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# Curious Multisection Identities by Index Factorization

Christophe Vignat
Department of Mathematics
Tulane University
6823 St Charles Avenue
New Orleans, LA 70118
USA
cvignat@tulane.edu

Michael Milgram
Geometrics Unlimited, Ltd.
Box 1484
Deep River, ON K0J 1P0
Canada
mike@geometrics-unlimited.com

**Abstract**

This work introduces general multisection identities expressed equivalently in terms of infinite double products and/or infinite double series, reordered by way of their indices. From this reordering, we derive new product and summation identities involving special functions including gamma, hyperbolic trigonometric, polygamma, and zeta functions. It is shown that a parametrized version of the multisection identity exists, a specialization of which coincides with the standard multisection identity.

# 1 Preamble

We begin with the most primitive integer sequence of all, OEIS [13] entry number A000027, that is,

$$\mathbb{N} \equiv \{1, 2, 3, 4, 5, \ldots\}, \tag{1}$$

by pointing out that the term "Integer Sequence", used in the OEIS sense, refers to a *specific* entry (or instance) in a collection of entries extracted from $\mathbb{N}$ in some recognizable order. In other words, entries in the OEIS database are chosen from selected elements of $\mathbb{N}$ and assigned meaning according to their correspondence with some mathematical or physical entity. For example, the subset (i.e., OEIS integer sequence A005843)

$$2\mathbb{N} \equiv \{2, 4, 6, \ldots\} \tag{2}$$

chosen from $\mathbb{N}$ obviously corresponds to the set of all even numbers, among other possibilities. In contrast to the mandate of OEIS to collect and maintain a database of identifiable sequences, here we treat $\mathbb{N}$ as an abstract entity in its own right, and demonstrate that much information can be gleaned by reordering its elements and utilizing each reordered element to label (i.e., index) an abstract mathematical object. An example can be found from an examination of the well-known summation theorem

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}, \tag{3}$$

where each element of the sum is equated to the inverse square of its index and thereby the statement of the sum is specified by an integer sequence (i.e., indices) selected from, and in this case, coinciding with, $\mathbb{N}$. By reordering the indices into two groups (i.e., dissection)—the even and odd integers—we effectively select two integer sequences from $\mathbb{N}$, those being $2\mathbb{N}$ and the OEIS entry A005408

$$2\mathbb{N} - 1 \equiv \{1, 3, 5, \ldots\}, \tag{4}$$

following which we then write

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \sum_{m=1}^{\infty} \frac{1}{(2m)^2} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}, \tag{5}$$

the two right-hand sums being indexed by the elements of $2\mathbb{N}$ and $2\mathbb{N}-1$ respectively. From this elementary example, we discover that

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8} \tag{6}$$

and learn something new. In this work, we carry this principle further (see especially examples (81) and (89) below relevant to the above example) by creatively re-ordering and weighting elements of $\mathbb{N}$ into many groups (i.e., multisection) such that no element is omitted and the sum of the weights is conserved. Each such (scrambled) group is an integer sequence in its own right in the OEIS sense, identified by the common factorization of its elements and employed to index an infinite product or sum of abstract entities. In so doing we show that simple sums and products can be re-ordered in such a way as to yield new and curious identities.

# 2 Introduction

The inspiration for the approach discussed in the preamble arose from a recent article where the second author [11] derived the curious identity

$$\prod_{j\geq 1}\left(\frac{\tan(\frac{a}{2^j})}{\frac{a}{2^j}}\right)^{2^{j-1}} = \frac{a}{\sin a}, \tag{7}$$

along with several other identities of the same flavour. Upon closer inspection, the first author noticed that this identity is the specialization of a more general relationship that can be stated as the even more curious identity

$$\prod_{j\geq 0, n\geq 1}\left(\frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}}\right)^{2^j} = \prod_{m\geq 1} a_m \tag{8}$$

that holds for every sequence of entities $(a_m)$ such that the infinite product $\prod_{m\geq 1} a_m$ is absolutely convergent. In (8) we have explicitly written each index $m$ as the product of the two components of its factored form. That is, $m = (2n)\cdot 2^j$ or $m = (2n-1)\cdot 2^j$ according to whether $m$ is respectively even or odd (see the Preamble). The identity (7) is the specialization $a_m = (1 - \frac{a^2}{m^2\pi^2})^{-1}$ of (8) (see (65) and (195) below). Identity (8) can be viewed as a *structural identity*: it holds as a consequence of one way the terms are grouped by the components of the factorization of their index in the product, rather than of the specific values of the individual elements $a_m$.

