: Primary 11B50; Secondary 11B75.
Keywords: Davenport constant, weighted zero-sum sequence.

(Concerned with sequence A371769.)

Received March 22 2024; revised version received August 1 2024. Published in *Journal of Integer Sequences*, August 1 2024.

Return to Journal of Integer Sequences home page.