Journal of Integer Sequences, Vol. 27 (2024),
Article 24.1.7

Zero-Sum Constants Related to the Jacobi Symbol

Santanu Mondal, Krishnendu Paul, and Shameek Paul
School of Mathematical Sciences
Ramakrishna Mission Vivekananda Educational and Research Institute
West Bengal 711202
India
santanu.mondal.math18@gm.rkmvu.ac.in
krishnendu.p.math18@gm.rkmvu.ac.in
shameek.paul@rkmvu.ac.in

Abstract

Let $A \subseteq \mathbb{Z}_n$ be a subset. A sequence $S = (x_1, \ldots, x_k)$ is said to be an A-weighted zero-sum sequence if there exist $a_1, \ldots, a_k \in A$ such that $a_1x_1 + \cdots + a_kx_k = 0$. We refer to A as a weight-set. The A-weighted Davenport constant D_A is defined to be the smallest natural number k such that every sequence of k elements in $\mathbb{Z}_n$ has an A-weighted zero-sum subsequence. The constant C_A is defined to be the smallest natural number k such that every sequence of k elements in $\mathbb{Z}_n$ has an A-weighted zero-sum subsequence having consecutive terms.

When n is odd, let $S(n)$ be the set of all units in $\mathbb{Z}_n$ whose Jacobi symbol with respect to n is 1. We compute the constants $C_{S(n)}$ and $D_{S(n)}$. For a prime divisor p of n, we also compute these constants for a related weight-set $L(n;p)$. This is the set of all units x in $\mathbb{Z}_n$ such that the Jacobi symbol of x with respect to n is the same as the Legendre symbol of x with respect to p. We show that even though these weight-sets A may have half the size of $U(n)$ (which is the set of units of $\mathbb{Z}_n$), the corresponding A-weighted constants are the same as those for the weight-set $U(n)$.

1 Introduction

For $a, b \in \mathbb{Z}$, we denote the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$ by $[a, b]$. Let $U(n)$ denote the group of units in the ring $\mathbb{Z}_n$, and $U(n)^2 = \{\, x^2 : x \in U(n) \,\}$. For an odd prime p, let Q_p denote the set $U(p)^2$. For $n = p_1 p_2 \cdots p_k$ where p_i is a prime for each $i \in [1, k]$, we define $\Omega(n) = k$.

Definition 1. Let $A \subseteq \mathbb{Z}_n$ be a subset. A sequence $S = (x_1, \ldots, x_k)$ is said to be an A-weighted zero-sum sequence if there exist $a_1, \ldots, a_k \in A$ such that $a_1 x_1 + \cdots + a_k x_k = 0$. We refer to A as a weight-set.

Definition 2. For a weight-set $A \subseteq \mathbb{Z}_n$, the *A-weighted Davenport constant* D_A is defined to be the least positive integer k, such that every sequence in $\mathbb{Z}_n$ of length k has an A-weighted zero-sum subsequence.

Adhikari and Rath [4] gave the previous definition. Chintamani and Moriya [5] showed that $D_{U(n)^2} = 2\,\Omega(n) + 1$ when every prime divisor of n is at least seven. Grynkiewicz and Hennecart [7] generalized this by showing that $D_{U(n)^2} \geq 2\Omega(n) + \min\{v_3(n), v_5(n)\} + 1$ when n is odd, with equality if either $3 \nmid n$ or $v_3(n) \geq v_5(n)$. Mazumdar and Sinha [10] made suitable modifications in the method of Griffiths [6] to consider the case when n is an even integer. (However, their result cannot be used to determine $D_{U(n)^2}$ when n is even.) Adhikari et al. [1, Lem. 2.1] showed that $D_{\{1,-1\}} = \lfloor \log_2 n \rfloor + 1$ for every positive integer n.

Mondal, Paul, and Paul [11] gave the following definition.

Definition 3. For a weight-set $A \subseteq \mathbb{Z}_n$, the *A-weighted constant* C_A is defined to be the least positive integer k, such that every sequence in $\mathbb{Z}_n$ of length k has an A-weighted zero-sum subsequence of consecutive terms.

Mondal, Paul, and Paul [11, Cor. 3, Cor. 6] showed that $C_{U(n)^2} = 3^{\Omega(n)}$ when every prime divisor of n is at least seven and $C_{\{1,-1\}} = n$ when n is a power of two. Mondal, Paul, and Paul [12] showed the next result.

Theorem 4. *For every positive integer n we have $D_{U(n)} = \Omega(n) + 1$ and $C_{U(n)} = 2^{\Omega(n)}$.*

When p is an odd prime such that $p \equiv 2 \pmod 3$, we can show that $U(p)^3 = U(p)$. Mondal, Paul, and Paul [11, Thm. 7, Lem. 2] showed that when $p \neq 7$ is a prime such that $p \equiv 1 \pmod 3$, we have $D_{U(p)^3} = C_{U(p)^3} = 3$, and also that $D_{U(7)^3} = 3$ and $C_{U(7)^3} = 4$. Adhikari and Rath [4, Thm. 2], and Mondal, Paul, and Paul [11, Thm. 4] showed the next result.

Theorem 5. *Let p be an odd prime. Then $C_{Q_p} = D_{Q_p} = 3$.*

Let m be a divisor of n. We refer to the ring homomorphism $f_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $a + n\mathbb{Z} \mapsto a + m\mathbb{Z}$ as the *natural map*. As this map sends units to units, we get a group homomorphism $U(n) \to U(m)$, which we also refer to as the natural map. When n is odd and $x \in \mathbb{Z}_n$, the Jacobi symbol $\left(\frac{x}{n}\right)$ is defined in Section 2.

The following are some of the results in this paper. We assume that n is an odd, squarefree number whose every prime divisor is at least seven.

- Let $S(n) = \left\{ x \in U(n) : \left(\frac{x}{n}\right) = 1 \right\}$.

  If n is prime, then $D_{S(n)} = 3$, and $D_{S(n)} = \Omega(n) + 1$ otherwise.

  If n is prime, then $C_{S(n)} = 3$, and $C_{S(n)} = 2^{\Omega(n)}$ otherwise.

- Let $L(n; p) = \left\{ x \in U(n) : \left(\frac{x}{n}\right) = \left(\frac{x}{p}\right) \right\}$ where p is a prime divisor of n.

  If $\Omega(n) = 2$, then $D_{L(n;p)} = 4$, and $D_{L(n;p)} = \Omega(n) + 1$ otherwise.

  If $\Omega(n) = 2$, then $C_{L(n;p)} = 6$, and $C_{L(n;p)} = 2^{\Omega(n)}$ otherwise.