Other examples of structural identities follow:

- for an arbitrary sequence $(a_m)$ with absolutely convergent infinite product,

$$\prod_{m\geq 1} a_{2m-1}a_{2m} = \prod_{m\geq 1} a_m, \tag{9}$$

  which is a simple application of the dissection principle;

- for an arbitrary summable double sequence $(a_{m,n})_{(m,n)\in\mathbb{Z}^2}$,

$$\sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} = \sum_{m<n} a_{m,n} + \sum_{m>n} a_{m,n} + \sum_{m\in\mathbb{Z}} a_{m,m}, \tag{10}$$

corresponding to an obvious multisection of the two-dimensional lattice $\mathbb{Z}^2$. For similar additive rather than multiplicative multisections, see Beauregard and Dobrushkin [1], Chu and Esposito [6], and Somos [17].

In the following, each extension or specialization of (8) will be provided along with its double series equivalent expression; in the case of (8), this is

$$\sum_{j\geq 0, n\geq 1} 2^j b_{(2n-1)\cdot 2^j} - 2^j b_{(2n)\cdot 2^j} = \sum_{m\geq 1} b_m, \tag{11}$$

where $b_m$ is an arbitrary summable sequence.

Let us now introduce some further notation:

- the $b$-adic valuation $\nu_b(m)$ of a positive integer $m$ is defined to be the integer

$$\nu_b(m) = \max\left\{k \in \mathbb{N} : m \equiv 0 \pmod{b^k}\right\} \tag{12}$$

  representing the largest exponent in the factorization of the integer $m$ relative to an integer $b$ (not to be confused with any of the element(s) $b_m$), which will be referred to as the *base*. For example:

  - $\nu_2(3) = 0$, because $3 = 3 \cdot 2^0$ is not an integral multiple of 2;
  - $\nu_3(6) = 1$ because 1 is the largest exponent of 3 in the factorization $6 = 2 \cdot 3^1$;
  - $\nu_5(100) = 2$ because of the decomposition $100 = 4 \cdot 5^2$ relative to base 5,
  - and $\nu_2(100) = 2$ because of the decomposition $100 = 25 \cdot 2^2$ relative to base 2.

  Notice that, as a consequence of this definition, for every positive integer $m$, there is a unique representation

$$m = n \cdot b^{\nu_b(m)}, b \nmid n. \tag{13}$$

- for a multiset $S$ that contains the integer $m$ repeated $j$ times, we use the notation

$$S = \{\ldots, m^{(j)}, \ldots\}. \tag{14}$$

All identities in this article are based on the following principle: define the multisets $C_b$ and $D_b$ as

$$C_b = \bigcup_{0<k<b} \{(bn-k) \cdot b^j, n \geq 1, j \geq 0\} \tag{15}$$

and

$$D_b = \{(bn) \cdot b^j, n \geq 1, j \geq 0\}. \tag{16}$$

For example, in the case $b = 2$, the set $C_2$ collects all integers, and

$$D_2 = \{2, 4^{(2)}, 6, 8^{(3)}, 10, 12^{(2)}, 14, 16^{(4)}, \ldots\}. \tag{17}$$

In the case $b=3$, the set $C_3$ collects all integers, and

$$D_3 = \{3, 6, 9^{(2)}, 12, 15, 18^{(2)}, 21, 24, 27^{(3)}, \ldots\}. \tag{18}$$

Defining the multiset

$$E_b = \{m^{(\nu_b(m))}, m \in \mathbb{N}\}, \tag{19}$$

we compute

$$E_2 = \{m^{(\nu_2(m))}, m \in \mathbb{N}\} = \{2, 4^{(2)}, 6, 8^{(3)}, 10, 12^{(2)}, 14, 16^{(4)}, \ldots\}, \tag{20}$$

and

$$E_3 = \{m^{(\nu_3(m))}, m \in \mathbb{N}\} = \{3, 6, 9^{(2)}, 12, 15, 18^{(2)}, 21, 24, 27^{(3)}, \ldots\}, \tag{21}$$

and it appears that $D_2 = E_2$ and $D_3 = E_3$, while $C_2 = C_3 = \mathbb{N}$.

We notice that the equality $C_b = \mathbb{N}$ is a simple consequence of the fact that each set $\{(bn-k)\cdot b^j, n \geq 1, j \geq 0\}$ is the set of integers having, in their base-$b$ representation, $j$ trailing zeros and their $j+1$st digit equal to $b-k$. Hence these sets form a partition of $\mathbb{N}$.

