Journal of Integer Sequences, Vol. 27 (2024),
Article 24.5.4

Symmetric Identities on Modified Degenerate Bernoulli Polynomials

Qi-Peng Su and Hao Pan
School of Applied Mathematics
Nanjing University of Finance and Economics
Nanjing, Jiangsu 210023
China
3093420060@qq.com
haopan79@zoho.com

Abstract

The modified degenerate Bernoulli polynomial $\mathfrak{B}_{n,\lambda}(x)$ introduced by Dolgy et al. is given by

$$\sum_{n=0}^{\infty} \frac{\mathfrak{B}_{n,\lambda}(x)}{n!} t^n = \frac{t(1+\lambda)^{\frac{xt}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}} - 1}.$$

In this paper, we prove some symmetric identities on modified degenerate Bernoulli polynomials, which generalize the result of Fu, Pan, and Zhang.

1 Introduction

The Bernoulli polynomial $B_n(x)$ ($n \in \mathbb{N}$) is given by

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{te^{xt}}{e^t - 1}.$$

In particular, $B_n := B_n(0)$ is called the n-th Bernoulli number. A curious identity of Miki [9] says that

$$\sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k(n-k)} = \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k(n-k)} + 2H_n \cdot \frac{B_n}{n} \tag{1}$$

for $n \geqslant 4$, where the n-th harmonic number $H_n := \sum_{i=1}^{n} 1/i$. Different proofs of (1) were given by Shirantani and Yokoyama [12], Gessel [6], Dunne and Schubert [4]. Furthermore, for $n \geqslant 4$, Dunne and Schubert also proved a similar identity

$$(n+2)\sum_{k=2}^{n-2} B_k B_{n-k} = 2\sum_{k=2}^{n-2}\binom{n+2}{k} B_k B_{n-k} + n(n+1)B_n, \tag{2}$$

which was conjectured by Matiyasevich [8].

In [10], Pan and Sun used the difference-differential method to establish the polynomial extensions of the Miki identity and the Matiyasevich-Dunne-Schubert identity. Moreover, Sun and Pan [14] proved the following general symmetric identity involving Bernoulli polynomials:

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0, \tag{3}$$

where $r+s+t=n$, $x+y+z=1$ and

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n = \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k} B_{n-k}(x)B_k(y).$$

Subsequently, Fu, Pan, and Zhang [5] gave a generalization of (3) involving sums of products of more Bernoulli polynomials as follows:

$$\begin{aligned} &r_{m+1}\sum_{\substack{k_1,\dots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\prod_{j=1}^{m}\binom{r_j}{k_j}B_{k_j}(x_j)\\ &= -\sum_{i=1}^{m} r_i \sum_{\substack{k_1,\dots,k_m\geq 0\\ k_1+\cdots+k_m=n}}\binom{r_{m+1}}{k_i}B_{k_i}(1-x_i)\prod_{\substack{1\leq j\leq m\\ j\neq i}}\binom{r_j}{k_j}B_{k_j}(x_j-x_i+\mathbf{1}_{j>i}), \end{aligned} \tag{4}$$

where $r_{m+1} = n - r_1 - \cdots - r_m$ and $\mathbf{1}_{j>i} = 1$ or 0 according to whether $j > i$ or not.

On the other hand, in [2], Dolgy, Kim, Kwon, and Seo introduced the modified degenerate Bernoulli polynomial, which is different from Carlitz's degenerate Bernoulli polynomial [1]. Define the modified degenerate Bernoulli polynomial $\mathfrak{B}_{n,\lambda}(x)$ by

$$\sum_{n=0}^{\infty}\frac{\mathfrak{B}_{n,\lambda}(x)}{n!}t^n = \frac{t(1+\lambda)^{\frac{xt}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1},$$

where $\lambda \in \mathbb{C}$. Moreover, set $\mathfrak{B}_{n,\lambda}(x) = 0$ if $n < 0$. Since $(1+\lambda)^{\frac{1}{\lambda}}$ tends to e as $\lambda \to 0$, clearly

$$\lim_{\lambda\to 0}\mathfrak{B}_{n,\lambda}(x) = B_n(x).$$

Various results related to the modified degenerate Bernoulli polynomials have been discussed in [3, 7, 11, 13].

In this short note, we extend (4) to the modified degenerate Bernoulli polynomials.

