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Some Identities Involving Stirling Numbers Arising from Matrix Decompositions

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Abstract

An infinite matrix $\mathsf{S} = [\mathsf{S}_{i,j}]_{i,j\geq 1}$ is said to be Stirling-like if its entries satisfy the recurrence $\mathsf{S}_{i,j} = \mathsf{S}_{i-1,j-1} + j\mathsf{S}_{i-1,j}$ for $i, j \geq 2$. The aim of the present study is twofold. Firstly, we find some matrix decompositions for certain Stirling-like matrices and specifically evaluate their determinants. Secondly, using the obtained matrix decompositions, we derive some new combinatorial identities involving Stirling numbers.

1 Introduction

For integers i and j with $i \geq j \geq 0$, the classical *Stirling numbers of the first kind* $s(i,j)$ and *of the second kind* $S(i,j)$ can be defined as the coefficients in the following expansion of a variable x:

$$(x)_i = \sum_{j=0}^{i} s(i,j)x^j \quad \text{and} \quad x^i = \sum_{j=0}^{i} S(i,j)(x)_j, \tag{1}$$

where

$$(x)_i = \begin{cases} 1, & \text{if } i = 0; \\ x(x-1)(x-2)\cdots(x-i+1), & \text{if } i \geq 1, \end{cases}$$

is the falling factorial. It follows immediately from the definition that

$$s(0,0) = S(0,0) = 1,$$

while for $i > 0$

$$s(i,0) = S(i,0) = 0, \; s(i,i) = S(i,i) = S(i,1) = 1 \text{ and } s(i,1) = (-1)^{i-1}(i-1)!.$$

Moreover, if $i < j$, then obviously $s(i,j) = S(i,j) = 0$. It is also known that the Stirling numbers of the second kind satisfy the following recurrence relation:

$$S(i,j) = S(i-1,j-1) + jS(i-1,j), \quad (i,j \geq 1). \tag{2}$$

The *(infinite) Stirling matrix of the first kind* $s(\infty) = [s_{i,j}]_{i,j\geq 1}$ and *of the second kind* $S(\infty) = [S_{i,j}]_{i,j\geq 1}$ are defined by

$$s_{i,j} = \begin{cases} s(i,j), & \text{if } i \geq j; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad S_{i,j} = \begin{cases} S(i,j), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have

$$s(\infty) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 2 & -3 & 1 & 0 & \cdots \\ -6 & 11 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } S(\infty) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & \cdots \\ 1 & 7 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Here, we index matrices starting at $(1,1)$. We let $s(n)$ (resp., $S(n)$) denote the submatrix consisting of the first n rows and columns of $s(\infty)$ (resp., $S(\infty)$). It is immediate from (1) that the Stirling matrices of the first kind and the second kind are inverses of each other, that is

$$S(n)^{-1} = s(n) \quad \text{and} \quad s(n)^{-1} = S(n). \tag{3}$$

There are scattered results in the literature on Stirling matrices, e.g., [1, 2, 3, 4, 5, 6, 10]. In the sequel, we propose a more general definition, as follows. Let $\alpha = (\alpha_0, \alpha_1, \dots)$ be a sequence of variables. Let $\mathsf{S}_\alpha(\infty) = [\mathsf{S}_{i,j}]_{i,j\geq 1}$ be an infinite matrix whose entries satisfy the recurrence relation

$$\mathsf{S}_{i,j} = \mathsf{S}_{i-1,j-1} + j\mathsf{S}_{i-1,j}, \quad (i, j \geq 2),$$

and the initial conditions $\mathsf{S}_{i,1} = \alpha_0$ and $\mathsf{S}_{1,j} = \alpha_{j-1}$ for $i, j \geq 1$. Thus, we have

$$\mathsf{S}_\alpha(\infty) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_0 & \alpha_0 + 2\alpha_1 & \alpha_1 + 3\alpha_2 & \alpha_2 + 4\alpha_3 & \cdots \\ \alpha_0 & 3\alpha_0 + 4\alpha_1 & \alpha_0 + 5\alpha_1 + 9\alpha_2 & \alpha_1 + 7\alpha_2 + 16\alpha_3 & \cdots \\ \alpha_0 & 7\alpha_0 + 8\alpha_1 & 6\alpha_0 + 19\alpha_1 + 27\alpha_2 & \alpha_0 + 9\alpha_1 + 37\alpha_2 + 64\alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We call this matrix $\mathsf{S}_\alpha(\infty)$ the *Stirling-like matrix* (associated with α). In the special case that $\alpha = (1, 0, 0, 0, \dots)$, $\mathsf{S}_\alpha(\infty) = S(\infty)$. As before, we let $\mathsf{S}_\alpha(n)$ denote the submatrix consisting of the first n rows and columns of $\mathsf{S}_\alpha(\infty)$.

The first part of the paper deals, among other matters, with two different decompositions of $\mathsf{S}_\alpha(n)$, see (7) and (8). The first decomposition (7) shows its connection with the Stirling matrix $S(n)$, while the second decomposition (8) indicates its connection with a Vandermonde matrix. We then prove the following determinant evaluation for $\mathsf{S}_\alpha(n)$, which will be achieved in Theorem 4:

$$\det \mathsf{S}_\alpha(n) = \prod_{i=0}^{n-1} \sum_{j=0}^{i} (i)_j \alpha_j.$$

In the second part of the paper, using the previous results, we will obtain several combinatorial identities involving Stirling numbers (of the first and second kind).

