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The Fourth Positive Element in the Greedy B_h-Set

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Abstract

For a positive integer h, a *B_h-set* is a set of integers A such that every integer n has at most one representation in the form $n = x_1 + \cdots + x_h$, where $x_r \in A$ for all $r = 1, \ldots, h$ and $x_1 \leq \cdots \leq x_h$. The *greedy B_h-set* is the infinite set of nonnegative integers $\{a_0(h), a_1(h), a_2(h), \ldots\}$ constructed as follows: $a_0(h) = 0$, and $a_{k+1}(h)$ is least integer greater than $a_k(h)$ for which $\{a_0(h), a_1(h), a_2(h), \ldots, a_k(h), a_{k+1}(h)\}$ is a B_h-set. Nathanson gave the formulas $a_1(h) = 1$, $a_2(h) = h+1$, and $a_3(h) = h^2+h+1$, valid for all h. This paper proves that $a_4(h)$, the fourth positive term of the greedy B_h-set, is $\left(h^3 + 3h^2 + 3h + 1\right)/2$ if h is odd and $\left(h^3 + 2h^2 + 3h + 2\right)/2$ if h is even. In particular, $a_4(h)$ is not a polynomial, but is a quasipolynomial.

1 The greedy algorithm

Let h be a positive integer. A finite or infinite set A of nonnegative integers is a *B_h-set* if no integer has two different representations as sums of h elements of A. Equivalently, the set A is a B_h-set if the equation

$$x_1 + \cdots + x_h = y_1 + \cdots \leq y_h$$

with $x_i, y_i \in A$ for all $r = 1, \ldots, h$ and

$$x_1 \leq \cdots \leq x_h \qquad \text{and} \qquad y_1 \leq \cdots y_h$$

implies that $x_i = y_i$ for all $i = 1, \ldots, h$. A B_2-set is also called a *Sidon set.*

For every positive integer h, we use a greedy algorithm to construct a B_h-set $\{a_k(h) : k = 0, 1, 2, \ldots\}$ as follows: $a_0(h) = 0$ and, if $\{a_0(h), a_1(h), \ldots, a_k(h)\}$ is a B_h-set, then $a_{k+1}(h)$ is the least positive integer such that $\{a_0(h), a_1(h), \ldots, a_k(h), a_{k+1}(h)\}$ is a B_h-set. The greedy B_1-set is simply the set $\mathbb{N}_0$ of nonnegative integers, that is, $a_k(1) = k$ for all $k \in \mathbb{N}_0$. The greedy B_2-set is the (shifted) Mian-Chowla sequence (Mian-Chowla [2], Guy [1, Section E28]). In the OEIS [4], the greedy B_h-set for $1 \leq h \leq 9$ are sequences (in order, with various offsets and shifts) A001477, A005282, A051912, A365300, A365301, A365302, A365303, A365304, A365305. The first, second, third, fourth, fifth and sixth positive terms of the greedy B_h-set are A000012, A020725, A002061 (offset), A369817, A369818, A369819, respectively. These are rows and columns of the table sequences A365515 and (shifted by 1) A347570.

For all $h \geq 1$ we have

$$a_0(h) = 0, \quad a_1(h) = 1, \quad \text{and} \quad a_2(h) = h + 1.$$

From computer calculations of initial segments of B_h-sets for small h, the second author conjectured on September 30 2023 that, for all positive integers h, we have

$$a_3(h) = h^2 + h + 1.$$

This formula for $a_3(h)$ was later proved by the first author [3]. In this paper we obtain an explicit quasi-polynomial for $a_4(h)$.