Theorem 1. *Suppose that $m \geq 2$ and $n \geq 1$ are integers. Let $r_1, \ldots, r_m, \lambda$ be arbitrary complex numbers and $r_0 = n - r_1 - \cdots - r_m$. We have*

$$\begin{aligned} & r_0 \sum_{\substack{k_1,\ldots,k_m \geq 0 \\ k_1+\cdots+k_m=n}} \prod_{j=1}^{m} \binom{r_j}{k_j} \mathfrak{B}_{k_j,\lambda}(x_j) \\ & = -\sum_{i=1}^{m} r_i \sum_{\substack{k_1,\ldots,k_m \geq 0 \\ k_1+\cdots+k_m=n}} \binom{r_0}{k_i} \mathfrak{B}_{k_i,\lambda}(1-x_i) \prod_{\substack{1\leq j\leq m \\ j\neq i}} \binom{r_j}{k_j} \mathfrak{B}_{k_j,\lambda}(x_j - x_i + \mathbf{1}_{j>i}). \end{aligned} \tag{5}$$

When $m = 2$, we have the following extension of (3).

Corollary 2. *For $s, t, \lambda \in \mathbb{C}$, let*

$$\mathfrak{D}_{s,t,\lambda}(x, y) = \sum_{k=0}^{n} (-1)^k \binom{s}{k} \binom{t}{n-k} \mathfrak{B}_{n-k,\lambda}(x) \mathfrak{B}_{k,\lambda}(y).$$

Then for each $n \geq 1$,

$$r \cdot \mathfrak{D}_{s,t,\lambda}(x, y) + s \cdot \mathfrak{D}_{t,r,\lambda}(y, z) + t \cdot \mathfrak{D}_{r,s,\lambda}(z, x) = 0, \tag{6}$$

where $r, s, t, x, y, z \in \mathbb{C}$ satisfy that $r + s + t = n$ and $x + y + z = 1$. In particular, we have the extensions of (1) and (2) as follows:

$$\sum_{k=1}^{n-1} \frac{\mathfrak{B}_{k,\lambda}}{k} \cdot \mathfrak{B}_{n-k,\lambda} - \sum_{k=1}^{n-1} \binom{n}{k} \frac{\mathfrak{B}_{k,\lambda}}{k} \cdot \mathfrak{B}_{n-k,\lambda} = \frac{\lambda}{\ln(1+\lambda)} \cdot H_n \mathfrak{B}_{n,\lambda} + \frac{n}{2} \cdot \mathfrak{B}_{n-1,\lambda}, \tag{7}$$

and

$$(n+2) \sum_{k=2}^{n-2} \mathfrak{B}_{k,\lambda} \mathfrak{B}_{n-k,\lambda} = 2 \sum_{k=2}^{n-2} \binom{n+2}{k} \mathfrak{B}_{k,\lambda} \mathfrak{B}_{n-k,\lambda} + \frac{\lambda}{\ln(1+\lambda)} \cdot n(n+1) \mathfrak{B}_{n,\lambda}, \tag{8}$$

where $\mathfrak{B}_{k,\lambda} = \mathfrak{B}_{k,\lambda}(0)$ and $n \geq 4$.

The proofs of Theorem 1 and Corollary 2 will be given in the next section.

2 Proofs of Theorem 1 and Corollary 2

First, we need the following lemma.

Lemma 3. *Suppose that $m \geq 2$ and $s_1, \ldots, s_m$ be non-negative integers. Then*

$$\sum_{i=1}^{m} s_i \mathfrak{B}_{s_i-1,\lambda}(x_i) \prod_{\substack{1\leq j\leq m\\ j\neq i}} \mathfrak{B}_{s_j,\lambda}(x_j)$$

$$= \sum_{i=1}^{m} s_i \sum_{\substack{k_1,k_2,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=s_1+\cdots+s_m}} \mathfrak{B}_{k_i-1,\lambda}(x_i) \prod_{\substack{1\leq j\leq m\\ j\neq i}} \binom{s_j}{k_j} \mathfrak{B}_{k_j,\lambda}(x_j - x_i + \mathbf{1}_{j>i}). \tag{9}$$