A few words about the contents. In Section 2, we introduce some definitions and notation, and also prove two auxiliary lemmas, namely Lemmas 1 and 2. In the same section we prove Theorems 3 and 4. In Section 3, as some applications of the obtained results, we present several combinatorial identities involving Stirling numbers.

The notation used in this paper is fairly standard; the reader is referred to [7, 9], for instance. As usual, for a matrix A, we use $A_{i,j}$ to denote its (i, j)-th entry.

2 Matrix decompositions of $\mathsf{S}_\alpha(n)$

Before stating our main result in this section (namely Theorem 3), we introduce some matrices that will be used in the rest of the paper. Let $\alpha = (\alpha_0, \alpha_1, \dots)$ be a sequence of variables and let n be a positive integer.

(a) Let $\mathsf{U}_\alpha(n) = [\mathsf{U}_{i,j}]_{1\leq i,j\leq n}$ be an $n \times n$ matrix, whose entries are given by the recurrence relation

$$\mathsf{U}_{i,j} = \mathsf{U}_{i-1,j-1} + (j - i + 1)\mathsf{U}_{i-1,j}, \quad \text{for } i, j \geq 2, \tag{4}$$

and the initial conditions $\mathsf{U}_{1,1} = \alpha_0$, $\mathsf{U}_{i,1} = 0$ and $\mathsf{U}_{1,j} = \alpha_{j-1}$, for $i, j \geq 2$. As an example, the matrix $\mathsf{U}_\alpha(4)$ is given by

$$\mathsf{U}_\alpha(4) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \alpha_0 + \alpha_1 & \alpha_1 + 2\alpha_2 & \alpha_2 + 3\alpha_3 \\ 0 & 0 & \alpha_0 + 2\alpha_1 + 2\alpha_2 & \alpha_1 + 4\alpha_2 + 6\alpha_3 \\ 0 & 0 & 0 & \alpha_0 + 3\alpha_1 + 6\alpha_2 + 6\alpha_3 \end{bmatrix}.$$

(b) Let $\mathsf{R}_\alpha(n) = [\mathsf{R}_{i,j}]_{1 \leq i,j \leq n}$ be an $n \times n$ matrix whose entries are given by

$$\mathsf{R}_{i,j} = \begin{cases} \frac{(-1)^{j-i}}{(j-i)!} \sum\limits_{l=0}^{i-1} \frac{\alpha_l}{(i-1-l)!}, & \text{if } j \geq i; \\ 0, & \text{otherwise.} \end{cases}$$

For instance, we have

$$\mathsf{R}_\alpha(4) = \begin{bmatrix} \alpha_0 & -\alpha_0 & \alpha_0/2 & -\alpha_0/6 \\ 0 & \alpha_0 + \alpha_1 & -(\alpha_0 + \alpha_1) & (\alpha_0 + \alpha_1)/2 \\ 0 & 0 & (\alpha_0 + 2\alpha_1 + 2\alpha_2)/2 & -(\alpha_0 + 2\alpha_1 + 2\alpha_2)/2 \\ 0 & 0 & 0 & (\alpha_0 + 3\alpha_1 + 6\alpha_2 + 6\alpha_3)/6 \end{bmatrix}.$$

Note that the sum of all entries on jth column of $\mathsf{R}_\alpha(n)$ is α_{j-1} (see Lemma 2).

(c) Let $V(n) = [V_{i,j}]_{1 \leq i,j \leq n}$ be a special *Vandermonde matrix* defined by $V_{i,j} = j^{i-1}$. For example, when $n = 4$, the corresponding Vandermonde matrix $V(4)$ is given by

$$V(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{bmatrix}.$$

We preserve the notation $\mathsf{S}_\alpha(n)$, $\mathsf{U}_\alpha(n)$, $\mathsf{R}_\alpha(n)$, $S(n)$, $V(n)$ in the rest of the paper. For the next purpose, we need the following lemma, which describes an explicit formula for entries $\mathsf{U}_{i,j}$.

Lemma 1. *For a sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ of variables, the matrix $\mathsf{U}_\alpha(n) = [\mathsf{U}_{i,j}]_{1 \leq i,j \leq n}$ is an upper triangular matrix whose entries $\mathsf{U}_{i,j}$ are given by*

$$\mathsf{U}_{i,j} = \sum_{l=0}^{i-1} \binom{i-1}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i}, \quad \textit{for } j \geq i \geq 1.$$

In particular, we obtain the following form of the diagonal entries of $\mathsf{U}_\alpha(n)$:

$$\mathsf{U}_{i,i} = \sum_{l=0}^{i-1} (i-1)_l \alpha_l, \quad \textit{for} \;\; i = 1, 2, \ldots, n.$$

Proof. Observe, first of all, that $\mathsf{U}_{2,1} = 0$ and $\mathsf{U}_{i+1,i} = \mathsf{U}_{i,i-1}$ for $i \geq 2$, which implies that the subdiagonal entries of $\mathsf{U}_\alpha(n)$ are all 0. Now from the recurrence relation (4), it easily follows that $\mathsf{U}_\alpha(n)$ is an upper triangular matrix.