The previous result is in fact true for every base $b \geq 2$. As will be shown in Section 4, for an arbitrary base $b \geq 2$,

$$C_b = \mathbb{N} \text{ and } D_b = E_b. \tag{22}$$

More precisely, every integer $m \geq 1$ appears once in $C_b$ (i.e., $j = \nu_b(m)$) and $\nu_b(m)$ times in $D_b$ (once for each value of $j$ such that $0 \leq j \leq \nu_b(m)-1$). Notice that the special case $\nu_b(m) = 0$ means that $m$ does not appear in the multiset $D_b$.

As a consequence, for two arbitrary functions $\varphi$ and $\chi$ such that the following (creatively chosen) infinite products exist, we have the general formula (see (52) below)

$$\prod_{j\geq 0, n\geq 1} \Big(\prod_{k=1}^{b-1} a_{(nb-k)\cdot b^j}^{\varphi(j)}\Big) a_{(nb)\cdot b^j}^{\chi(j)} = \prod_{m\geq 1} a_m^{\varphi(\nu_b(m)) + \sum_{k=0}^{\nu_b(m)-1}\chi(k)}, \tag{23}$$

whose validity follows from the equality of the collected exponent of each element $a_m$ on both sides of the identity. Since this identity is structural, i.e., a consequence of the fact that the two multisets of indices $C_b \cup D_b$ and $\mathbb{N} \cup E_b$ coincide, it also translates into a sum form; for an arbitrary sequence $(b_m)$ such that the corresponding sums exist, it is given by

$$\sum_{j\geq 0, n\geq 1} \left(\sum_{k=1}^{b-1} \varphi(j) b_{(nb-k)\cdot b^j} + \chi(j) b_{(nb)\cdot b^j}\right) = \sum_{m\geq 1} \left(\varphi(\nu_b(m)) + \sum_{k=0}^{\nu_b(m)-1} \chi(k)\right) b_m \tag{24}$$

and proven in Section 4 (below). Notice that the structural form of (23) allows us to deduce (24) without the assumption that $b_m = \log(a_m)$. For example, different choices of the functions $\varphi$ and $\chi$, such that

$$\varphi(\nu_b(m)) + \sum_{k=0}^{\nu_b(m)-1} \chi(k) = 1, \tag{25}$$

produce the identities that will be studied in this article.

We close this introduction by noting the following:

1. the product form
$$\prod_{n\in C_b\cup D_b} a_n = \prod_{n\in\mathbb{N}\cup E_b} a_n \tag{26}$$
and the sum form
$$\sum_{n\in C_b\cup D_b} b_n = \sum_{n\in\mathbb{N}\cup E_b} b_n \tag{27}$$
in formulas (23) and (24) respectively can be replaced by any symmetric form, such as, for example,
$$\sum_{\substack{n_1<n_2\\ n_1,n_2\in C_b\cup D_b}} b_{n_1}b_{n_2} = \sum_{\substack{n_1<n_2\\ n_1,n_2\in\mathbb{N}\cup E_b}} b_{n_1}b_{n_2}; \tag{28}$$

2. at a fundamental level, we are effectively introducing a form of multiplicative telescoping (i.e., cancellation between different elements in a product) similar to that noted by Milgram [11, Eq. (2.40)].

# 3 A first generalization

Playing with identity (8) suggested the following generalization.

**Proposition 1.** *For an arbitrary value $q\in\mathbb{C}$ and for an arbitrary sequence $(a_m)$ such that $\prod_{m\geq 1} a_m$ exists, we have*
$$\prod_{j\geq 0,n\geq 1}\left(\frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}^{q-1}}\right)^{q^j} = \prod_{m\geq 1} a_m \tag{29}$$
*and its series version: for an arbitrary sequence $(b_m)$ such that $\sum_{m\geq 1} b_m$ exists,*
$$\sum_{j\geq 0,n\geq 1} q^j b_{(2n-1)\cdot 2^j} - q^j(q-1)b_{(2n)\cdot 2^j} = \sum_{m\geq 1} b_m. \tag{30}$$

A proof of this identity is provided in Section 4 below. A proof in the particular case $b_m = t^m$ can be found in Section 7 below. Specializations of identity (29) are given next.