Theorem 1. *The fourth positive element in the greedy B_h-set is*

$$a_4(h) = \begin{cases} \frac{1}{2}\left(h^3 + 3h^2 + 3h + 1\right), & \text{if } h \text{ is odd;} \\ \frac{1}{2}\left(h^3 + 2h^2 + 3h + 2\right), & \text{if } h \text{ is even.} \end{cases}$$

Using the floor function, we may also write

$$a_4(h) = \left\lfloor \frac{h+3}{2} \right\rfloor h^2 + \left\lfloor \frac{3h}{2} \right\rfloor + 1.$$

It is an open problem to compute exact values or even asymptotic estimates for $a_k(h)$ for $k \geq 5$. Indeed, it is not even known if $a_k(h) < a_k(h+1)$ for all integers $h \geq 1$ and $k \geq 2$. We have the upper bound [3]

$$a_k(h) \leq \sum_{i=0}^{k-1} h^i < h^{k-1} + 2h^{k-2}$$

for all positive integers h and k.

2 Lower bound for $a_4(h)$

In this section we compute a lower bound for $a_4(h)$.

For $u, v \in \mathbb{R}$, define the interval of integers $[u, v] = \{n \in \mathbb{Z} : u \leq n \leq v\}$.

Lemma 2. *For all integers $h \geq 2$ we have*

$$a_4(h) \geq \begin{cases} \frac{1}{2}\left(h^3 + 3h^2 + 3h + 1\right), & \text{if } h \text{ is odd;} \\ \frac{1}{2}\left(h^3 + 2h^2 + 3h + 2\right), & \text{if } h \text{ is even.} \end{cases}$$

Proof. Let $\mathcal{B}$ be the set of positive integers b such that $a_4(h) \neq b$. A sufficient condition that $b \in \mathcal{B}$ is the existence of nonnegative integers x_1, x_2, y_1, y_2, y_3 with

$$x_1 + x_2 \leq h - 1, \qquad y_1 + y_2 + y_3 \leq h, \qquad x_1 y_1 = x_2 y_2 = 0$$

such that

$$b + x_1 + x_2(h+1) = y_1 + y_2(h+1) + y_3(h^2 + h + 1)$$

or, equivalently,

$$b = y_3(h^2 + h + 1) + (y_2 - x_2)(h+1) + (y_1 - x_1).$$

We consider the following two cases separately: $x_2 = 0$ and $y_2 = 0$.

Let $x_2 = 0$ and let $1 \leq y_3 \leq h$ and $0 \leq y_2 \leq h - y_3$. Because

$$0 \leq x_1 \leq h - 1 \qquad \text{and} \qquad 0 \leq y_1 \leq h - y_3 - y_2,$$

the set $\mathcal{B}$ contains the interval

$$\begin{aligned} y_3(h^2{+}h + 1) &+ y_2(h+1) + [-(h-1), h - y_3 - y_2] \\ &= y_3(h^2 + h + 1) + [y_2(h+1) - h + 1, y_2 h + h - y_3]. \end{aligned} \tag{1}$$

If

$$0 \leq y_2 \leq h - y_3 - 1 \tag{2}$$

then $\mathcal{B}$ also contains the interval

$$\begin{aligned} &y_3(h^2{+}h+1) + [(y_2+1)(h+1) - h + 1, (y_2+1)h + h - y_3] \\ &\quad = y_3(h^2+h+1) + [y_2(h+1) + 2, y_2 h + 2h - y_3]\,. \end{aligned} \tag{3}$$

Inequality (2) implies that

$$y_2(h+1) + 2 \le y_2 h + h - y_3 + 1,$$

and so intervals (1) and (3) overlap. Therefore, $\mathcal{B}$ contains the interval

$$\begin{aligned} &y_3(h^2+h+1) + \bigcup_{y_2=0}^{h-y_3} [y_2(h+1) - h + 1, y_2 h + h - y_3] \\ &= y_3(h^2+h+1) + [-h+1, (h-y_3)h + h - y_3] \\ &= \left[y_3(h^2+h+1) - h + 1, y_3 h^2 + h^2 + h\right]. \end{aligned} \tag{4}$$

Let $y_2 = 0$ and let $1 \le y_3 \le h$ and $0 \le x_2 \le h - y_3$. Note that $h - y_3 \le h - 1$. Because

$$0 \le x_1 \le h - 1 - x_2 \qquad \text{and} \qquad 0 \le y_1 \le h - y_3,$$

the set $\mathcal{B}$ contains the interval

$$\begin{aligned} &y_3(h^2{+}h+1) - x_2(h+1) + [-(h-1-x_2), h - y_3] \\ &\quad = y_3(h^2+h+1) + [-x_2 h - h + 1, -x_2(h+1) + h - y_3]\,. \end{aligned} \tag{5}$$