Now let $\mathsf{Q}_\alpha(n) = [\mathsf{Q}_{i,j}]_{1\leq i,j\leq n}$ be the upper triangular matrix given by

$$\mathsf{Q}_{i,j} = \sum_{l=0}^{i-1} \binom{i-1}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} \quad \text{for } j \geq i \geq 1.$$

To prove the lemma we need only to show that $\mathsf{U}_\alpha(n) = \mathsf{Q}_\alpha(n)$. In fact, we will show that the first row of $\mathsf{Q}_\alpha(n)$ is equal to the first row of $\mathsf{U}_\alpha(n)$, and for $1 < i \leq j$ the entries $\mathsf{Q}_{i,j}$ of the matrix $\mathsf{Q}_\alpha(n)$ satisfy the recursion relation $\mathsf{Q}_{i,j} = \mathsf{Q}_{i-1,j-1} + (j-i+1)\mathsf{Q}_{i-1,j}$.

If $i = 1$, then for $j \geq 1$ a direct computation gives

$$\mathsf{Q}_{1,j} = \sum_{l=0}^{1-1} \binom{1-1}{l} \frac{(l+j-1)!}{(j-1)!} \alpha_{l+j-1} = \binom{0}{0} \frac{(j-1)!}{(j-1)!} \alpha_{0+j-1} = \alpha_{j-1} = \mathsf{U}_{1,j}.$$

If $1 < i \leq j$, then using the Pascal identity, we obtain

$$\begin{aligned}
\mathsf{Q}_{i,j} &= \sum_{l=0}^{i-1} \binom{i-1}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} = \alpha_{j-i} + \sum_{l=1}^{i-2} \binom{i-1}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + \frac{(j-1)!}{(j-i)!} \alpha_{j-1} \\
&= \alpha_{j-i} + \sum_{l=1}^{i-2} \left(\binom{i-2}{l} + \binom{i-2}{l-1} \right) \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + \frac{(j-1)!}{(j-i)!} \alpha_{j-1} \\
&= \sum_{l=0}^{i-2} \binom{i-2}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + \sum_{l=1}^{i-2} \binom{i-2}{l-1} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + \frac{(j-1)!}{(j-i)!} \alpha_{j-1} \\
&= \sum_{l=0}^{i-2} \binom{i-2}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + \sum_{l=0}^{i-3} \binom{i-2}{l} \frac{(l+1+j-i)!}{(j-i)!} \alpha_{l+1+j-i} + \frac{(j-1)!}{(j-i)!} \alpha_{j-1} \\
&= \sum_{l=0}^{i-2} \binom{i-2}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + \sum_{l=0}^{i-2} \binom{i-2}{l} \frac{(l+1+j-i)!}{(j-i)!} \alpha_{l+1+j-i} \\
&= \sum_{l=0}^{i-2} \binom{i-2}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i} + (j-i+1) \sum_{l=0}^{i-2} \binom{i-2}{l} \frac{(l+1+j-i)!}{(j-i+1)!} \alpha_{l+1+j-i} \\
&= \mathsf{Q}_{i-1,j-1} + (j-i+1)\mathsf{Q}_{i-1,j},
\end{aligned}$$

as desired.

The second statement follows from the first. The proof is complete. □

Lemma 2. *We have* $\sum_{i=1}^{j} \mathsf{R}_{i,j} = \alpha_{j-1}$.

Proof. First of all, by the definition of $\mathsf{R}_\alpha(n)$, we have

$$\sum_{i=1}^{j} \mathsf{R}_{i,j} = \sum_{i=1}^{j} \left(\frac{(-1)^{j-i}}{(j-i)!} \sum_{l=0}^{i-1} \frac{\alpha_l}{(i-1-l)!} \right).$$

By interchanging the order of summation, the sum on the right-hand side becomes

$$\sum_{i=1}^{j} \left(\frac{(-1)^{j-i}}{(j-i)!} \sum_{l=0}^{i-1} \frac{\alpha_l}{(i-1-l)!} \right) = \sum_{l=0}^{j-1} \left(\sum_{i=l}^{j-1} \frac{(-1)^{j-1-i}}{(j-1-i)!(i-l)!} \right) \alpha_l. \tag{5}$$

If $l = j-1$ on the right-hand side of (5), then obviously the coefficient of α_{j-1} is 1. Hence, from now on we assume $l < j-1$. Let l be fixed with $l < j-1$. To complete our proof we must show that

$$\sum_{i=l}^{j-1} \frac{(-1)^{j-1-i}}{(j-1-i)!(i-l)!} = 0. \tag{6}$$

First, a direct computation shows that

$$\frac{1}{(j-1-i)!(i-l)!} = \binom{j-1}{l} \binom{j-1-l}{i-l} \frac{l!}{(j-1)!},$$

and by substituting this into the left-hand side of (6) and applying the index shifting $i \to i+l$, we get

$$\sum_{i=l}^{j-1} \frac{(-1)^{j-1-i}}{(j-1-i)!(i-l)!} = \frac{(-1)^{j-1-l} l!}{(j-1)!} \binom{j-1}{l} \sum_{i=0}^{j-1-l} (-1)^i \binom{j-1-l}{i}.$$

Now the result follows from the fact that $\sum_{i=0}^{j-1-l} (-1)^i \binom{j-1-l}{i} = 0$. □

We are now ready to state and prove the main result of this section which gives two different decompositions of the Stirling-like matrix $\mathsf{S}_\alpha(n)$. In the first decomposition the first factor is the Stirling matrix $S(n)$, and in the second decomposition the first factor is the Vandermonde matrix $V(n)$. We will use the resulting decompositions to obtain some combinatorial identities concerning Stirling numbers of the first and second kind.