**Example 2.** For $q=0$, identity (29) reduces to the usual dissection identity
$$\prod_{n\geq 1} a_{2n-1}a_{2n} = \prod_{n\geq 1} a_n. \tag{31}$$

*Proof.* Notice that

$$q^j = \begin{cases} 1, & \text{if } j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

so that the right-hand side reduces to

$$\prod_{j\geq 0, n\geq 1} \left( \frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}^{q-1}} \right)^{q^j} = \prod_{n\geq 1} \frac{a_{(2n-1)}}{a_{2n}^{-1}} = \prod_{n\geq 1} a_{2n-1} a_{2n} = \prod_{n\geq 1} a_n. \tag{32}$$

□

*Remark* 3. The fact that we recover the usual dissection identity for $q = 0$ shows that identity (29) can be considered as a parameterized version of the usual dissection formula.

**Example 4.** The case $q = 1$ of identity (29) produces

$$\prod_{j\geq 1, n\geq 1} a_{(2n-1)\cdot 2^j} = \prod_{n\geq 1} a_{2n}. \tag{33}$$

*Proof.* Start from

$$\begin{aligned} \prod_{m\geq 1} a_m = \prod_{j\geq 0, n\geq 1} \left( \frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}^{q-1}} \right)^{q^j} &= \prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j} \\ &= \prod_{n\geq 1} a_{2n-1} \prod_{j\geq 1, n\geq 1} a_{(2n-1)\cdot 2^j} , \end{aligned} \tag{34}$$

from which we deduce

$$\prod_{j\geq 1, n\geq 1} a_{(2n-1)\cdot 2^j} = \prod_{n\geq 1} a_{2n}. \tag{35}$$

□

The interpretation of this result follows: consider an arbitrary even number $m$. By (13), there is a unique way to write

$$m = p \cdot 2^{\nu_2(m)} \tag{36}$$

with $p$ an odd number: first compute its valuation $\nu_2(m)$ according to (12) and then consider

$$p = \frac{m}{2^{\nu_2(m)}} \tag{37}$$

which by definition is an odd number.

**Example 5.** In the case $q = -1$, (29) produces

$$\prod_{j\geq 0, n\geq 1} \frac{a_{(2n-1)\cdot 2^{2j}} a_{2n\cdot 2^{2j}}^2}{a_{(2n-1)\cdot 2^{2j+1}} a_{(2n)\cdot 2^{2j+1}}^2} = \prod_{m\geq 1} a_m. \tag{38}$$

*Proof.* For $q=-1$, we have

$$\prod_{j\geq 0, n\geq 1}\left(\frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}^{q-1}}\right)^{q^j} = \prod_{j\geq 0, n\geq 1}(a_{(2n-1)\cdot 2^j}a_{(2n)\cdot 2^j}^2)^{(-1)^j}. \tag{39}$$

Separating the terms with even values of $j$ (with a $+1$ exponent) from those with odd values of $j$ (with a $-1$ power) produces the result. □

These results, extended to the base-3 case, follow.

**Corollary 6.** *For all $q\in\mathbb{C}$,*

$$\prod_{n\geq 1, j\geq 0}\left(\frac{a_{(3n-1)\cdot 3^j}a_{(3n-2)\cdot 3^j}}{a_{(3n)\cdot 3^j}^{q-1}}\right)^{q^j} = \prod_{m\geq 1} a_m. \tag{40}$$

*The case $q=1$ produces*

$$\prod_{n\geq 1, j\geq 0} a_{(3n-1)\cdot 3^j}a_{(3n-2)\cdot 3^j} = \prod_{m\geq 1} a_m, \tag{41}$$

*from which we deduce the identity*

$$\prod_{n\geq 1, j\geq 1} a_{(3n-1)\cdot 3^j}a_{(3n-2)\cdot 3^j} = \prod_{m\geq 1} a_{3m}. \tag{42}$$

*The case $q=0$ produces, as previously, the usual multisection identity*

$$\prod_{n\geq 1} a_{3n}a_{3n-1}a_{3n-2} = \prod_{m\geq 1} a_m. \tag{43}$$

The arbitrary base-$b$ case follows.

**Proposition 7.** *For every integer $n\geq 2$ and $q\in\mathbb{C}$, we have*

$$\prod_{n\geq 1, j\geq 0}\left(\frac{a_{(bn-1)\cdot b^j}a_{(bn-2)\cdot b^j}\cdots a_{(bn-(b-1))\cdot b^j}}{a_{(bn)\cdot b^j}^{q-1}}\right)^{q^j} = \prod_{m\geq 1} a_m. \tag{44}$$

*The case $q=1$ produces*

$$\prod_{n\geq 1, j\geq 0} a_{(bn-1)\cdot b^j}a_{(bn-2)\cdot b^j}\cdots a_{(bn-(b-1))\cdot b^j} = \prod_{m\geq 1} a_m, \tag{45}$$

*from which we deduce the identity*

$$\prod_{n\geq 1, j\geq 1} a_{(bn-1)\cdot b^j}a_{(bn-2)\cdot b^j}\cdots a_{(bn-(b-1))\cdot b^j} = \prod_{m\geq 1} a_{bm}. \tag{46}$$

*The case $q=0$ produces, as previously, the usual multisection identity*

$$\prod_{n\geq 1} a_{(bn)\cdot b^j}a_{(bn-1)\cdot b^j}a_{(bn-2)\cdot b^j}\cdots a_{(bn-(b-1))\cdot b^j} = \prod_{m\geq 1} a_m. \tag{47}$$

# 4 A general case

A general version of identity (8) is given next.