If

$$0 \le x_2 \le h - y_3 - 1 \tag{6}$$

then $\mathcal{B}$ also contains the interval

$$\begin{aligned} &y_3(h^2{+}h+1) + [-(x_2+1)h - h + 1, -(x_2+1)(h+1) + h - y_3] \\ &\quad = y_3(h^2+h+1) + [-x_2 h - 2h + 1, -x_2(h+1) - y_3 - 1]\,. \end{aligned} \tag{7}$$

Inequality (6) implies that

$$-x_2 h - h \le -x_2(h+1) - y_3 - 1,$$

and so intervals (5) and (7) overlap. Therefore, $\mathcal{B}$ contains the interval

$$\begin{aligned} &y_3(h^2{+}h+1) + \bigcup_{x_2=0}^{h-y_3} [-x_2 h - h + 1, -x_2(h+1) + h - y_3] \\ &= y_3(h^2+h+1) + [-(h-y_3)h - h + 1, h - y_3] \\ &= \left[y_3(h^2+2h+1) - h^2 - h + 1, y_3(h^2+h) + h\right]. \end{aligned} \tag{8}$$

Intervals (4) and (8) overlap because $y_3 \leq h \leq 2h$, and so $\mathcal{B}$ contains the interval

$$I(y_3) = \left[y_3(h^2+2h+1) - h^2 - h + 1, y_3h^2 + h^2 + h\right] \tag{9}$$

for $y_3 \in [1, h]$. These intervals move to the right as y_3 increases. Intervals $(I(y_3)$ and $I(y_3+1)$ overlap if

$$(y_3+1)(h^2+2h+1) - h^2 - h + 1 \leq y_3h^2 + h^2 + h + 1$$

or, equivalently, if $(2h+1)y_3 \leq h^2 - 1$ or

$$y_3 \leq \left\lfloor \frac{h^2-1}{2h+1} \right\rfloor = \begin{cases} \frac{h-1}{2}, & \text{if } h \text{ is odd;} \\ \frac{h-2}{2}, & \text{if } h \text{ is even.} \end{cases}$$

For odd $h \geq 3$, the set $\mathcal{B}$ contains the interval

$$\begin{aligned} \bigcup_{y_3=1}^{(h+1)/2} I(y_3) &= \bigcup_{y_3=1}^{(h+1)/2} \left[y_3(h^2+2h+1) - h^2 - h + 1, y_3h^2 + h^2 + h\right] \\ &= \left[h+2, \left(\frac{h+1}{2}\right)h^2 + h^2 + h\right] \\ &= \left[h+2, \frac{h^3+3h^2+2h}{2}\right]. \end{aligned} \tag{10}$$

For even $h \geq 2$, the set $\mathcal{B}$ contains the interval

$$\begin{aligned} \bigcup_{y_3=1}^{h/2} I(y_3) &= \left[y_3(h^2+2h+1) - h^2 - h + 1, y_3h^2 + h^2 + h\right] \\ &= \left[h+2, \left(\frac{h}{2}\right)h^2 + h^2 + h\right] \\ &= \left[h+2, \frac{h^3+2h^2+2h}{2}\right]. \end{aligned} \tag{11}$$

For odd $h \geq 3$, let

$$y_2 = 0, \qquad y_3 = \frac{h+3}{2}, \qquad x_2 = \frac{h+1}{2}.$$

The set $\mathcal{B}$ contains the interval

$$\begin{aligned} & y_3(h^2+h+1) - x_2(h+1) + [-(h-1-x_2), h - y_3] \\ &= \left(\frac{h+3}{2}\right)(h^2+h+1) - \left(\frac{h+1}{2}\right)(h+1) + \left[-\frac{h-3}{2}, \frac{h-3}{2}\right] \\ &= \left[\frac{h^3+3h^2+h+5}{2}, \frac{h^3+3h^2+3h-1}{2}\right]. \end{aligned} \tag{12}$$