Proof. Clearly

$$(t_1 + \cdots + t_m) \prod_{j=1}^{m} \frac{t_j(\lambda+1)^{\frac{x_j t_j}{\lambda}}}{(\lambda+1)^{\frac{t_j}{\lambda}} - 1}$$

$$= \frac{(t_1 + \cdots + t_m)}{(\lambda+1)^{\frac{t_1+\cdots+t_m}{\lambda}} - 1} \left(\sum_{i=1}^{m} ((\lambda+1)^{\frac{t_i}{\lambda}} - 1)(\lambda+1)^{\sum_{i<j\leq m} \frac{t_j}{\lambda}} \right) \prod_{j=1}^{m} \frac{t_j(\lambda+1)^{\frac{x_j t_j}{\lambda}}}{(\lambda+1)^{\frac{t_j}{\lambda}} - 1}$$

$$= \sum_{i=1}^{m} \frac{(t_1 + \cdots + t_m)}{(\lambda+1)^{\frac{t_1+\cdots+t_m}{\lambda}} - 1} \cdot (\lambda+1)^{\sum_{i<j\leq m} \frac{t_j}{\lambda}} \cdot t_i(\lambda+1)^{\frac{x_i t_i}{\lambda}} \prod_{\substack{j=1\\ j\neq i}}^{m} \frac{t_j(\lambda+1)^{\frac{x_j t_j}{\lambda}}}{(\lambda+1)^{\frac{t_j}{\lambda}} - 1}$$

$$= \sum_{i=1}^{m} \frac{t_i(t_1 + \cdots + t_m)(\lambda+1)^{\frac{x_i(t_1+\cdots+t_m)}{\lambda}}}{(\lambda+1)^{\frac{t_1+\cdots+t_m}{\lambda}} - 1} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{t_j(\lambda+1)^{\frac{(x_j - x_i + \mathbf{1}_{j>i})t_j}{\lambda}}}{(\lambda+1)^{\frac{t_j}{\lambda}} - 1}. \tag{10}$$

Let $[t_1^{s_1} \cdots t_m^{s_m}] f(t_1, \ldots, t_m)$ denote the coefficient of $t_1^{s_1} \cdots t_m^{s_m}$ in the formal power series $f(t_1, \ldots, t_m)$. Then

$$[t_1^{s_1} \cdots t_m^{s_m}](t_1 + \cdots + t_m) \prod_{j=1}^{m} \frac{t_j(\lambda+1)^{\frac{x_j t_j}{\lambda}}}{(\lambda+1)^{\frac{t_j}{\lambda}} - 1}$$

$$= [t_1^{s_1} \cdots t_m^{s_m}](t_1 + \cdots + t_m) \sum_{k_j=0}^{\infty} \frac{\mathfrak{B}_{s_j,\lambda}(x)}{s_j!} t_j^{s_j}$$

$$= \sum_{i=1}^{m} \frac{\mathfrak{B}_{s_i-1,\lambda}(x_i, \lambda)}{(s_i - 1)!} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{\mathfrak{B}_{s_j,\lambda}(x_j)}{s_j!}. \tag{11}$$

On the other hand, we have

$$[t_1^{s_1} \cdots t_m^{s_m}] \sum_{i=1}^{m} \frac{t_i(t_1 + \cdots + t_m)(\lambda+1)^{\frac{x_i(t_1+\cdots+t_m)}{\lambda}}}{(\lambda+1)^{\frac{t_1+\cdots+t_m}{\lambda}} - 1} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{t_j(\lambda+1)^{\frac{t_j(x_j - x_i + \mathbf{1}_{j>i})}{\lambda}}}{(\lambda+1)^{\frac{t_j}{\lambda}} - 1} \tag{12}$$

$$= [t_1^{s_1}\cdots t_m^{s_m}]\sum_{i=1}^m t_i \sum_{k_1,\ldots,k_m\geq 0}^{\infty} \frac{\mathfrak{B}_{k_1+\cdots+k_m,\lambda}(x_i)}{k_1!k_2!\cdots k_m!} t_1^{k_1}\cdots t_m^{k_m} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \sum_{s_j=0}^{\infty} \frac{\mathfrak{B}_{s_j,\lambda}(x_j - x_i + \mathbf{1}_{j>i})}{s_j!} t_j^{s_j}$$