**Proposition 8.** *For two functions* $\varphi$ *and* $\chi$*, with* $\nu_2(m)$ *being the 2-adic valuation of* $m$*, and assuming that the infinite products are absolutely convergent, then*

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}^{\varphi(j)} a_{(2n)\cdot 2^j}^{\chi(j)} = \prod_{m\geq 1} a_m^{\varphi(\nu_2(m))+\sum_{k=0}^{\nu_2(m)-1}\chi(k)}, \tag{48}$$

*or, equivalently,*

$$\sum_{j\geq 0, n\geq 1} \varphi(j) b_{(2n-1)\cdot 2^j} + \chi(j) b_{(2n)\cdot 2^j} = \sum_{m\geq 1} (\varphi(\nu_2(m)) + \chi(0) + \cdots + \chi(\nu_2(m)-1)) b_m. \tag{49}$$

*The base-3 case is*

$$\prod_{j\geq 0, n\geq 1} a_{(3n-2)\cdot 3^j}^{\varphi(j)} a_{(3n-1)\cdot 3^j}^{\varphi(j)} a_{(3n)\cdot 3^j}^{\chi(j)} = \prod_{m\geq 1} a_m^{\varphi(\nu_3(m))+\sum_{k=0}^{\nu_3(m)-1}\chi(k)}, \tag{50}$$

*or*

$$\sum_{j\geq 0, n\geq 1} \varphi(j) b_{(3n-2)\cdot 3^j} + \varphi(j) b_{(3n-1)\cdot 3^j} + \chi(j) b_{(3n)\cdot 3^j}$$
$$= \sum_{m\geq 1} (\varphi(\nu_3(m)) + \chi(0) + \cdots + \chi(\nu_3(m)-1)) b_m, \tag{51}$$

*and the arbitrary base-*$b$ *case, with* $b \geq 2$*, is*

$$\prod_{j\geq 0, n\geq 1} (\prod_{k=1}^{b-1} a_{(nb-k)\cdot b^j}^{\varphi(j)}) a_{(nb)\cdot b^j}^{\chi(j)} = \prod_{m\geq 1} a_m^{\varphi(\nu_b(m))+\sum_{k=0}^{\nu_b(m)-1}\chi(k)}, \tag{52}$$

*or equivalently*

$$\sum_{j\geq 0, n\geq 1} \left( \sum_{k=1}^{b-1} \varphi(j) b_{(nb-k)\cdot b^j} + \chi(j) b_{(nb)\cdot b^j} \right) = \sum_{m\geq 1} \left( \varphi(\nu_b(m)) + \sum_{k=0}^{\nu_b(m)-1} \chi(k) \right) b_m. \tag{53}$$

*Proof.* The term $a_m$ appears once in the sequence

$$\left(a_{(2n-1)\cdot 2^j}\right)_{j\geq 0, n\geq 1} \tag{54}$$

for $j = \nu_2(m)$ and appears $\nu_2(m)$ times in the sequence

$$\left(a_{(2n)\cdot 2^j}\right)_{j\geq 0, n\geq 1} \tag{55}$$

for $j = 0, 1, \ldots, \nu_2(m)-1$ successively, and therefore appears with equal cumulative exponent (or coefficient) on both sides of (52) or (53). More generally, the term $a_m$ appears once in the sequence

$$\left(a_{(bn-k)\cdot b^j}\right)_{j\geq 0, n\geq 1} \tag{56}$$

for $j = \nu_b(m)$ and appears $\nu_b(m)$ times in the sequence

$$\left(a_{(bn)\cdot b^j}\right)_{j\geq 0, n\geq 1} \tag{57}$$

for $j = 0, 1, \ldots, \nu_b(m) - 1$ successively. □

**Example 9.** Here are a few specializations of the previous formula:

1. In the case $\varphi(j) = \chi(j) = j$,

$$\prod_{j\geq 0, n\geq 1} (a_{(2n-1)\cdot 2^j} a_{(2n)\cdot 2^j})^j = \prod_{m\geq 1} a_m^{\frac{\nu_2(m)(\nu_2(m)+1)}{2}}, \tag{58}$$

   or equivalently

$$\sum_{j\geq 0, n\geq 1} j(b_{(2n-1)\cdot 2^j} + b_{(2n)\cdot 2^j}) = \sum_{m\geq 1} \frac{\nu_2(m)(\nu_2(m)+1)}{2} b_m. \tag{59}$$