Theorem 3. *For a sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ of variables and positive integer n, the Stirling-like matrix $\mathsf{S}_\alpha(n)$ has the following decompositions:*

$$\mathsf{S}_\alpha(n) = S(n) \cdot \mathsf{U}_\alpha(n), \tag{7}$$

and

$$\mathsf{S}_\alpha(n) = V(n) \cdot \mathsf{R}_\alpha(n). \tag{8}$$

Proof. For the proof of the claimed decomposition (7), we show that both sides satisfy the same recurrence and the same initial conditions. To simplify the notation, we write S and U for $S(n)$ and $\mathsf{U}_\alpha(n)$, respectively. Let us do the required calculations.

If $i = 1$ and $1 \le j \le n$, then we obtain

$$(S \cdot \mathsf{U})_{1,j} = \sum_{k=1}^{n} S_{1,k}\mathsf{U}_{k,j} = S_{1,1}\mathsf{U}_{1,j} = 1 \cdot \alpha_{j-1} = \alpha_{j-1} = \mathsf{S}_{1,j}.$$

Similarly, if $j = 1$ and $1 \le i \le n$, then we obtain

$$(S \cdot \mathsf{U})_{i,1} = \sum_{k=1}^{n} S_{i,k}\mathsf{U}_{k,1} = S_{i,1}\mathsf{U}_{1,1} = 1 \cdot \alpha_0 = \alpha_0 = \mathsf{S}_{i,1}.$$

We may therefore assume that $2 \le i, j \le n$. By simple computations, we get

$$\begin{aligned}
(S \cdot \mathsf{U})_{i,j} &= \sum_{k=1}^{n} S_{i,k}\mathsf{U}_{k,j} = S_{i,1}\mathsf{U}_{1,j} + \sum_{k=2}^{n} S_{i,k}\mathsf{U}_{k,j} \\
&= 1 \cdot \alpha_{j-1} + \sum_{k=2}^{n} \left[S_{i-1,k-1} + kS_{i-1,k}\right]\mathsf{U}_{k,j} \text{ (by (2))} \\
&= \alpha_{j-1} + \sum_{k=2}^{n} S_{i-1,k-1}\mathsf{U}_{k,j} + \sum_{k=2}^{n} kS_{i-1,k}\mathsf{U}_{k,j} \\
&= \alpha_{j-1} + \sum_{k=2}^{n} S_{i-1,k-1}\left[\mathsf{U}_{k-1,j-1} + (j-k+1)\mathsf{U}_{k-1,j}\right] + \sum_{k=2}^{n} kS_{i-1,k}\mathsf{U}_{k,j} \text{ (by (4))} \\
&= \alpha_{j-1} + \sum_{k=1}^{n} S_{i-1,k}\left[\mathsf{U}_{k,j-1} + (j-k)\mathsf{U}_{k,j}\right] + \sum_{k=2}^{n} kS_{i-1,k}\mathsf{U}_{k,j} \text{ (since } S_{i-1,n} = 0) \\
&= \alpha_{j-1} + (S \cdot \mathsf{U})_{i-1,j-1} + \sum_{k=1}^{n} (j-k)S_{i-1,k}\mathsf{U}_{k,j} + \sum_{k=2}^{n} kS_{i-1,k}\mathsf{U}_{k,j} \\
&= \alpha_{j-1} + (S \cdot \mathsf{U})_{i-1,j-1} + (j-1)S_{i-1,1}\mathsf{U}_{1,j} + \sum_{k=2}^{n} jS_{i-1,k}\mathsf{U}_{k,j} \\
&= \alpha_{j-1} + (S \cdot \mathsf{U})_{i-1,j-1} - S_{i-1,1}\mathsf{U}_{1,j} + j\sum_{k=1}^{n} S_{i-1,k}\mathsf{U}_{k,j} \\
&= \alpha_{j-1} + (S \cdot \mathsf{U})_{i-1,j-1} - 1 \cdot \alpha_{j-1} + j\sum_{k=1}^{n} S_{i-1,k}\mathsf{U}_{k,j} \\
&= (S \cdot \mathsf{U})_{i-1,j-1} + j(S \cdot \mathsf{U})_{i-1,j},
\end{aligned}$$

which completes the proof of the decomposition given in (7).