$$= [t_1^{s_1}\cdots t_m^{s_m}]\sum_{i=1}^m t_i \sum_{k_1,k_2,\ldots,k_m\geq 0}^{\infty} \frac{\mathfrak{B}_{k_1+\cdots+k_m,\lambda}(x_i)}{k_1!k_2!\cdots k_m!} t_1^{k_1} t_2^{k_2}\cdots t_m^{k_m}\cdot$$

$$\cdot \prod_{\substack{1\leq j\leq m\\ j\neq i}} \sum_{s_j-k_j=0}^{\infty} \frac{\mathfrak{B}_{s_j-k_j,\lambda}(x_j - x_i + \mathbf{1}_{j>i})}{(s_j-k_j)!} t_j^{s_j-k_j}$$

$$= \sum_{i=1}^m \sum_{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_m\geq 0} \frac{\mathfrak{B}_{k_1+\cdots+k_{i-1}+s_i-1+k_{i+1}+\cdots+k_m,\lambda}(x_i)}{k_1!\cdots k_{i-1}!(s_i-1)!k_{i+1}!\cdots k_m!} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{\mathfrak{B}_{s_j-k_j,\lambda}(x_j - x_i + \mathbf{1}_{j>i})}{(s_j-k_j)!}. \tag{13}$$

Combining (10), (11), and (12), we obtain that

$$\sum_{i=1}^m s_i \mathfrak{B}_{s_i-1,\lambda}(x_i) \prod_{\substack{1\leq j\leq m\\ j\neq i}} \mathfrak{B}_{s_j,\lambda}(x_j) = s_1!s_2!\cdots s_m! \sum_{i=1}^m \frac{\mathfrak{B}_{i-1,\lambda}(x_i)}{(s_i-1)!} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{\mathfrak{B}_{s_j,\lambda}(x_j)}{s_j!}$$

$$= s_1!\cdots s_m! \sum_{i=1}^m \sum_{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_m\geq 0} \frac{\mathfrak{B}_{k_1+\cdots+k_{i-1}+s_i-1+k_{i+1}+\cdots+k_m}(x_i, \lambda_1\lambda_2\cdots\lambda_m)}{k_1!k_2!\cdots k_{i-1}!(s_i-1)!k_{i+1}!\cdots k_m!}\cdot$$

$$\cdot \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{\mathfrak{B}_{s_j-k_j}(x_j - x_i + \mathbf{1}_{j>i}, \lambda_j)}{(s_j-k_j)!}$$

$$= \sum_{i=1}^m s_i \sum_{\substack{k_1,k_2,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=s_1+\cdots+s_m}} \mathfrak{B}_{k_i-1,\lambda}(x_i) \prod_{\substack{1\leq j\leq m\\ j\neq i}} \binom{s_j}{k_j} \mathfrak{B}_{k_j,\lambda}(x_j - x_i + \mathbf{1}_{j>i}).$$

Thus we get the desired result. □

Now we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. Clearly

$$(n - r_1 - \cdots - r_m) \sum_{\substack{k_1,k_2,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}} \prod_{j=1}^m \binom{r_j}{k_j} \mathfrak{B}_{k_j,\lambda}(x_j)$$

$$= \sum_{\substack{k_1,k_2,\ldots,k_m\geq 0\\ k_1+\cdots+k_m=n}} \sum_{i=1}^m (k_i - r_i)\binom{r_i}{k_i} \mathfrak{B}_{k_i,\lambda}(x_i) \prod_{\substack{1\leq j\leq m\\ j\neq i}} \binom{r_j}{k_j} \mathfrak{B}_{k_j,\lambda}(x_j)$$

$$= -\sum_{\substack{k_1,k_2,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n}} \sum_{i=1}^{m} (k_i+1)\binom{r_i}{k_i+1}\mathfrak{B}_{k_i,\lambda}(x_i) \prod_{\substack{1\le j\le m\\ j\ne i}} \binom{r_j}{k_j}\mathfrak{B}_{k_j,\lambda}(x_j)$$

$$= -\sum_{\substack{k_1,k_2,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}} \sum_{i=1}^{m} k_i\binom{r_i}{k_i}\mathfrak{B}_{k_i-1,\lambda}(x_i) \prod_{\substack{1\le j\le m\\ j\ne i}} \binom{r_j}{k_j}\mathfrak{B}_{k_j,\lambda}(x_j).$$