Intervals (10) and (12) overlap and so $\mathcal{B}$ contains the interval

$$\left[h+2, \frac{h^3+3h^2+3h-1}{2}\right].$$

It follows that, for odd $h \geq 3$, we have the lower bound

$$a_4(h) \geq \frac{h^3+3h^2+3h-1}{2} + 1 = \frac{h^3+3h^2+3h+1}{2}.$$

For even $h \geq 2$, let

$$y_2 = 0, \qquad y_3 = \frac{h+2}{2}, \qquad x_2 = \frac{h}{2}.$$

The set $\mathcal{B}$ contains the interval

$$\begin{aligned} &y_3(h^2+h+1) - x_2(h+1) + [-(h-1-x_2), h-y_3] \\ &= \left(\frac{h+2}{2}\right)(h^2+h+1) - \left(\frac{h}{2}\right)(h+1) + \left[-\frac{h-2}{2}, \frac{h-2}{2}\right] \\ &= \left[\frac{h^3+2h^2+h+4}{2}, \frac{h^3+2h^2+3h}{2}\right]. \end{aligned} \tag{13}$$

Intervals (11) and (13) overlap and so $\mathcal{B}$ contains the interval

$$\left[h+2, \frac{h^3+2h^2+3h}{2}\right].$$

It follows that, for even $h \geq 2$, we have the lower bound

$$a_4(h) \geq \frac{h^3+2h^2+3h}{2} + 1 = \frac{h^3+2h^2+3h+2}{2}.$$

This completes the proof of Lemma 2. □

3 The upper bound for $a_4(h)$

In this section we compute an upper bound for $a_4(h)$.

Lemma 3. *For all positive integers h we have*

$$a_4(h) \leq \begin{cases} \frac{1}{2}\left(h^3+3h^2+3h+1\right), & \text{if } h \text{ is odd;} \\ \frac{1}{2}\left(h^3+2h^2+3h+2\right), & \text{if } h \text{ is even.} \end{cases}$$

Proof. Let

$$H = \begin{cases} \frac{1}{2}(h^2+2h+1), & \text{if } h \text{ is odd;} \\ \frac{1}{2}(h^2+h+2), & \text{if } h \text{ is even.} \end{cases}$$

We must prove that

$$a_4(h) \leq (h+1)H.$$

This inequality is equivalent to the statement that there do not exist nonnegative integers x_0, x_1, x_2, x_3 and y_1, y_2, y_3 such that

$$\begin{aligned} x_0(h+1)H + x_1 + x_2(h+1) + x_3(h^2+h+1) \\ = y_1 + y_2(h+1) + y_3(h^2+h+1) \end{aligned} \tag{14}$$

with

$$x_0 + x_1 + x_2 + x_3 \leq h, \qquad y_1 + y_2 + y_3 \leq h \tag{15}$$

and

$$x_0 \geq 1, \qquad x_1y_1 = x_2y_2 = x_3y_3 = 0. \tag{16}$$

Suppose that equation (14) has a solution satisfying conditions (15) and (16). Note that

$$h^2 + 1 < 2H.$$

If $x_0 \geq 2$, then

$$\begin{aligned} (h+1)(h^2+1) &< 2(h+1)H \leq x_0(h+1)H \\ &\leq y_1 + y_2(h+1) + y_3(h^2+h+1) \\ &\leq h(h^2+h+1) \\ &< (h+1)(h^2+1), \end{aligned}$$

which is absurd. Therefore,

$$x_0 = 1.$$

If $y_3 = 0$, then

$$\frac{(h+1)(h^2+1)}{2} < (h+1)H \leq y_1 + y_2(h+1) \leq h(h+1)$$

and so $(h-1)^2 < 0$, which is absurd. Therefore,

$$y_3 \geq 1 \qquad \text{and} \qquad x_3 = 0.$$

With $x_0 = 1$ and $x_3 = 0$, equation (14) becomes

$$\begin{aligned} x_1 + (H + x_2)(h+1) &= y_1 + y_2(h+1) + y_3(h^2+h+1) \\ &= y_1 + y_3 + (y_2 + y_3h)(h+1). \end{aligned} \tag{17}$$