We now prove the claimed decomposition (8). To simplify the notation, we write V and R for $V(n)$ and $\mathsf{R}_\alpha(n)$, respectively. Observe, first of all, that since R is an upper triangular matrix, for $1 \le i, j \le n$, we have

$$(V \cdot \mathsf{R})_{i,j} = \sum_{k=1}^{j} V_{i,k}\mathsf{R}_{k,j} = \sum_{k=1}^{j} k^{i-1}\frac{(-1)^{j-k}}{(j-k)!}\sum_{l=0}^{k-1}\frac{\alpha_l}{(k-1-l)!}. \tag{9}$$

Now, if $j = 1$, then for $i \ge 1$, we have

$$(V \cdot \mathsf{R})_{i,1} = \sum_{k=1}^{1} k^{i-1}\frac{(-1)^{1-k}}{(1-k)!}\sum_{l=0}^{k-1}\frac{\alpha_l}{(k-1-l)!} = \alpha_0 = \mathsf{S}_{i,1}. \tag{10}$$

Also, if $j \ge 2$, we have, using Lemma 2, that

$$(V \cdot \mathsf{R})_{1,j} = \sum_{k=1}^{j} V_{1,k}\mathsf{R}_{k,j} = \sum_{k=1}^{j} \mathsf{R}_{k,j} = \alpha_{j-1} = \mathsf{S}_{1,j}. \tag{11}$$

By (10) and (11), the first row and first column of $V \cdot \mathsf{R}$ are equal to the first row and first column of $\mathsf{S}_\alpha(n)$, respectively. To complete the proof, it is enough to show that the remaining entries of $V \cdot \mathsf{R}$ satisfy the same recurrence relation as $\mathsf{S}_\alpha(n)$, that is,

$$(V \cdot \mathsf{R})_{i,j} = (V \cdot \mathsf{R})_{i-1,j-1} + j(V \cdot \mathsf{R})_{i-1,j} \quad \text{for} \quad 2 \le i, j \le n.$$

In what follows, for the sake of convenience, we write

$$\Psi_\alpha(k) = \sum_{l=0}^{k-1}\frac{\alpha_l}{(k-1-l)!}.$$

Now, using (9), for all $2 \le i, j \le n$, we have

$$(V \cdot \mathsf{R})_{i-1,j-1} = \sum_{k=1}^{j-1} k^{i-2}\frac{(-1)^{j-1-k}}{(j-1-k)!}\Psi_\alpha(k) \;\text{ and }\; (V \cdot \mathsf{R})_{i-1,j} = \sum_{k=1}^{j} k^{i-2}\frac{(-1)^{j-k}}{(j-k)!}\Psi_\alpha(k),$$

and, therefore,

$$\begin{aligned}
(V \cdot \mathsf{R})_{i-1,j-1} + j(V \cdot \mathsf{R})_{i-1,j} &= \sum_{k=1}^{j-1} k^{i-2}\frac{(-1)^{j-1-k}}{(j-1-k)!}\Psi_\alpha(k) + j\sum_{k=1}^{j} k^{i-2}\frac{(-1)^{j-k}}{(j-k)!}\Psi_\alpha(k) \\
&= \sum_{k=1}^{j-1} k^{i-2}(-1)^{j-k}\left(\frac{j}{(j-k)!} - \frac{1}{(j-1-k)!}\right)\Psi_\alpha(k) + j^{i-1}\Psi_\alpha(j) \\
&= \sum_{k=1}^{j-1} k^{i-1}\frac{(-1)^{j-k}}{(j-k)!}\Psi_\alpha(k) + j^{i-1}\Psi_\alpha(j) \\
&= \sum_{k=1}^{j} k^{i-1}\frac{(-1)^{j-k}}{(j-k)!}\Psi_\alpha(k) = (V \cdot \mathsf{R})_{i,j}, \text{ (by (9))}
\end{aligned}$$

as desired. The proof of the decomposition (8) is now complete. □

Theorem 4. *Suppose that $\alpha = (\alpha_0, \alpha_1, \dots)$ is a sequence of variables. Then for every positive integer n we have*

$$\det \mathsf{S}_\alpha(n) = \prod_{i=0}^{n-1} \sum_{j=0}^{i} (i)_j \alpha_j.$$

Before proving Theorem 4, we point out that if $\alpha = (1, 1, 1, \dots)$, then we obtain

$$\det \mathsf{S}_\alpha(n) = \prod_{i=0}^{n-1} \sum_{j=0}^{i} (i)_j,$$

which shows that the sequence of leading principal minors of Stirling-like matrix $\mathsf{S}_\alpha(\infty)$ is a well-known sequence of integers, that is the product of first n terms of the binomial transform of the factorial (A294352 in the OEIS [8]). Also note that the sequence

$$\beta = (\beta_i)_{i\geq 0} = \left(\sum_{j=0}^{i} (i)_j\right)_{i\geq 0} = (1, 2, 5, 16, 65, \dots),$$

is the sequence A000522 in the OEIS [8] which satisfies the recurrence relation $\beta_i = i\beta_{i-1} + 1$ with $\beta_0 = 1$.