By Lemma 2.1, we have

$$\sum_{\substack{k_1,k_2,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}} \sum_{i=1}^{m} k_i\binom{r_i}{k_i}\mathfrak{B}_{k_i-1,\lambda}(x_i) \prod_{\substack{1\le j\le m\\ j\ne i}} \binom{r_j}{k_j}\mathfrak{B}_{k_j,\lambda}(x_j)$$

$$= \sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}} \prod_{j=1}^{m}\binom{r_j}{k_j}\cdot\Bigg(\sum_{i=1}^{m} k_i\mathfrak{B}_{k_i-1,\lambda}(x_i)\prod_{\substack{1\le j\le m\\ j\ne i}}\mathfrak{B}_{k_j,\lambda}(x_j)\Bigg)$$

$$= \sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}} \prod_{j=1}^{m}\binom{r_j}{k_j}\cdot\Bigg(\sum_{i=1}^{m} k_i\sum_{\substack{l_1,\dots,l_m\ge 0\\ l_1+\cdots+l_m=n}}\mathfrak{B}_{l_i,\lambda}(x_i)\prod_{\substack{1\le j\le m\\ j\ne i}}\binom{k_j}{l_j}\mathfrak{B}_{l_j,\lambda}(x_j-x_i+\mathbf{1}_{j>i})\Bigg)$$

$$= \sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}} \sum_{i=1}^{m} r_i\binom{r_i-1}{k_i-1}\sum_{\substack{l_1,\dots,l_m\ge 0\\ l_1+\cdots+l_m=n}}\mathfrak{B}_{l_i,\lambda}(x_i)\prod_{\substack{1\le j\le m\\ j\ne i}}\binom{r_j}{l_j}\binom{r_j-l_j}{k_j-l_j}\mathfrak{B}_{l_j,\lambda}(x_j-x_i+\mathbf{1}_{j>i})$$

$$= \sum_{i=1}^{m} r_i\sum_{\substack{l_1,\dots,l_m\ge 0\\ l_1+\cdots+l_m=n}}\mathfrak{B}_{l_i,\lambda}(x_i)\prod_{\substack{1\le j\le m\\ j\ne i}}\binom{r_j}{l_j}\mathfrak{B}_{l_j,\lambda}(x_j-x_i+\mathbf{1}_{j>i})\cdot$$

$$\cdot\Bigg(\sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}}\binom{r_i-1}{k_i-1}\prod_{\substack{1\le j\le m\\ j\ne i}}\binom{r_j-l_j}{k_j-l_j}\Bigg).$$

According to the Chu-Vandermonde identity,

$$\sum_{\substack{k_1,\dots,k_m\ge 0\\ k_1+\cdots+k_m=n+1}}\binom{r_i-1}{k_i-1}\prod_{\substack{1\le j\le m\\ j\ne i}}\binom{r_j-l_j}{k_j-l_j}$$

$$= \binom{r_1+\cdots+r_m-1-l_1-\cdots-l_{i-1}-l_{i+1}-\cdots-l_m}{k_1+\cdots+k_m-1-l_1-\cdots-l_{i-1}-l_{i+1}-\cdots-l_m}$$

$$= \binom{r_1+\cdots+r_m-1-n+l_i}{l_i} = (-1)^{l_i}\binom{n-r_1-\cdots-r_m}{l_i}.$$

Finally, since

$$\frac{t(1+\lambda)^{\frac{(1-x)t}{\lambda}}}{(1+\lambda)^{\frac{t}{\lambda}}-1} = \frac{(-t)(1+\lambda)^{\frac{x(-t)}{\lambda}}}{(1+\lambda)^{\frac{-t}{\lambda}}-1},$$

we have

$$\mathfrak{B}_{l,\lambda}(1-x) = (-1)^l \mathfrak{B}_{l,\lambda}(x) \tag{14}$$

for each $l \geq 0$. All are done. □

Proof of Corollary 2. Using (14) and applying to Theorem 1 with $m = 2$, $r_1 = s$, $r_2 = t$, $x_1 = 1 - y$ and $x_2 = x$, we have

$$\begin{aligned} r \cdot \mathfrak{D}_{s,t,\lambda}(x,y) &= r \sum_{k=0}^{n} \binom{s}{k}\binom{t}{n-k} \mathfrak{B}_{k,\lambda}(1-y)\mathfrak{B}_{n-k,\lambda}(x) \\ &= -s \sum_{k=0}^{n} \binom{r}{k}\binom{t}{n-k} \mathfrak{B}_{k,\lambda}(y)\mathfrak{B}_{n-k,\lambda}(y+x) \\ &\quad - t \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} \mathfrak{B}_{k,\lambda}(1-x)\mathfrak{B}_{n-k,\lambda}(1-x-y) \\ &= -s \cdot \mathfrak{D}_{t,r,\lambda}(y, 1-x-y) - t \cdot \mathfrak{D}_{r,s,\lambda}(1-x-y, x). \end{aligned}$$

Thus, Eq. (6) is concluded.