Remark 6. Adhikari and Hegde [3] showed that if $A = \mathbb{Z}_n \setminus \{0\}$ and $B = \{1, 2, \ldots, \lceil n/2 \rceil\}$, we have $D_A = D_B$. We make a similar observation in this paper. In Proposition 11, we show that $S(n)$ is a subgroup of $U(n)$ having index two when n is not a square. Theorem 4 shows that, when n is odd, we have $D_{U(n)} = \Omega(n) + 1$ and $C_{U(n)} = 2^{\Omega(n)}$. In addition, if n is not a prime, Theorems 23 and 24 show that $D_{S(n)} = D_{U(n)}$ and $C_{S(n)} = C_{U(n)}$. Thus, even though these weight-sets may have different sizes, they can have the same constants. If $\Omega(n) \neq 2$, Theorems 33 and 35 show that $D_{L(n;\,p)} = D_{U(n)}$ and $C_{L(n;\,p)} = C_{U(n)}$.

If p is a prime divisor of n, we use the notation $v_p(n) = r$ to mean that $p^r \mid n$ and $p^{r+1} \nmid n$. Let p be a prime divisor of n and $v_p(n) = r$. We denote the image in $U(p^r)$ of $x \in U(n)$ under $f_{n,\,p^r}$ by $x^{(p)}$. For a sequence $S = (x_1, \ldots, x_l)$ in $\mathbb{Z}_n$, let $S^{(p)}$ denote the sequence $(x_1^{(p)}, \ldots, x_l^{(p)})$ in $\mathbb{Z}_{p^r}$, which is the image of S under $f_{n,\,p^r}$. Griffiths [6, Obs. 2.2] made the following observation.

Observation 7. Let $n = p_1^{r_1} \cdots p_k^{r_k}$ where the p_i's are distinct primes and $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$. Suppose for every $i \in [1, k]$ there exist $c_{i,1}, \ldots, c_{i,j}, \ldots, c_{i,l} \in U(p_i^{r_i})$ such that $c_{i,1}x_1^{(p_i)} + \cdots + c_{i,j}x_j^{(p_i)} + \cdots + c_{i,l}x_l^{(p_i)} = 0$. Then there exist $a_1, \ldots, a_j, \ldots, a_l \in U(n)$ such that for every $(i, j) \in [1, k] \times [1, l]$ we have $a_j^{(p_i)} = c_{i,j}$ and $a_1x_1 + \cdots + a_jx_j + \cdots + a_lx_l = 0$.

Proof. Let $j \in [1, l]$. By the Chinese remainder theorem, there exists $a_j \in U(n)$ such that for every $i \in [1, k]$ we have that $a_j^{(p_i)} = c_{i,j}$. Let $x = a_1x_1 + \cdots + a_jx_j + \cdots + a_lx_l$. For each $i \in [1, k]$ we see that $f_{n,\,p_i^{r_i}}(x) = x^{(p_i)} = c_{i,1}x_1^{(p_i)} + \cdots + c_{i,j}x_j^{(p_i)} + \cdots + c_{i,l}x_l^{(p_i)} = 0$. So by using the Chinese remainder theorem once again, we see that $x = 0$. □

Mondal, Paul, and Paul [11, Lem. 3] showed the next result, which will be used in Theorem 36. In the next two results, for a subset A of $\mathbb{Z}_n$, we use the notation $C_A(n)$ and $D_A(n)$ for the constants C_A and D_A respectively.

Lemma 8. *Let $n = mq$. Let A, B, C be subsets of $\mathbb{Z}_n, \mathbb{Z}_m, \mathbb{Z}_q$ respectively. Suppose $f_{n,m}(A) \subseteq B$ and $f_{n,q}(A) \subseteq C$. Then we have $C_A(n) \geq C_B(m)\, C_C(q)$.*

We now prove a similar result for the weighted Davenport constant, which we will use in Theorem 34. Grynkiewicz, Marchan, and Ordaz [8, Lem. 3.1] proved a generalization of this result for abelian groups.

Lemma 9. *Let $n = mq$. Let A, B, C be subsets of $\mathbb{Z}_n, \mathbb{Z}_m, \mathbb{Z}_q$ respectively. Suppose $f_{n,m}(A) \subseteq B$ and $f_{n,q}(A) \subseteq C$. Then we have $D_A(n) \geq D_B(m) + D_C(q) - 1$.*

Proof. Let $D_B(m) = k$ and $D_C(q) = l$. If $k = 1$, we let S'_1 be the empty sequence, and if $l = 1$, we let S'_2 be the empty sequence. Otherwise, there exists a sequence $S'_1 = (u_1, \ldots, u_{k-1})$ of length $k - 1$ in $\mathbb{Z}_m$, which has no B-weighted zero-sum subsequence, and there exists a sequence $S'_2 = (v_1, \ldots, v_{l-1})$ of length $l - 1$ in $\mathbb{Z}_q$, which has no C-weighted zero-sum subsequence.

As $f_{n,m}$ is onto, for every $i \in [1, k-1]$ there exists $x_i \in \mathbb{Z}_n$ such that $f_{n,m}(x_i) = u_i$. As $f_{n,q}$ is onto, for every $j \in [1, l-1]$ there exists $y_j \in \mathbb{Z}_n$ such that $f_{n,q}(y_j) = v_j$. Consider the following sequence of length $k + l - 2$ in $\mathbb{Z}_n$:

$$S = (qx_1, \ldots, qx_{k-1}, y_1, \ldots, y_{l-1}).$$

Let $S_1 = (qx_1, \ldots, qx_{k-1})$ and $S_2 = (y_1, \ldots, y_{l-1})$. Suppose S has an A-weighted zero-sum subsequence T. If the sequence T contains some term of S_2, by taking the image of T under $f_{n,q}$ we get the contradiction that S'_2 has a C-weighted zero-sum subsequence, as $f_{n,q}(qx_i) = 0$ and as $f_{n,q}(A) \subseteq C$.

Thus, no term of S_2 is a term of T, and so T is a subsequence of S_1. Let T' be the subsequence of S'_1, such that u_i is a term of T' if and only if qx_i is a term of T. As $f_{n,m}(A) \subseteq B$, by dividing the A-weighted zero-sum which is obtained from T by q and by taking the image under $f_{n,m}$ we get the contradiction that T' is a B-weighted zero-sum subsequence of S'_1.

Hence, we see that S does not have a A-weighted zero-sum subsequence. As S has length $k + l - 2$, it follows that $D_A(n) \geq k + l - 1$. □

2 Some results about the weight-set $S(n)$

From this point onwards, we will assume that n is odd.

Definition 10. For an odd prime p and $a \in U(p)$, the symbol $\left(\frac{a}{p}\right)$ is the *Legendre symbol with respect to* p, which is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \in Q_p; \\ -1, & \text{if } a \notin Q_p. \end{cases}$$

For a prime divisor p of n, we use the notation $\left(\frac{a}{p}\right)$ to denote $\left(\frac{f_{n,p}(a)}{p}\right)$ where $a \in U(n)$. Let $n = p_1^{r_1} \cdots p_k^{r_k}$ where the p_i's are distinct primes.