Proof. We give two proofs for the theorem. By (7) in Theorem 3, we have the LU-decomposition $\mathsf{S}_\alpha(n) = S(n) \cdot \mathsf{U}_\alpha(n)$ of the Stirling-like matrix $\mathsf{S}_\alpha(n)$. The result is now immediate, since $S(n)$ is a lower triangular matrix with 1's on the diagonal, whereas, by Lemma 1, $\mathsf{U}_\alpha(n)$ is an upper triangular matrix with diagonal entries $\sum_{j=0}^{i-1} (i-1)_j \alpha_j$, $i = 1, 2, \dots, n$.

As an alternative proof, we use the decomposition (8) of Theorem 3, $\mathsf{S}_\alpha(n) = V(n) \cdot \mathsf{R}_\alpha(n)$, which implies that $\det \mathsf{S}_\alpha(n) = \det V(n) \det \mathsf{R}_\alpha(n)$. Note that the determinant of $V(n)$ (which is called the *Vandermonde determinant*) has the following simple form:

$$\det V(n) = \prod_{1\leq i<j\leq n} (j - i) = \prod_{i=1}^{n} (i-1)!.$$

Also, $\mathsf{R}_\alpha(n)$ is an upper triangular matrix with diagonal entries $\sum_{j=0}^{i-1} \frac{\alpha_j}{(i-1-j)!}$, $i = 1, 2, \dots, n$, and hence

$$\det \mathsf{R}_\alpha(n) = \prod_{i=1}^{n} \sum_{j=0}^{i-1} \frac{\alpha_j}{(i-1-j)!}.$$

Therefore,

$$\begin{aligned}\det \mathsf{S}_\alpha(n) = \det V(n) \det \mathsf{R}_\alpha(n) &= \prod_{i=1}^{n} (i-1)! \prod_{i=1}^{n} \sum_{j=0}^{i-1} \frac{\alpha_j}{(i-1-j)!} \\ &= \prod_{i=1}^{n} \sum_{j=0}^{i-1} \frac{(i-1)!}{(i-1-j)!} \alpha_j = \prod_{i=0}^{n-1} \sum_{j=0}^{i} \frac{i!}{(i-j)!} \alpha_j = \prod_{i=0}^{n-1} \sum_{j=0}^{i} (i)_j \alpha_j,\end{aligned}$$

which completes the proof. □

3 Identities involving Stirling numbers

In this section, as some applications of the results obtained in the previous section, we give several new combinatorial identities involving Stirling numbers. We start with the following result.

Proposition 5. *Let $i, j \geq 1$ be integers and let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence of variables. Then we have*

$$\sum_{k=1}^{\min(i,j)} S(i,k) \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{(l+j-k)!}{(j-k)!} \alpha_{l+j-k} = \sum_{k=1}^{j} k^{i-1} \frac{(-1)^{j-k}}{(j-k)!} \sum_{l=0}^{k-1} \frac{\alpha_l}{(k-1-l)!}. \tag{12}$$

Proof. Let i and j be fixed, and choose the integer $n \geq 1$ large enough so that $1 \leq i, j \leq n$. By the decompositions of the matrix $\mathsf{S}_\alpha(n)$ given in Theorem 3, we have

$$S(n) \cdot \mathsf{U}_\alpha(n) = \mathsf{S}_\alpha(n) = V(n) \cdot \mathsf{R}_\alpha(n). \tag{13}$$

On the one hand, since $S(n)$ and $\mathsf{U}_\alpha(n)$ are triangular matrices, we obtain for the (i,j)-th entry

$$(S(n) \cdot \mathsf{U}_\alpha(n))_{i,j} = \sum_{k=1}^{\min(i,j)} S_{i,k} \mathsf{U}_{k,j} = \sum_{k=1}^{\min(i,j)} S(i,k) \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{(l+j-k)!}{(j-k)!} \alpha_{l+j-k},$$

where we used Lemma 1 to justify the last equation. On the other hand, by (9), we have

$$(V(n) \cdot \mathsf{R}_\alpha(n))_{i,j} = \sum_{k=1}^{j} k^{i-1} \frac{(-1)^{j-k}}{(j-k)!} \sum_{l=0}^{k-1} \frac{\alpha_l}{(k-1-l)!}.$$

Now, the result follows from the equality of the (i,j)-th entries on both sides of (13). $\square$

As a special case of the above theorem, we can obtain the following result which is the classical formula for the Stirling numbers of the second kind.

Corollary 6. *Let $i \geq j \geq 1$ be integers. Then we have*

$$S(i,j) = \frac{1}{j!} \sum_{k=1}^{j} \binom{j}{k} (-1)^{j-k} k^i. \tag{14}$$

Proof. Let $\alpha = (1, 0, 0, \ldots)$. If we apply Proposition 5 to α, the right-hand side of Equation (12) simplifies to

$$\sum_{k=1}^{j} k^{i-1} \frac{(-1)^{j-k}}{(j-k)!(k-1)!} = \frac{1}{j!} \sum_{k=1}^{j} \binom{j}{k} (-1)^{j-k} k^i. \tag{15}$$

On the other hand, since $i \geq j$, $\min(i,j) = j$, and the left-hand side of Equation (12) simplifies to

$$\sum_{k=1}^{j} S(i,k) \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{(l+j-k)!}{(j-k)!} \alpha_{l+j-k} = S(i,j), \tag{16}$$

since if $k < j$, then $l+j-k \geq 1$ and so $\alpha_{l+j-k} = 0$, and if $k = j$, the only nonzero term in the second sum occurs when $l = 0$, in which case Equation (16) reduces to $S(i,j)$. Therefore, Equation (14) follows from the equality of Equations (15) and (16). $\square$