2. In the case $\varphi(j) = j, \chi(j) = 2j$,

$$\prod_{j\geq 0, n\geq 1} (a_{(2n-1)\cdot 2^j} a^2_{(2n)\cdot 2^j})^j = \prod_{m\geq 1} a_m^{\nu_2^2(m)}, \tag{60}$$

   or equivalently

$$\sum_{j\geq 0, n\geq 1} j(b_{(2n-1)\cdot 2^j} + 2b_{(2n)\cdot 2^j}) = \sum_{m\geq 1} \nu_2^2(m) b_m. \tag{61}$$

3. In the case $\varphi(j) = \chi(j) = 1$,

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j} a_{(2n)\cdot 2^j} = \prod_{m\geq 1} a_m^{\nu_2(m)+1}, \tag{62}$$

   or equivalently

$$\sum_{j\geq 0, n\geq 1} b_{(2n-1)\cdot 2^j} + b_{(2n)\cdot 2^j} = \sum_{m\geq 1} (\nu_2(m)+1) b_m. \tag{63}$$

4. The case $\varphi(j) = q^j, \chi(j) = (1-q)q^j$ corresponds to the parameterized identity

$$\prod_{j\geq 0, n\geq 1} \left(\frac{a_{(2n-1)\cdot 2^j}}{a^{q-1}_{(2n)\cdot 2^j}}\right)^{q^j} = \prod_{m\geq 1} a_m, \tag{64}$$

which is (29), and the case $\varphi(j) = 2^j, \chi(j) = -2^j$ corresponds to the original identity

$$\prod_{j\geq 0, n\geq 1} \left(\frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}}\right)^{2^j} = \prod_{m\geq 1} a_m, \tag{65}$$

which is (8).

5. A last case is the choice $\chi(j) = j^{2p}$ for an integer $p \geq 1$ and $\varphi(j) = -\frac{B_{2p+1}(j)}{2p+1}$ with $B_{2p+1}(x)$ the Bernoulli polynomial of degree $2p+1$, which produces the identity

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}^{\frac{B_{2p+1}(j)}{2p+1}} = \prod_{j\geq 0, n\geq 1} a_{(2n)\cdot 2^j}^{j^{2p}}, \tag{66}$$

or

$$\sum_{j\geq 0, n\geq 1} \frac{B_{2p+1}(j)}{2p+1} b_{(2n-1)\cdot 2^j} = \sum_{j\geq 0, n\geq 1} j^{2p} b_{(2n)\cdot 2^j}. \tag{67}$$

**Example 10.** Some more examples follow.

1. The case $\varphi(j) = -j^2 + j + 1$, $\chi(j) = 2j$ produces

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}^{-j^2+j+1} a_{(2n)\cdot 2^j}^{2j} = \prod_{m\geq 1} a_m, \tag{68}$$

or

$$\sum_{n\geq 1, j\geq 0} (-j^2 + j + 1) b_{(2n-1)\cdot 2^j} + (2j) b_{(2n)\cdot 2^j} = \sum_{m\geq 1} b_m; \tag{69}$$

2. The telescoping choice $\varphi(j) = \frac{1}{j+1}$, $\chi(j) = \frac{1}{(j+1)(j+2)}$ produces

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}^{\frac{1}{j+1}} a_{(2n)\cdot 2^j}^{\frac{1}{(j+1)(j+2)}} = \prod_{m\geq 1} a_m, \tag{70}$$

or

$$\sum_{j\geq 0, n\geq 1} \frac{1}{j+1} b_{(2n-1)\cdot 2^j} + \frac{1}{(j+1)(j+2)} b_{(2n)\cdot 2^j} = \sum_{m\geq 1} b_m; \tag{71}$$