For $a \in U(n)$, we define the Jacobi symbol $\left(\frac{a}{n}\right)$ to be $\left(\frac{a}{p_1}\right)^{r_1} \cdots \left(\frac{a}{p_k}\right)^{r_k}$. Observe that we have $\left(\frac{a}{n}\right) = \left(\frac{a^{(p_1)}}{p_1^{r_1}}\right) \cdots \left(\frac{a^{(p_k)}}{p_k^{r_k}}\right)$.

Let $S(n)$ denote the kernel of the homomorphism $U(n) \to \{1, -1\}$ given by $a \mapsto \left(\frac{a}{n}\right)$.

Adhikari, David, and Urroz [2, Sec. 3] considered the set $S(n)$ as a weight-set.

Proposition 11. *$S(n)$ is a subgroup having index two in $U(n)$ when n is a non-square, and $S(n) = U(n)$ when n is a square.*

Proof. Let $n = p_1^{r_1} \cdots p_k^{r_k}$ where the p_i's are distinct primes. If n is a square, then all the r_i are even, and so $S(n) = U(n)$. If n is not a square, there exists j such that r_j is odd. As for every $i \in [1, k]$ the map $f_{p_i^{r_i}, p_i}$ is onto, by the Chinese Remainder theorem we see that there is a unit $b \in U(n)$ such that $\left(\frac{b}{p_i}\right) = 1$ when $i \neq j$, and $\left(\frac{b}{p_j}\right) = -1$. It follows that $\left(\frac{b}{n}\right) = -1$ and so the homomorphism $U(n) \to \{1, -1\}$ given by $a \mapsto \left(\frac{a}{n}\right)$ is onto. Hence, we see that $S(n)$ has index two in $U(n)$. □

Remark 12. In particular, if n is squarefree, then $S(n)$ has index two in $U(n)$. It follows that when p is an odd prime we have $S(p) = Q_p$.

Observation 13. Let $n = p_1 \cdots p_k$ where the p_i's are distinct prime numbers. For $a \in U(n)$, let $\mu(a)$ denote the cardinality of $\{\, j \in [1, k] : f_{n, p_j}(a) = a^{(p_j)} \notin Q_{p_j} \,\}$. As we have that

$$\left(\frac{a}{n}\right) = \left(\frac{a^{(p_1)}}{p_1}\right) \cdots \left(\frac{a^{(p_j)}}{p_j}\right) \cdots \left(\frac{a^{(p_k)}}{p_k}\right),$$

it follows that $a \in S(n)$ if and only if $\mu(a)$ is even.

Lemma 14. *Let d be a proper divisor of n such that d is not a square. Suppose d is coprime with n' where $n' = n/d$. Then we have that $U(n') \subseteq f_{n,n'}(S(n))$.*

Proof. Let $a' \in U(n')$. By the Chinese remainder theorem, there is an isomorphism $\psi : U(n) \to U(n') \times U(d)$. As d is not a square, by Proposition 11 there exists $b \in U(d)$ such that $b \notin S(d)$. If $a' \in S(n')$, let $a \in U(n)$ be a unit such that $\psi(a) = (a', 1)$. If $a' \notin S(n')$, let $a \in U(n)$ be a unit such that $\psi(a) = (a', b)$. Then we have $a \in S(n)$ and $f_{n,n'}(a) = a'$. □

Lemma 15. *Let S be a sequence in $\mathbb{Z}_n$ and d be a proper divisor of n which divides every term of S. Let $n' = n/d$ and d be coprime with n'. Let S' be the sequence in $\mathbb{Z}_{n'}$ which is the image of the sequence S under $f_{n,n'}$. Let $A \subseteq \mathbb{Z}_n$ and $A' \subseteq \mathbb{Z}_{n'}$ be subsets such that $A' \subseteq f_{n,n'}(A)$. Suppose S' is an A'-weighted zero-sum sequence. Then S is an A-weighted zero-sum sequence.*

Proof. Let $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$ and $S' = (x_1', \ldots, x_k')$ where $x_i' = f_{n,n'}(x_i)$ for every $i \in [1, k]$. Suppose S' is an A'-weighted zero-sum sequence. Then for every $i \in [1, k]$ there exist $a_i' \in A'$ such that $a_1'x_1' + \cdots + a_k'x_k' = 0$. Since $A' \subseteq f_{n,n'}(A)$, for every $i \in [1, k]$ there exist $a_i \in A$ such that $f_{n,n'}(a_i) = a_i'$. As $a_1'x_1' + \cdots + a_k'x_k' = 0$ in $\mathbb{Z}_{n'}$, it follows that

$f_{n,n'}(a_1x_1 + \cdots + a_kx_k) = 0$. Let $x = a_1x_1 + \cdots + a_kx_k \in \mathbb{Z}_n$. As $f_{n,n'}(x) = 0$, we see that $n' \mid x$, and as every term of S is divisible by d, we see that $d \mid x$. As d is coprime with n', it follows that x is divisible by $n = n'd$, and so $x = 0$. Thus, we see that S is an A-weighted zero-sum sequence. □

Griffiths [6, Lem. 2.1] proved the next result, which we restate here using our terminology.

Lemma 16. *Let p be an odd prime. If a sequence S in $\mathbb{Z}_{p^r}$ has at least two terms coprime to p, then S is a $U(p^r)$-weighted zero-sum sequence.*

Chintamani and Moriya [5, Lem. 1] proved the next result.

Lemma 17. *Let $A = U(n)^2$ where $n = p^r$ and p is a prime which is at least seven. Suppose we have elements $x_1, x_2, x_3 \in U(n)$. Then we get that $Ax_1 + Ax_2 + Ax_3 = \mathbb{Z}_n$.*

We will use the next result in Lemma 22.

Lemma 18. *Let $n = p^r$ where p is a prime which is at least seven. Let $A_1 = U(n)^2$ and $A_2 = U(n) \setminus U(n)^2$. Suppose $x_1, x_2, x_3 \in U(n)$ and $f : \{1, 2, 3\} \to \{1, 2\}$ is a function. Then $A_{f(1)}x_1 + A_{f(2)}x_2 + A_{f(3)}x_3 = \mathbb{Z}_n$.*

Proof. From [9, Thm. 2, p. 43] we see that when n is a power of an odd prime, the group $U(n)$ is cyclic. So it follows that -1 is the unique element in $U(n)$ of order 2. Thus, the map $U(n) \to U(n)$ given by $x \mapsto x^2$ has kernel $\{1, -1\}$. Hence, the image of this map is a subgroup of $U(n)$ having index 2 and so there exists $c \in U(n)$ such that $A_2 = cA_1$.

For every $i \in [1, 3]$ let

$$y_i = \begin{cases} x_i, & \text{if } f(i) = 1; \\ cx_i, & \text{if } f(i) = 2. \end{cases}$$

Let $x \in \mathbb{Z}_n$. By Lemma 17 there exist $b_1, b_2, b_3 \in U(n)^2$ with $x = b_1y_1 + b_2y_2 + b_3y_3$.