For a sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$ of variables we define the $n \times n$ upper triangular matrix $\tilde{\mathsf{R}}_\alpha(n) = [\tilde{\mathsf{R}}_{i,j}]_{1 \leq i,j \leq n}$ as follows:

$$\tilde{\mathsf{R}}_{i,j} = \begin{cases} (j-1)_{i-1} \left(\sum_{l=0}^{j-1} (j-1)_l \alpha_l \right)^{-1}, & \text{if } j \geq i; \\ 0, & \text{otherwise.} \end{cases}$$

For instance, if $n = 4$, the matrix $\tilde{\mathsf{R}}_\alpha(4)$ is

$$\tilde{\mathsf{R}}_\alpha(4) = \begin{bmatrix} 1/\alpha_0 & 1/(\alpha_0+\alpha_1) & 1/(\alpha_0+2\alpha_1+2\alpha_2) & 1/(\alpha_0+3\alpha_1+6\alpha_2+6\alpha_3) \\ 0 & 1/(\alpha_0+\alpha_1) & 2/(\alpha_0+2\alpha_1+2\alpha_2) & 3/(\alpha_0+3\alpha_1+6\alpha_2+6\alpha_3) \\ 0 & 0 & 2/(\alpha_0+2\alpha_1+2\alpha_2) & 6/(\alpha_0+3\alpha_1+6\alpha_2+6\alpha_3) \\ 0 & 0 & 0 & 6/(\alpha_0+3\alpha_1+6\alpha_2+6\alpha_3) \end{bmatrix}.$$

Lemma 7. *Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence of variables and let n be a positive integer. Then we have $\tilde{\mathsf{R}}_\alpha(n) = \mathsf{R}_\alpha(n)^{-1}$ provided*

$$\prod_{j=1}^{n} \sum_{i=0}^{j-1} (j-1)_i \alpha_i \neq 0. \tag{17}$$

Proof. To simplify the notation, we write R instead of $\mathsf{R}_\alpha(n)$ and $\tilde{\mathsf{R}}$ instead of $\tilde{\mathsf{R}}_\alpha(n)$. Let I be the identity matrix of order n. Note that Condition (17) guarantees that the matrices R and $\tilde{\mathsf{R}}$ are invertible. We need to show that $\tilde{\mathsf{R}} \cdot \mathsf{R} = I$. Since $\tilde{\mathsf{R}}$ and R are both upper triangular matrices, their product is also upper triangular. Without loss of generality, we may assume that $j \geq i$. Now we have

$$\begin{aligned} (\tilde{\mathsf{R}} \cdot \mathsf{R})_{i,j} &= \sum_{k=i}^{j} \tilde{\mathsf{R}}_{i,k} \mathsf{R}_{k,j} = \sum_{k=i}^{j} \left((k-1)_{i-1} \left(\sum_{r=0}^{k-1} (k-1)_r \alpha_r \right)^{-1} \frac{(-1)^{j-k}}{(j-k)!} \sum_{s=0}^{k-1} \frac{\alpha_s}{(k-1-s)!} \right) \\ &= \sum_{k=i}^{j} \left((k-1)_{i-1} \left(\sum_{r=0}^{k-1} (k-1)_r \alpha_r \right)^{-1} \frac{(-1)^{j-k}}{(k-1)!(j-k)!} \sum_{s=0}^{k-1} (k-1)_s \alpha_s \right) \\ &= \sum_{k=i}^{j} (-1)^{j-k} \frac{(k-1)_{i-1}}{(k-1)!(j-k)!} = \sum_{k=i}^{j} (-1)^{j-k} \frac{1}{(k-i)!(j-k)!} \end{aligned}$$

$$= \sum_{k=i}^{j} (-1)^{j-k} \binom{j-i}{k-i} \frac{1}{(j-i)!} = \frac{(-1)^{j-i}}{(j-i)!} \sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k} = \begin{cases} 0, & \text{if } j > i; \\ 1, & \text{if } j = i. \end{cases}$$

The proof is now complete. □

Proposition 8. *Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence of variables and let $i, j \geq 1$ be two integers such that*

$$\prod_{k=1}^{\max(i,j)} \sum_{l=0}^{k-1} (k-1)_l \alpha_l \neq 0. \tag{18}$$

Then we have

$$\sum_{k_1=1}^{i} \sum_{k_2=k_1}^{j} S(i, k_1)(j-1)_{k_2-1} \Lambda(\alpha, k_1, k_2) = j^{i-1} \Omega(\alpha, j), \tag{19}$$

where

$$\Lambda(\alpha, k_1, k_2) = \sum_{l_1=0}^{k_1-1} \binom{k_1-1}{l_1} \frac{(l_1+k_2-k_1)!}{(k_2-k_1)!} \alpha_{l_1+k_2-k_1},$$