3. The base-3 case of the previous identity is

$$\prod_{j\geq 0, n\geq 1} a_{(3n-1)\cdot 3^j}^{\frac{1}{j+1}} a_{(3n-2)\cdot 3^j}^{\frac{1}{j+1}} a_{(3n)\cdot 3^j}^{\frac{1}{(j+1)(j+2)}} = \prod_{m\geq 1} a_m, \tag{72}$$

or

$$\sum_{j\geq 0, n\geq 1} \frac{1}{j+1} b_{(3n-1)\cdot 3^j} + \frac{1}{j+1} b_{(3n-2)\cdot 3^j} + \frac{1}{(j+1)(j+2)} b_{(3n)\cdot 3^j} = \sum_{m\geq 1} b_m; \tag{73}$$

4. The choice $\varphi(j) = 1 - \frac{j(j+3)}{4(j+1)(j+2)}$, $\chi(j) = \frac{1}{(j+1)(j+2)(j+3)}$ produces

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}^{1-\frac{j(j+3)}{4(j+1)(j+2)}} a_{(2n)\cdot 2^j}^{\frac{1}{(j+1)(j+2)(j+3)}} = \prod_{m\geq 1} a_m \, , \tag{74}$$

or

$$\sum_{j\geq 0, n\geq 1} (1 - \frac{j(j+3)}{4(j+1)(j+2)}) b_{(2n-1)\cdot 2^j} + \frac{1}{(j+1)(j+2)(j+3)} b_{(2n)\cdot 2^j} = \sum_{m\geq 1} b_m. \tag{75}$$

*Remark* 11. We close this section with two specializations:

- the specialization $\chi(k) = 0$ produces generating functions for the valuation function $\nu_b(n)$, of the type

$$\prod_{j\geq 0, n\geq 1} \prod_{k=1}^{b-1} a_{(nb-k)\cdot b^j}^{\varphi(j)} = \prod_{m\geq 1} a_m^{\varphi(\nu_b(m))}, \tag{76}$$

or

$$\sum_{j\geq 0, n\geq 1} \varphi(j) \sum_{k=1}^{b-1} b_{(nb-k)\cdot b^j} = \sum_{m\geq 1} \varphi(\nu_b(m)) b_m. \tag{77}$$

For example, in the case $b = 2$,

$$\prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}^{\varphi(j)} = \prod_{m\geq 1} a_m^{\varphi(\nu_2(m))}, \tag{78}$$

or equivalently

$$\sum_{j\geq 0, n\geq 1} \varphi(j) b_{(2n-1)\cdot 2^j} = \sum_{m\geq 1} \varphi(\nu_2(m)) b_m. \tag{79}$$

  – The choice $b_m = t^m$ produces a generating function for the sequence $(\varphi(\nu_2(m)))$ in the form

$$\sum_{j\geq 0} \varphi(j) \frac{t^{2^j}}{1 - t^{2^{j+1}}} = \sum_{m\geq 1} \varphi(\nu_2(m)) t^m; \tag{80}$$

  – The choice $b_m = \frac{1}{m^s}$ produces the identity between Dirichlet series

$$(1 - 2^{-s})\zeta(s) \sum_{j\geq 0} \frac{\varphi(j)}{2^{sj}} = \sum_{m\geq 1} \frac{\varphi(\nu_2(m))}{m^s}. \tag{81}$$

  Choosing $\varphi(j) = \cos(2\pi j x)$ produces

$$\sum_{m\geq 1} \frac{\cos(2\pi x \nu_2(m))}{m^s} = \frac{2^s - 1}{2^{s+1}} \frac{\cos(2\pi x) - 2^s}{\cos(2\pi x) - \frac{2^s + 2^{-s}}{2}} \zeta(s). \tag{82}$$

The specializations $x = \frac{1}{4}$ and $x = \frac{1}{2}$ successively produce, for $s > 1$, the identities

$$\sum_{m\geq 1} \frac{\cos(\frac{\pi}{2}\nu_2(m))}{m^s} = \frac{4^s - 2^s}{4^s + 1}\zeta(s) \tag{83}$$

and

$$\sum_{m\geq 1} \frac{(-1)^{\nu_2(m)}}{m^s} = \frac{2^s - 1}{2^s + 1}\zeta(s). \tag{84}$$