For every $i \in [1, 3]$ let

$$a_i = \begin{cases} b_i, & \text{if } f(i) = 1; \\ b_ic, & \text{if } f(i) = 2. \end{cases}$$

For every $i \in [1, 3]$ it follows that $a_i \in A_{f(i)}$ and $b_iy_i = a_ix_i$. Thus, we see that $x = a_1x_1 + a_2x_2 + a_3x_3$. □

The next result follows immediately from Lemma 18.

Corollary 19. *Let $n = p^r$ where p is a prime which is at least seven. Suppose S is a sequence in $\mathbb{Z}_n$ such that at least three terms of S are in $U(n)$. Then S is a $U(n)^2$-weighted zero-sum sequence.*

Remark 20. The conclusion of Corollary 19 may not hold when $p \leq 5$. One can check that the sequence $(1, 1, 1)$ in $\mathbb{Z}_n$ is not a $U(n)^2$-weighted zero-sum sequence when $n = 2, 5$. Also, the sequence $(1, 2, 1)$ in $\mathbb{Z}_3$ is not a $U(3)^2$-weighted zero-sum sequence.

3 The constants $D_{S(n)}$ and $C_{S(n)}$

Lemma 21. *Let n be squarefree and $S = (x_1, \dots, x_l)$ be a sequence in $\mathbb{Z}_n$. Suppose for every prime divisor p of n, at least two terms of S are coprime to p. If at most one term of S is a unit, then S is an $S(n)$-weighted zero-sum sequence.*

Proof. As we have assumed that n is odd and for every prime divisor p of n at least two terms of S are coprime to p, by Lemma 16 we see that for every prime divisor p of n the sequence $S^{(p)} = (x_1^{(p)}, \dots, x_j^{(p)}, \dots, x_l^{(p)})$ is a $U(p)$-weighted zero-sum sequence. Let $n = p_1 \cdots p_i \cdots p_k$ where the p_i's are distinct primes. For every $i \in [1, k]$ there exist $c_{i,1}, \dots, c_{i,j}, \dots, c_{i,l} \in U(p_i)$ such that $c_{i,1}x_1^{(p_i)} + \cdots + c_{i,j}x_j^{(p_i)} + \cdots + c_{i,l}x_l^{(p_i)} = 0$. We will refer to this $U(p_i)$-weighted zero-sum in $\mathbb{Z}_{p_i}$ as the i^{th} sum.

By Observation 7 we see that for every $j \in [1, l]$ there exists $a_j \in U(n)$ such that

$$a_1x_1 + \cdots + a_jx_j + \cdots + a_lx_l = 0, \tag{1}$$

and for every $i \in [1, k]$ we have $(a_1^{(p_i)}, \dots, a_j^{(p_i)}, \dots, a_l^{(p_i)}) = (c_{i,1}, \dots, c_{i,j}, \dots, c_{i,l})$. We observe that for some $i \in [1, k]$, a different choice for the i^{th} sum will give us a different l-tuple $(a_1, \dots, a_l)$ in (1). For example, if for some $i \in [1, k]$ there exists $j \in [1, l]$ such that $x_j^{(p_i)}$ is zero, we can make an arbitrary choice for $c_{i,j}$ in the i^{th} sum. For every $i \in [1, k]$ we want to choose the i^{th} sum so that all the a_j's in (1) are in $S(n)$. Consider the following matrices:

$$C = \begin{pmatrix} c_{1,1} & \cdots & c_{1,j} & \cdots & c_{1,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i,1} & \cdots & c_{i,j} & \cdots & c_{i,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{k,1} & \cdots & c_{k,j} & \cdots & c_{k,l} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_1^{(p_1)} & \cdots & x_j^{(p_1)} & \cdots & x_l^{(p_1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{(p_i)} & \cdots & x_j^{(p_i)} & \cdots & x_l^{(p_i)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{(p_k)} & \cdots & x_j^{(p_k)} & \cdots & x_l^{(p_k)} \end{pmatrix}.$$

Suppose some entry $x_j^{(p_i)}$ of X is 0. From Proposition 11 and Observation 13 we see that by making a suitable choice for $c_{i,j}$ we can ensure that in (1) we have $a_j \in S(n)$. Thus, if the j^{th} column of X has a zero, we can get a $U(n)$-weighted zero-sum (1) in which $a_j \in S(n)$.

We observe that a term x_j of S is a unit if and only if the j^{th} column of X does not have a zero. Hence, if no term of S is a unit, then every column of X has a zero. So in this case S is an $S(n)$-weighted zero-sum sequence.

Suppose exactly one term of S is a unit, say x_{j_0}. Then the j_0^{th} column of X does not have a zero and there is a zero in all the other columns of X. By multiplying the 1^{st} row of C by a suitable element of $U(p_1)$, we can modify c_{1,j_0} so that $a_{j_0} \in S(n)$. As the other columns of X have a zero, we can modify those columns of C suitably so that $a_j \in S(n)$ for $j \neq j_0$. Thus, it follows that S is an $S(n)$-weighted zero-sum sequence. □

Lemma 22. *Let n be a squarefree integer with every prime divisor of n at least seven. Let $S = (x_1, \dots, x_l)$ be a sequence in $\mathbb{Z}_n$ such that, for every prime divisor of n, at least two* *terms of S are coprime to it. Suppose there is a prime divisor p of n such that at least three terms of S are coprime to p. Then S is an $S(n)$-weighted zero-sum sequence.*

Proof. If $\Omega(n) = 1$, then n is a prime say p. As at least three terms of S are coprime to p, Corollary 19 implies S is a Q_p-weighted zero-sum sequence with $Q_p = S(p)$.

Suppose $\Omega(n) \geq 2$. Let $n = p_1 \cdots p_k$ where the p_i's are distinct primes. By Lemma 16 for every $i \in [1, k]$ there exist $c_{i,1}, \ldots, c_{i,l} \in U(p_i)$ such that $c_{i,1}x_1^{(p_i)} + \cdots + c_{i,l}x_l^{(p_i)} = 0$. By Observation 7 there exist $a_1, \ldots, a_l \in U(n)$ such that

$$a_1x_1 + \cdots + a_lx_l = 0. \tag{2}$$

Assume that $p = p_1$ and that $x_1^{(p)}$, $x_2^{(p)}$, and $x_3^{(p)}$ are units. A similar argument will work in the general case. Let us denote $c_{1,1}, \ldots, c_{1,l} \in U(p_1)$ by $b_1, \ldots, b_l$. We want to choose the b_i's in $U(p)$ so that the corresponding a_i's in (2) are in $S(n)$.