However, the proof of (7) is a little more complicated. Note that $\binom{-1}{k} = (-1)^k$ and

$$\frac{d}{ds}\binom{s}{k}\bigg|_{s=0} = \frac{(-1)^k}{k}.$$

Taking the derivative with respect to s on both sides of (6) and letting $s = 0$ and $t = -1$, we obtain that

$$\begin{aligned} 0 &= -(n+1)\sum_{k=1}^{n}(-1)^{n-k}\frac{\mathfrak{B}_{k,\lambda}(y)}{k} \cdot \mathfrak{B}_{n-k,\lambda}(x) - (-1)^n \mathfrak{B}_{n,\lambda}(x)\mathfrak{B}_{0,\lambda}(y) \\ &\quad + \sum_{k=0}^{n}\binom{n+1}{n-k}\mathfrak{B}_{n-k,\lambda}(y)\mathfrak{B}_{k,\lambda}(z) + (-1)^n \sum_{k=0}^{n-1}\binom{n+1}{k}\frac{\mathfrak{B}_{n-k,\lambda}(z)}{n-k} \cdot \mathfrak{B}_{k,\lambda}(x) \\ &\quad + (-1)^n\left(1 + \frac{n+1}{n} + \cdots + \frac{n+1}{2}\right)\mathfrak{B}_{0,\lambda}(z)\mathfrak{B}_{n,\lambda}(x). \end{aligned} \tag{15}$$

Note that $\mathfrak{B}_{n,\lambda}(1) = (-1)^n \mathfrak{B}_{n,\lambda}$ by (14). Substituting $x = 1$ and $y = z = 0$ in (15), we get

$$\begin{aligned} 0 &= -(n+1)\sum_{k=1}^{n}\frac{\mathfrak{B}_{k,\lambda}}{k} \cdot \mathfrak{B}_{n-k,\lambda} - \mathfrak{B}_{n,\lambda}\mathfrak{B}_{0,\lambda} \\ &\quad + \sum_{k=0}^{n}\binom{n+1}{n-k}\mathfrak{B}_{n-k,\lambda}\mathfrak{B}_{k,\lambda} + \sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+1}{k}\frac{\mathfrak{B}_{n-k,\lambda}}{n-k} \cdot \mathfrak{B}_{k,\lambda} \\ &\quad + \left(1 + \frac{n+1}{n} + \cdots + \frac{n+1}{2}\right)\mathfrak{B}_{0,\lambda}(z)\mathfrak{B}_{n,\lambda}. \end{aligned} \tag{16}$$

Since

$$t = \frac{t(1+\lambda)^{\frac{t}{\lambda}} - t}{(1+\lambda)^{\frac{t}{\lambda}} - 1} = \sum_{k=0}^{\infty} \frac{\mathfrak{B}_{k,\lambda}(1) - \mathfrak{B}_{k,\lambda}}{k!} t^k,$$

we have $\mathfrak{B}_{k,\lambda}(1) = \mathfrak{B}_{k,\lambda}$ for each $k \geq 2$. Thus by (14), $\mathfrak{B}_{k,\lambda} = 0$ if $k \geq 3$ is odd, i.e., $(-1)^k \mathfrak{B}_{k,\lambda} = 0$ for each $k \geq 2$. It follows from (16) that