and

$$\Omega(\alpha, j) = \sum_{l_2=0}^{j-1} (j-1)_{l_2} \alpha_{l_2}.$$

Proof. Let i and j be fixed and let $n = \max(i, j) \geq 1$. Since

$$\prod_{k=1}^{n} \sum_{l=0}^{k-1} (k-1)_l \alpha_l \neq 0,$$

we can apply Lemma 7 to conclude that the matrices $\tilde{\mathsf{R}}_\alpha(n)$ and $\mathsf{R}_\alpha(n)$ are invertible and $\mathsf{R}_\alpha(n)^{-1} = \tilde{\mathsf{R}}_\alpha(n)$. To simplify the notation, as before, we write S, V, S, R, U, and $\tilde{\mathsf{R}}$, instead of $S(n)$, $V(n)$, $\mathsf{S}_\alpha(n)$, $\mathsf{R}_\alpha(n)$, $\mathsf{U}_\alpha(n)$, $\tilde{\mathsf{R}}_\alpha(n)$, respectively. We start by the decompositions of Stirling-like matrix S given in Theorem 3, that is $S \cdot \mathsf{U} = \mathsf{S} = V \cdot \mathsf{R}$. Multiplying both sides of this equation from right by $\tilde{\mathsf{R}} = \mathsf{R}^{-1}$, we get $V = S \cdot \mathsf{U} \cdot \tilde{\mathsf{R}}$. Since the matrices S, U and $\tilde{\mathsf{R}}$ are triangular, the (i, j)-th entry is

$$\left(S \cdot \mathsf{U} \cdot \tilde{\mathsf{R}} \right)_{i,j} = \sum_{k_1=1}^{i} \sum_{k_2=k_1}^{j} S_{i,k_1} \mathsf{U}_{k_1,k_2} \tilde{\mathsf{R}}_{k_2,j} = \Omega(\alpha, j)^{-1} \sum_{k_1=1}^{i} \sum_{k_2=k_1}^{j} S(i, k_1) \Lambda(\alpha, k_1, k_2)(j-1)_{k_2-1}.$$

where the second equality follows from Lemma 1. Now from $V_{i,j} = (S \cdot \mathsf{U} \cdot \tilde{\mathsf{R}})_{i,j}$, we deduce that

$$j^{i-1} = \Omega(\alpha, j)^{-1} \sum_{k_1=1}^{i} \sum_{k_2=k_1}^{j} S(i, k_1)(j-1)_{k_2-1} \Lambda(\alpha, k_1, k_2),$$

and the result follows. □

Proposition 9. *Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ be a sequence of variables and let $i, j \geq 1$ be two integers. Then we have*

$$\sum_{k_1=1}^{i} \sum_{k_2=1}^{j} s(i,k_1) {k_2}^{k_1-1} \frac{(-1)^{j-k_2}}{(j-k_2)!} \Psi_\alpha(k_2) = \begin{cases} \sum_{l=0}^{i-1} \binom{i-1}{l} \frac{(l+j-i)!}{(j-i)!} \alpha_{l+j-i}, & \text{if } i \leq j; \\ 0, & \text{if } i > j, \end{cases}$$

where

$$\Psi_\alpha(k_2) = \sum_{l_1=0}^{k_2-1} \frac{\alpha_{l_1}}{(k_2-1-l_1)!}.$$

Proof. Let i, j be fixed. Choose the integer $n \geq 1$ to be such that $1 \leq i, j \leq n$. We use the notation as in Proposition 8, and also we write s instead of $s(n)$. From Theorem 3 and (3), we observe that $\mathsf{U} = s \cdot V \cdot \mathsf{R}$, which implies that $\mathsf{U}_{i,j} = (s \cdot V \cdot \mathsf{R})_{i,j}$. Using the fact that s and R are triangular matrices, we find that

$$(s \cdot V \cdot \mathsf{R})_{i,j} = \sum_{k_1=1}^{i} \sum_{k_2=1}^{j} s_{i,k_1} V_{k_1,k_2} \mathsf{R}_{k_2,j} = \sum_{k_1=1}^{i} \sum_{k_2=1}^{j} \left(s(i,k_1) {k_2}^{k_1-1} \frac{(-1)^{j-k_2}}{(j-k_2)!} \Psi_\alpha(k_2) \right),$$

and the result follows by Lemma 1. □

The above result, in a special case, gives the following familiar identity involving Stirling numbers of the first kind.

Corollary 10. *Let $i, j \geq 1$ be integers. Then we have*

$$\sum_{k_1=1}^{i} \sum_{k_2=1}^{j} (-1)^{j-k_2} \frac{s(i,k_1) {k_2}^{k_1-1}}{(j-k_2)!(k_2-1)!} = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta.

Proof. We may apply Proposition 9 with $\alpha = (\alpha_j)_{j \geq 0} = (1, 0, 0, \ldots)$. In this situation, we see that

$$\Psi_\alpha(k_2) = \sum_{l_1=0}^{k_2-1} \frac{\alpha_{l_1}}{(k_2-1-l_1)!} = \frac{\alpha_0}{(k_2-1)!} = \frac{1}{(k_2-1)!}.$$

Moreover, since $\mathsf{U}_\alpha(n) = I_n$, the identity matrix of order n, we conclude that $\mathsf{U}_{i,j} = \delta_{i,j}$. The proof is complete. □

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