Using Observation 13 we can choose $b_4, \ldots, b_l \in U(p)$ so that $a_4, \ldots, a_l \in S(n)$. Let $y = -(b_4x_4^{(p)} + \cdots + b_lx_l^{(p)})$. By using Observation 13 and Lemma 18 we can choose $b_1, b_2, b_3 \in U(p)$ so that $a_1, a_2, a_3 \in S(n)$ and $b_1x_1^{(p)} + b_2x_2^{(p)} + b_3x_3^{(p)} = y$. Thus, S is an $S(n)$-weighted zero-sum sequence. □

Theorem 23. *Let n be squarefree. If n is prime we have $D_{S(n)} = 3$. If n is not a prime and every prime divisor of n is at least seven, we have $D_{S(n)} = \Omega(n) + 1$.*

Proof. From Theorem 4 we have $D_{U(n)} = \Omega(n) + 1$. As $S(n) \subseteq U(n)$ it follows that $D_{S(n)} \geq D_{U(n)}$ and so $D_{S(n)} \geq \Omega(n) + 1$. If $\Omega(n) = 1$, then n is a prime and $S(n) = Q_n$. So by Theorem 5, we have $D_{S(n)} = 3$.

Suppose $\Omega(n) \geq 2$. We claim that $D_{S(n)} \leq \Omega(n) + 1$. Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ of length $l = k + 1$ where $k = \Omega(n)$. We have to show that S has an $S(n)$-weighted zero-sum subsequence. If at least one term of S is zero, then that term will give us an $S(n)$-weighted zero-sum subsequence of length 1.

Case 1: There is a prime divisor p of n such that at most one term of S is coprime to p.

Let us assume without loss of generality that x_i is divisible by p for every $i \in [2, l]$. Let T denote the subsequence $(x_2, \ldots, x_l)$ of S. Let $n' = n/p$ and let T' be the sequence in $\mathbb{Z}_{n'}$ which is the image of T under $f_{n,n'}$. From Theorem 4, we see that $D_{U(n')} = \Omega(n') + 1$. As T' has length $l - 1 = \Omega(n) = \Omega(n') + 1$, it follows that T' has a $U(n')$-weighted zero-sum subsequence. As n is squarefree, p is coprime to n'. Thus, by Lemmas 14 and 15 we see that S has an $S(n)$-weighted zero-sum subsequence.

Case 2: For every prime divisor p of n, exactly two terms of S are coprime to p.

Suppose S has at most one unit. By Lemma 21, we see that S is an $S(n)$-weighted zero-sum sequence. So we can assume that S has at least two units. By the assumption in this subcase, we see that S will have exactly two units and the other terms of S will be zero. As S has length $k + 1$ and as $k \geq 2$, some term of S is zero.

Case 3: For every prime divisor p of n at least two terms of S are coprime to p, and there is a prime divisor p' of n such that at least three terms of S are coprime to p'.

In this case, we are done by Lemma 22. □

Theorem 24. *Let n be squarefree. If n is a prime, then $C_{S(n)} = 3$. If n is not a prime and every prime divisor of n is at least seven, then $C_{S(n)} = 2^{\Omega(n)}$.*

Proof. If $n = p$ where p is a prime, then $S(n) = Q_p$. As p is odd, from Theorem 5 we get that $C_{S(n)} = 3$. Let $n = p_1 \cdots p_k$ where $k \geq 2$. As $S(n) \subseteq U(n)$, it follows that $C_{S(n)} \geq C_{U(n)}$. As n is odd, from Theorem 4 we have $C_{S(n)} \geq 2^k$.

Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ of length $l = 2^k$. If we show that S has an $S(n)$-weighted zero-sum subsequence of consecutive terms, it will follow that $C_{S(n)} \leq 2^k$. If at least one term of S is zero, we get an $S(n)$-weighted zero-sum subsequence of S of length 1.

Case 1: There is a prime divisor p of n such that at most one term of S is coprime to p.

We will get a subsequence T of consecutive terms of S of length $l/2$ with all its terms divisible by p. Let $n' = n/p$ and let T' be the image of T under $f_{n,n'}$. From Theorem 4, we have $C_{U(n')} = 2^{\Omega(n')}$. As the length of T' is $2^{\Omega(n')}$, it follows that T' has a $U(n')$-weighted zero-sum subsequence of consecutive terms. As n' is coprime with p, by Lemmas 14 and 15 we get that T (and hence S) has an $S(n)$-weighted zero-sum subsequence of consecutive terms.

Case 2: For every prime divisor p of n exactly two terms of S are coprime to p.

In this case, as $\Omega(n) = k$, there are at most $2k$ non-zero terms in S. Suppose $k \geq 3$. As S has length 2^k and as $2^k > 2k$, some term of S is zero and we are done. Now assume that $k = 2$. Then S has length four. If S has at most one unit, by Lemma 21 this sequence S is an $S(n)$-weighted zero-sum sequence. So we can assume that S has at least two units. By the assumption in this subcase, we see that S has exactly two units and so the other two terms of S are zero.

Case 3: For every prime divisor p of n at least two terms of S are coprime to p, and there is a prime divisor p' of n such that at least three terms of S are coprime to p'.

In this case, we are done by Lemma 22. □

4 Some results about the weight-set $L(n; p)$

To determine the constant $D_{S(n)}$ for some non-squarefree n, we consider the following subset of $\mathbb{Z}_n$ as a weight-set.

Definition 25. Let p be a prime divisor of n where n is odd. We define

$$L(n;p) = \left\{\, a \in U(n)\ :\ \left(\frac{a}{n}\right) = \left(\frac{a}{p}\right) \,\right\}.$$

Consider the homomorphism $\varphi : U(n) \to \{1, -1\}$ given by $\varphi(a) = \left(\frac{a}{n}\right)\left(\frac{a}{p}\right)$. Then the kernel of φ is $L(n;p)$. It follows that $L(n;p)$ is a subgroup having an index at most two in $U(n)$.

Proposition 26. *Let p be a prime divisor of n. Then $L(n;p)$ has index two in $U(n)$ unless p is the unique prime divisor of n such that $v_p(n)$ is odd.*

Proof. Let $n = p^r m$ where m is coprime to p. Let $\psi : U(n) \to U(p^r) \times U(m)$ be the isomorphism that is given by the Chinese remainder theorem. If we show that -1 is in the image of the homomorphism $\varphi : U(n) \to \{1, -1\}$ which was defined above, then the kernel of φ will be a subgroup of index two in $U(n)$.

Case 1: r is odd.

Suppose m is a square. For every $a \in U(n)$, we have $\varphi(a) = \left(\frac{a}{m}\right)\left(\frac{a}{p^{r+1}}\right) = 1$. Thus, φ is the trivial map, and so $L(n;p) = U(n)$.

Suppose m is not a square. By Proposition 11 we see that $S(m)$ has index two in $U(m)$. For $c \in U(m) \setminus S(m)$, there exists $a \in U(n)$ such that $\psi(a) = (1, c)$. Thus $\left(\frac{a}{p}\right) = \left(\frac{1}{p}\right) = 1$ and so $\varphi(a) = \left(\frac{a}{n}\right) = \left(\frac{a}{m}\right) = \left(\frac{c}{m}\right) = -1$.