$$\begin{aligned}
&(n+1)\sum_{k=1}^{n-1} \frac{\mathfrak{B}_{k,\lambda}}{k} \cdot \mathfrak{B}_{n-k,\lambda} + n(n+1)\mathfrak{B}_{1,\lambda}\mathfrak{B}_{n-1,\lambda} \\
&= (n+1)\sum_{k=1}^{n-1} \left(\binom{n}{k+1} + \frac{1}{n+1}\binom{n+1}{k} \right) \frac{\mathfrak{B}_{n-k,\lambda}}{n-k} \cdot \mathfrak{B}_{k,\lambda} + (n+1)H_n \mathfrak{B}_{n,\lambda}\mathfrak{B}_{0,\lambda} \\
&= (n+1)\sum_{k=1}^{n-1} \binom{n}{k} \frac{\mathfrak{B}_{k,\lambda}}{k} \cdot \mathfrak{B}_{n-k,\lambda} + (n+1)H_n \mathfrak{B}_{n,\lambda}\mathfrak{B}_{0,\lambda}. \qquad (17)
\end{aligned}$$

Clearly,

$$\mathfrak{B}_{0,\lambda} = \lim_{t\to 0} \frac{t}{(1+\lambda)^{\frac{t}{\lambda}} - 1} = \frac{\lambda}{\ln(1+\lambda)}, \qquad \mathfrak{B}_{1,\lambda} = \lim_{t\to 0} \frac{d}{dt}\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}} - 1} \right) = -\frac{1}{2}.$$

Thus (7) is concluded.

Finally, substituting $r = n+2$, $s = t = -1$, $x = 1$ and $y = z = 0$ in (6), we can easily get (8). □

3 Acknowledgment

The authors thank the anonymous referee for very helpful suggestions. The authors are supported by the National Natural Science Foundation of China (Grant No. 12071208).

References

[1] L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Util. Math.* **15** (1979), 51–88.

[2] D. V. Dolgy, T. Kim, H.-I. Kwon, and J. J. Seo, On the modified degenerate Bernoulli polynomials, *Adv. Stud. Contemp. Math., Kyungshang* **26** (2016), 1–9.

[3] D. V. Dolgy, T. Kim, and J. J. Seo, On the symmetric identities of modified degenerate Bernoulli polynomials, *Proc. Jangjeon Math. Soc.* **19** (2016), 301–308.

[4] G. V. Dunne and C. Schubert, Bernoulli number identities from quantum field theory and topological string theory, *Commun. Number Theory Phys.* **7** (2013), 225–249.

[5] Amy M. Fu, H. Pan, and I. F. Zhang, Symmetric identities on Bernoulli polynomials, *J. Number Theory* **129** (2009), 2696–2701.

[6] I. M. Gessel, On Miki's identity for Bernoulli numbers, *J. Number Theory* **110** (2005), 75–82.

[7] J. G. Lee and J. Kwon, The modified degenerate q-Bernoulli polynomials arising from p-adic invariant integral on $\mathbb{Z}_p$, *Adv. Difference Equ.* **2017**, Paper No. 29, 9 p.

[8] Y. Matiyasevich, Identities with Bernoulli numbers, 1997, `http://logic.pdmi.ras.ru/~yumat/Journal/Bernoulli/bernulli.htm`.

[9] H. Miki, A relation between Bernoulli numbers, *J. Number Theory* **10** (1978), 297–302.

[10] H. Pan and Z.-W. Sun, New identities involving Bernoulli and Euler polynomials, *J. Combin. Theory Ser. A* **113** (2006), 156–175.

[11] J.-W. Park, B. M. Kim, and J. Kwon, On a modified degenerate Daehee polynomials and numbers, *J. Nonlinear Sci. Appl.* **10** (2017), 1108–1115.

[12] K. Shirantani and S. Yokoyama, An application of p-adic convolutions, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **36** (1982), 73–83.

[13] H. M. Srivastava, B. Kurt, and V. Kurt, Identities and relations involving the modified degenerate Hermite-based Apostol-Bernoulli and Apostol-Euler polynomials, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM* **113** (2019), 1299–1313.

[14] Z.-W. Sun and H. Pan, Identities concerning Bernoulli and Euler polynomials, *Acta Arith.* **125** (2006), 21–39.

2020 *Mathematics Subject Classification*: Primary 11B68; Secondary 05A10, 11B65.
Keywords: modified degenerate Bernoulli polynomial, symmetric identity, Miki's identity, Matiyasevich's identity.

(Concerned with sequences A027641 and A027642.)

Received August 27 2023; revised version received April 11 2024; April 24 2024. Published in *Journal of Integer Sequences*, May 12 2024.

Return to Journal of Integer Sequences home page.