We obtain the congruence

$$x_1 \equiv y_1 + y_3 \pmod{h+1}.$$

If $x_1 = 0$, then $y_1 + y_3 \equiv 0 \pmod{h+1}$. The inequalities $y_1 \geq 0$ and $y_3 \geq 1$ imply $y_1 + y_3 \geq h + 1$, which contradicts condition (15). Therefore,

$$x_1 \geq 1 \qquad \text{and} \qquad y_1 = 0$$

and

$$x_1 \equiv y_3 \pmod{h+1}.$$

Because $1 \leq y_3 \leq h$, if $x_1 \neq y_3$, then $x_1 \geq y_3 + h + 1 > h$, which is absurd. Therefore,

$$x_1 = y_3$$

and equation (17) becomes, simply,

$$H + x_2 = y_2 + y_3 h, \tag{18}$$

where x_2, y_2, y_3 are nonnegative integers such that

$$x_2 + y_3 \leq h - 1, \quad y_2 + y_3 \leq h, \qquad \text{and} \qquad x_2 y_2 = 0.$$

We consider separately the two cases: h odd and h even.

Let h be odd. If $y_3 \leq (h+1)/2$, then equation (18) gives

$$\begin{aligned} \frac{h^2 + 2h + 1}{2} &\leq y_2 + y_3 h \leq (h - y_3) + y_3 h \\ &= h + y_3(h-1) \leq h + \frac{h^2 - 1}{2} \\ &= \frac{h^2 + 2h - 1}{2} \end{aligned}$$

which is absurd. It follows that $y_3 > (h+1)/2$ and, because h is odd, that

$$y_3 \geq \frac{h+3}{2}.$$

If $y_2 = 0$, then

$$x_2 = y_3 h - H \geq \frac{h^2 + 3h}{2} - \frac{h^2 + 2h + 1}{2} \geq \frac{h-1}{2},$$

and so

$$h + 1 = \frac{h-1}{2} + \frac{h+3}{2} \leq x_2 + y_3 \leq h - 1$$

which is absurd. Therefore,

$$y_2 \geq 1 \qquad \text{and} \qquad x_2 = 0.$$

Equation (18) becomes

$$H = y_2 + y_3 h.$$

Equivalently,

$$h^2 + 2h + 1 = 2H = 2y_2 + 2y_3 h \geq 2y_2 + h^2 + 3h,$$

and so

$$1 \geq 2y_2 + h,$$

which is absurd. This completes the proof in the odd case.

Let h be even. If $y_3 \leq h/2$, then equation (18) gives

$$\begin{aligned} \frac{h^2+h+2}{2} &\leq y_2 + y_3 h \leq (h - y_3) + y_3 h \\ &= h + y_3(h-1) \leq h + \frac{h^2-h}{2} \\ &= \frac{h^2+h}{2} \end{aligned}$$

which is absurd. It follows that $y_3 > h/2$ and, because h is even, that

$$y_3 \geq \frac{h+2}{2}.$$

If $y_2 = 0$, then

$$x_2 = y_3 h - H \geq \frac{h^2+2h}{2} - \frac{h^2+h+2}{2} = \frac{h-2}{2},$$

and so

$$h = \frac{h+2}{2} + \frac{h-2}{2} \leq y_3 + x_2 \leq h - 1,$$

which is absurd. Therefore,

$$y_2 \geq 1 \qquad \text{and} \qquad x_2 = 0.$$

Equation (18) becomes

$$H = y_2 + y_3 h$$

Equivalently,

$$h^2 + h + 2 = 2H = 2y_2 + 2y_3 h \geq 2y_2 + h^2 + 2h,$$

and so

$$2 \geq 2y_2 + h,$$

which is absurd. This completes the proof in the even case. □

Now Theorem 1, the exact formula for $a_4(h)$, follows immediately from Lemmas 2 and 3.

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