Case 2: r is even.

Suppose $m = 1$. Then $\left(\frac{a}{n}\right) = \left(\frac{a}{p}\right)^r = 1$ and so $\varphi(a) = \left(\frac{a}{p}\right)$. Let $b \in U(p) \setminus Q_p$. There exists $a \in U(n)$ such that $f_{n,p}(a) = b$. Thus $\varphi(a) = \left(\frac{b}{p}\right) = -1$.

Suppose $m > 1$. Let $b \in U(p) \setminus Q_p$. There exists $b' \in U(p^r)$ such that $f_{p^r,p}(b') = b$. For $c \in S(m)$, there exists $a \in U(n)$ such that $\psi(a) = (b', c)$. Thus $\left(\frac{a}{n}\right) = \left(\frac{b}{p}\right)^r\left(\frac{c}{m}\right) = 1$ and so $\varphi(a) = \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = -1$. □

Remark 27. In particular, if n is a prime p, then $L(n;p) = U(p)$.

The remaining results in this section are technical results, which will be used in the next section.

Lemma 28. *Let p and p' be prime divisors of n such that p is coprime with $n' = n/p$. Then $S(n') \subseteq f_{n,n'}\big(L(n;p')\big)$.*

Proof. Let $b \in S(n')$ where $n' = n/p$. As p is coprime with n', by the Chinese remainder theorem we have an isomorphism $\psi : U(n) \to U(n') \times U(p)$.

Suppose $p = p'$. Let $a \in U(n)$ be a unit such that $\psi(a) = (b, 1)$. Thus $f_{n,n'}(a) = b$. We have $a \in L(n;p')$ as

$$\left(\frac{a}{n}\right) = \left(\frac{b}{n'}\right)\left(\frac{1}{p}\right) = \left(\frac{1}{p}\right) = \left(\frac{a}{p}\right) = \left(\frac{a}{p'}\right).$$

Suppose $p \neq p'$. Then p' divides n'. Let $c \in U(p)$ be a unit such that $\left(\frac{c}{p}\right) = \left(\frac{b}{p'}\right)$. Let $a \in U(n)$ be a unit such that $\psi(a) = (b, c)$. Thus $f_{n,n'}(a) = b$. We have $a \in L(n; p')$ as

$$\left(\frac{a}{n}\right) = \left(\frac{b}{n'}\right)\left(\frac{c}{p}\right) = \left(\frac{c}{p}\right) = \left(\frac{b}{p'}\right) = \left(\frac{a}{p'}\right). \qquad \square$$

Lemma 29. *Let p' be a prime divisor of n which is coprime to $n' = n/p'$. Then we have that $U(p') \subseteq f_{n,p'}\big(L(n; p')\big)$.*

Proof. Let $b \in U(p')$. As $n' = n/p'$ is coprime to p', by the Chinese remainder theorem we have an isomorphism $\psi : U(n) \to U(n') \times U(p')$. There exists $a \in U(n)$ such that $\psi(a) = (1, b)$. Thus $f_{n,p'}(a) = b$. We have $a \in L(n; p')$ as

$$\left(\frac{a}{n}\right) = \left(\frac{1}{n'}\right)\left(\frac{b}{p'}\right) = \left(\frac{b}{p'}\right) = \left(\frac{a}{p'}\right). \qquad \square$$

The next result follows from a similar argument as in the proof of Observation 7.

Observation 30. Let $n = m_1 m_2$ where m_1 and m_2 are coprime. Let $A \subseteq \mathbb{Z}_n$ be a subset and let S be a sequence in $\mathbb{Z}_n$. For every $i \in [1, 2]$ let $A_i \subseteq U(m_i)$ be given and S_i denote the image of the sequence S under f_{n,m_i}. Suppose $A_1 \times A_2 \subseteq \psi(A)$ where $\psi : U(n) \to U(m_1) \times U(m_2)$ is the isomorphism given by the Chinese remainder theorem. If S_1 is an A_1-weighted zero-sum sequence in $\mathbb{Z}_{m_1}$ and S_2 is an A_2-weighted zero-sum sequence in $\mathbb{Z}_{m_2}$, then S is an A-weighted zero-sum sequence in $\mathbb{Z}_n$.

Lemma 31. *Let n be a squarefree integer and let $n' = n/p'$, where p' is a prime divisor of n. Suppose $\psi : U(n) \to U(n') \times U(p')$ is the isomorphism given by the Chinese remainder theorem. Then we have that $S(n') \times U(p') \subseteq \psi\big(L(n; p')\big)$.*

Proof. Let $(b, c) \in S(n') \times U(p')$. There exists $a \in U(n)$ such that $\psi(a) = (b, c)$. Then we see that $a \in L(n; p')$ as

$$\left(\frac{a}{n}\right) = \left(\frac{b}{n'}\right)\left(\frac{c}{p'}\right) = \left(\frac{c}{p'}\right) = \left(\frac{a}{p'}\right). \qquad \square$$

5 The constants $D_{L(n;\,p)}$ and $C_{L(n;\,p)}$

Lemma 32. *Let n be a squarefree integer with every prime divisor of n at least seven. Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ such that for every prime divisor p of n at least two terms of S are coprime to p. Assume that S' denotes the image of S under $f_{n,n'}$, where $n' = n/p'$ with p' a prime divisor of n. Suppose at most one term of S' is a unit, or there is a prime divisor p of n/p' such that at least three terms of S are coprime to p. Then S is an $L(n; p')$-weighted zero-sum sequence.*

Proof. Let $n' = n/p'$ and let S' denote the image of the sequence S under $f_{n,n'}$. As at least two terms of $S^{(p')}$ are coprime to p', Lemma 16 implies that $S^{(p')}$ is a $U(p')$-weighted zero-sum sequence.

If at most one term of S' is a unit, by Lemma 21 we see that S' is an $S(n')$-weighted zero-sum sequence in $\mathbb{Z}_{n'}$. This is because n' is squarefree and for every prime divisor p of n' at least two terms of S' are coprime to p.

If there is a prime divisor p of n/p' such that at least three terms of S are coprime to p, by Lemma 22 we see that S' is an $S(n')$-weighted zero-sum sequence since at least three terms of S' are coprime to p.

As n is squarefree, n' is coprime to p'. Let $\psi : U(n) \to U(n') \times U(p')$ be the isomorphism given by the Chinese remainder theorem. By Lemma 31 we see that $S(n') \times U(p') \subseteq \psi\big(L(n; p')\big)$. Hence, by Observation 30 we see that S is an $L(n; p')$-weighted zero-sum sequence. □

Theorem 33. *Let n be a squarefree number such that every prime divisor of n is at least seven and $\Omega(n) \neq 2$. Suppose p' is a prime divisor of n. Then $D_{L(n;p')} = \Omega(n) + 1$.*

Proof. Let p' be a prime divisor of n. We have $D_{U(n)} \leq D_{L(n;p')}$, as $L(n; p') \subseteq U(n)$. From Theorem 4 we have $D_{U(n)} = \Omega(n) + 1$ and so $D_{L(n;p')} \geq \Omega(n) + 1$. If $\Omega(n) = 1$, then $L(n; p') = U(n)$ and so by Theorem 4 we have $D_{L(n;p')} = 2$.

Let n be a squarefree number such that every prime divisor is at least seven and $\Omega(n) \geq 3$. Suppose $S = (x_1, \ldots, x_l)$ is a sequence in $\mathbb{Z}_n$ of length $\Omega(n) + 1$. It suffices to show that S has an $L(n; p')$-weighted zero-sum subsequence.

Case 1: There is a prime divisor p of n such that at most one term of S is coprime to p.

Let us assume without loss of generality that x_i is divisible by p for $i > 1$. Let T denote the subsequence $(x_2, \ldots, x_l)$ of S. Let $n' = n/p$ and let T' denote the sequence in $\mathbb{Z}_{n'}$ which is the image of T under $f_{n,n'}$. We see that n' is a squarefree number, which is not a prime, every prime divisor of n' is at least seven, and T' has length $\Omega(n') + 1$.

So it follows from Theorem 23 that T' has an $S(n')$-weighted zero-sum subsequence. As n is squarefree, it follows that p is coprime to n'. So by Lemmas 15 and 28 we see that T has an $L(n; p')$-weighted zero-sum subsequence.

Case 2: For every prime divisor p of n/p', there are exactly two terms of S which are coprime to p, and at least two terms of S are coprime to p'.

Let $n' = n/p'$ and let $S' = (x'_1, \ldots, x'_l)$ be the image of S under $f_{n,n'}$. Suppose at most one term of S' is a unit. By Lemma 32 we see that S is an $L(n; p')$-weighted zero-sum sequence. Suppose at least two terms of S' are units. Under the assumptions in this case, two terms x'_{j_1} and x'_{j_2} of S' are units, and the other terms of S' are zero. It follows that all the terms of S are divisible by n' except x_{j_1} and x_{j_2}.

Hence, if some term $f_{n,p'}(x_j)$ of $S^{(p')}$ is zero for $j \neq j_1, j_2$, then $x_j = 0$. So we can assume that all the terms of $S^{(p')}$ are non-zero except possibly two terms. As $\Omega(n) \geq 3$, the sequence

S has length at least four. Let T be a subsequence of S of length at least two which does not contain the terms x_{j_1} and x_{j_2}.

As all the terms of $T^{(p')}$ are non-zero and as $T^{(p')}$ has length at least 2, by Lemma 16 we see that $T^{(p')}$ is a $U(p')$-weighted zero-sum sequence. Also, all the terms of T are divisible by n'. Hence, by Lemmas 15 and 29 we see that T is an $L(n;p')$-weighted zero-sum subsequence of S.

Case 3: For every prime divisor p of n, there are at least two terms of S which are coprime to p, and there is a prime divisor p of n/p' such that at least three terms of S are coprime to p.

In this case, we are done by Lemma 32. □

Theorem 34. *Let $n = p'q$ where p' and q are distinct primes which are at least seven. Then $D_{L(n;p')} = 4$.*

Proof. Let n be as in the statement of the theorem. As $L(n;p') \subseteq U(n)$, we have that $f_{n,p'}\big(L(n;p')\big) \subseteq U(p')$. Also observe that $f_{n,q}\big(L(n;p')\big) \subseteq Q_q$. As from Theorem 4 we have $D_{U(p')} = 2$ and from Theorem 5 we have $D_{Q_q} = 3$, by Lemma 9 it follows that $D_{L(n;p')} \geq 4$.

Let $S = (x_1, x_2, x_3, x_4)$ be a sequence in $\mathbb{Z}_n$. We will show that S has an $L(n;p')$-weighted zero-sum subsequence. It will follow that $D_{L(n;p')} = 4$. If some term of S is zero, then we are done. So we can assume that all the terms of S are non-zero. We continue with the notation and terminology that were used in the proof of Theorem 33.

Case 1: There is a prime divisor p of n such that at most one term of S is coprime to p.

We can find a subsequence T of S of length three such that all the terms of T are divisible by p. Let $n' = n/p$ and let T' be the sequence in $\mathbb{Z}_{n'}$ which is the image of T under $f_{n,n'}$. As all the terms of S are non-zero, no term of T can be divisible by n'. So T' is a sequence of non-zero terms of length three. As n' is a prime, we have $S(n') = Q_{n'}$. By Corollary 19 we see that T' is a $Q_{n'}$-weighted zero-sum subsequence. Thus, by Lemmas 15 and 28 we see that T is an $L(n;p')$-weighted zero-sum subsequence of S.

Case 2: Exactly two terms of S are coprime to q.

Let us assume that x_1 and x_2 are coprime to q. If $T = (x_3, x_4)$, the sequence $T^{(q)}$ has both terms zero. Hence, we get that $T^{(q)}$ is an $S(q)$-weighted zero-sum sequence. As S has all terms non-zero, we see that both the terms of $T^{(p')}$ are non-zero. So by Lemma 16 we get that $T^{(p')}$ is a $U(p')$-weighted zero-sum sequence. Let $\psi : U(n) \to U(q) \times U(p')$ be the isomorphism given by the Chinese remainder theorem. By Lemma 31 we have $S(q) \times U(p') \subseteq \psi\big(L(n;p')\big)$. Thus, by Observation 30 we see that T is an $L(n;p')$-weighed zero-sum subsequence of S.

Case 3: At least three terms of S are coprime to q, and at least two terms of S are coprime to p'.

In this case, we are done by Lemma 32. □

Theorem 35. *Let n be a squarefree number such that every prime divisor of n is at least seven and $\Omega(n) \neq 2$. Suppose p' is a prime divisor of n. Then $C_{L(n;p')} = 2^{\Omega(n)}$.*

Proof. If n is a prime, then $n = p'$ and $L(n;p') = U(n)$. So from Theorem 4 we have $C_{L(n;p')} = 2$. Let $p' = p_k$ and $n = p_1 \cdots p_k$ where $k \geq 3$. As $L(n;p') \subseteq U(n)$, we have $C_{L(n;p')} \geq C_{U(n)}$. So from Theorem 4, we have $C_{L(n;p')} \geq 2^{\Omega(n)}$. Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ of length $l = 2^{\Omega(n)}$. If we show that S has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms, it will follow that $C_{L(n;p')} \leq 2^{\Omega(n)}$. If at least one term of S is zero, we get an $L(n;p')$-weighted zero-sum subsequence of S of length one.

Case 1: There is a prime divisor p of n such that at most one term of S is coprime to p.

We can find a subsequence T of consecutive terms of S of length $l/2$ such that all the terms of T are divisible by p. Let $n' = n/p$ and let T' be the image of T under $f_{n,n'}$. As $\Omega(n') = \Omega(n) - 1 \geq 2$ and T' has length $2^{\Omega(n')}$, by Theorem 24 we see that T' has an $S(n')$-weighted zero-sum subsequence of consecutive terms. By Lemma 28 we get $S(n') \subseteq f_{n,n'}\big(L(n;p')\big)$. So by Lemma 15 we see that T (and hence S) has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms.

Case 2: For every prime divisor p of n/p', there are exactly two terms of S which are coprime to p, and at least two terms of S are coprime to p'.

In this case, we can use a slight modification of the argument which was used in the same case of the proof of Theorem 33. We just observe that if S is a sequence of length at least eight such that at most two terms of S are not divisible by n', then we can find a subsequence T of consecutive terms of S having length at least two such that all the terms of T are divisible by n'.

Case 3: For every prime divisor p of n, there are at least two terms of S which are coprime to p, and there is a prime divisor p of n/p' such that at least three terms of S are coprime to p.

In this case, we are done by Lemma 32. □

Theorem 36. *Let $n = p'q$ where p' and q are distinct primes which are at least seven. Then $C_{L(n;p')} = 6$.*

Proof. Let n be as in the statement of the theorem. By Theorems 4 and 5 we see that $C_{U(p')} = 2$ and $C_{Q_q} = 3$. As $f_{n,p'}\big(L(n;p')\big) \subseteq U(p')$ and $f_{n,q}\big(L(n;p')\big) \subseteq Q_q$, by Lemma 8 it follows that $C_{L(n;p')} \geq 6$.

Let $S = (x_1, \ldots, x_6)$ be a sequence in $\mathbb{Z}_n$. It is enough to show that S has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms. We can assume that all the terms of S are non-zero.

Case 1: There is a prime divisor p of n such that at most one term of S is coprime to p.

In this case, we can find a subsequence T of S of consecutive terms of length three whose all terms are divisible by p. As all the terms of S are non-zero, all the terms of T are coprime to n' where $n' = n/p$. If T' is the image of T under $f_{n,n'}$, then T' is a sequence of non-zero terms of length three in $\mathbb{Z}_{n'}$. As n' is a prime, it follows that $S(n') = Q_{n'}$. By Corollary 19 we get that T' is a $Q_{n'}$-weighted zero-sum sequence. By using Lemmas 15 and 28 it follows that T is an $L(n; p')$-weighted zero-sum subsequence of S of consecutive terms.

Case 2: Exactly two terms of S are coprime to q.

Let the terms x_{j_1} and x_{j_2} be coprime to q. As S has length six, we can find a subsequence T of consecutive terms of S of length two, such that neither x_{j_1} nor x_{j_2} is a term of T. As x_j is divisible by q when $j \neq j_1, j_2$, all the terms of T are divisible by q. As S has all terms non-zero, all the terms of T are coprime to p'.

By Lemma 16 we get that $T^{(p')}$ is a $U(p')$-weighted zero-sum sequence. So by Lemmas 15 and 29 it follows that T is an $L(n; p')$-weighted zero-sum subsequence of consecutive terms of S.

Case 3: At least three terms of S are coprime to q, and at least two terms of S are coprime to p'.

In this case, we are done by Lemma 32. □

6 Concluding remarks

We have $S(15) = \{1, 2, 4, 8\}$. We can check that the sequence $S = (1, 1, 1)$ does not have a $S(15)$-weighted zero-sum subsequence. So it follows that $D_{S(15)} \geq 4$ and hence $D_{S(15)} \geq \Omega(15) + 2$. This shows that the statement of Theorem 23 is not true in general if some prime divisor of n is smaller than seven. It will be interesting to find the Davenport constant $D_{S(n)}$ for non-squarefree n.

Adhikari et al. [1] proposed to characterize when two weight-sets $A \subseteq \mathbb{Z}_n$ have the same value of D_A. In this paper, we have seen that if $A \subseteq \mathbb{Z}_n$ is such that $S(n) \subseteq A \subseteq U(n)$ and if n is not a prime, then $D_A = D_{U(n)}$. We have also seen that if $A \subseteq \mathbb{Z}_n$ is such that $L(n; p) \subseteq A \subseteq U(n)$ and if $\Omega(n) \neq 2$, then again $D_A = D_{U(n)}$. We can investigate whether there are other weight-sets $A \subseteq \mathbb{Z}_n$ such that $D_A = D_{U(n)}$. We can also ask similar questions regarding the constant C_A.

7 Acknowledgments

Santanu Mondal would like to acknowledge CSIR, Govt. of India for a research fellowship whose file number is 09/934(0013)/2019-EMR-I. We thank the referee for the corrections and suggestions.

References

[1] S. D. Adhikari, Y. G. Chen, J. B. Friedlander, S. V. Konyagin, and F. Pappalardi, Contributions to zero-sum problems, *Discrete Math.* **306** (2006), 1–10.

[2] S. D. Adhikari, C. David, and J. J. Urroz, Generalizations of some zero-sum theorems, *Integers* **8** (2008), #A52.

[3] S. D. Adhikari and S. Hegde, Zero-sum constants involving weights, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **137** (2021), #A37.

[4] S. D. Adhikari and P. Rath, Davenport constant with weights and some related questions, *Integers* **6** (2006), #A30.

[5] M. N. Chintamani and B. K. Moriya, Generalizations of some zero sum theorems, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **122** (2012), 15–21.

[6] S. Griffiths, The Erdős-Ginzberg-Ziv theorem with units, *Discrete Math.* **308** (2008), 5473–5484.

[7] D. J. Grynkiewicz and F. Hennecart, A weighted zero-sum problem with quadratic residues, *Uniform Dist. Theory* **10** (2015), 69–105.

[8] D. J. Grynkiewicz, L. E. Marchan, and O. Ordaz, A weighted generalization of two theorems of Gao, *Ramanujan J.* **28** (2012), 323–340.

[9] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory* (*Second Edition*), Springer, 1992.

[10] E. Mazumdar and S. B. Sinha, Modification of Griffiths' result for even integers, *Electron. J. Combin.* **23(4)** (2016), #P4.18.

[11] S. Mondal, K. Paul, and S. Paul, On a different weighted zero-sum constant, *Discrete Math.* **346** (2023), 113350.

[12] S. Mondal, K. Paul, and S. Paul, On unit-weighted zero-sum constants of $\mathbb{Z}_n$, accepted by *Integers*, arxiv preprint arXiv:2111.14477v3 [math.NT], 2023. Available at https://arxiv.org/abs/2111.14477.

[13] P. Yuan and X. Zeng, Davenport constant with weights, *European J. Combin.* **31** (2010), 677–680.

2020 *Mathematics Subject Classification*: Primary 11B50; Secondary 11B75.
Keywords: Davenport constant, zero-sum sequence, Jacobi symbol.

Received May 18 2023; revised versions received May 21 2023; November 29 2023. Published in *Journal of Integer Sequences*, January 14 2024.

Return to Journal of Integer Sequences home page.