This identity should be compared to the Dirichlet series (see OEIS entry A007814)

$$\sum_{m\geq 1} \frac{\nu_2(m)}{m^s} = \frac{\zeta(s)}{2^s - 1}.$$

- the specialization $\varphi(k) = 0$ produces identities of the type

$$\prod_{j\geq 0, n\geq 1} a_{(bn)\cdot b^j}^{\chi(j)} = \prod_{m\geq 1} a_m^{\sum_{k=0}^{\nu_b(m)-1} \chi(k)}, \tag{85}$$

or

$$\sum_{j\geq 0, n\geq 1} \chi(j) b_{(bn)\cdot b^j} = \sum_{m\geq 1} b_m \sum_{k=0}^{\nu_b(m)-1} \chi(k); \tag{86}$$

for example, if $b = 2$,

$$\prod_{j\geq 0, n\geq 1} a_{(2n)\cdot 2^j}^{\chi(j)} = \prod_{m\geq 1} a_m^{\sum_{k=0}^{\nu_2(m)-1} \chi(k)}, \tag{87}$$

or equivalently

$$\sum_{j\geq 0, n\geq 1} \chi(j) b_{(2n)\cdot 2^j} = \sum_{m\geq 1} b_m \sum_{k=0}^{\nu_2(m)-1} \chi(k). \tag{88}$$

The specialization $b_m = \frac{1}{m^s}$ and $\chi(j) = t^j$ yields the Dirichlet series

$$\sum_{m\geq 1} \frac{t^{\nu_2(m)}}{m^s} = (\frac{2^s - 1}{2^s - t})\zeta(s), \tag{89}$$

of which the special case $t = -1$ reduces to (84) while the case $t = e^{2\pi \imath x}$ reduces to (82).

*Remark* 12. Identity (84) is in fact easy to recover directly from the observation that $\nu_2(m) = 0$ for $m$ odd while $\nu_2(m)$ is even or odd according to $j = 0$ or not in the factorization $m = (2m - 1) \cdot 2^j$. Employing this fact allows the odd terms of the sum in (82) to be evaluated, leading to

$$\sum_{m=1}^{\infty} \frac{\cos(2\pi x \nu_2(2m))}{(2m)^s} = -\frac{(\cos(2\pi x)(2^s - 1) - 1 + 2^{-s})\zeta(s)}{2\cos(2\pi x)2^s - 4^s - 1}. \tag{90}$$

Finally, take the limit $s \to 1$, with $x = 1/6$ where the series converges, to find

$$\sum_{m=1}^{\infty} \frac{\cos(\pi \nu_2(2m)/3)}{m} = \frac{\ln(2)}{3}. \tag{91}$$

# 5 A finite version

A sum or product over a finite range for the index $j$ is stated next.

**Proposition 13.** *For every integer $J \geq 1$, the following identity holds:*

$$\prod_{j \geq J, n \geq 1} a_{(2n-1)\cdot 2^j}^{\varphi(j)} = \prod_{p \geq 1} a_{p \cdot 2^J}^{\varphi(J+\nu_2(p))}. \tag{92}$$

*Proof.* Choosing an integer $J \geq 1$ and replacing the function $\varphi(j)$ with its truncated version

$$\varphi_J(j) = \begin{cases} \varphi(j), & j \geq J; \\ 0, & \text{otherwise.} \end{cases}$$

in

$$\prod_{j \geq 1, n \geq 1} a_{(2n-1)\cdot 2^j}^{\varphi(j)} = \prod_{m \geq 1} a_m^{\varphi(\nu_2(m))}, \tag{93}$$

produces the identity

$$\prod_{j \geq J, n \geq 1} a_{(2n-1)\cdot 2^j}^{\varphi(j)} = \prod_{p \geq 1} a_{p \cdot 2^J}^{\varphi(\nu_2(p \cdot 2^J))}. \tag{94}$$

Indeed, using

$$\prod_{j \geq 0, n \geq 1} a_{(2n-1)\cdot 2^j}^{\varphi_J(j)} = \prod_{m \geq 1} a_m^{\varphi_J(\nu_2(m))} \tag{95}$$

we look for the values of the index $m$ such that $\nu_2(m) \geq J$, which are exactly

$$m = 2^J p, p \in \mathbb{N} \tag{96}$$

so that

$$\prod_{j \geq J, n \geq 1} a_{(2n-1)\cdot 2^j}^{\varphi(j)} = \prod_{p \geq 1} a_{p \cdot 2^J}^{\varphi(\nu_2(p \cdot 2^J))}. \tag{97}$$

Since moreover $\nu_2(p \cdot 2^J) = J + \nu_2(p)$, we deduce the result. □

**Corollary 14.** *The specialization $\varphi(j) = 1$ produces*

$$\prod_{j \geq J, n \geq 1} a_{(2n-1)\cdot 2^j} = \prod_{p \geq 1} a_{p \cdot 2^J}, \tag{98}$$

*or equivalently*

$$\prod_{0\le j<J,n\ge 1} a_{(2n-1)\cdot 2^j} = \prod_{p\ge 1, p\not\equiv 0 \pmod{2^J}} a_p, \tag{99}$$

*or its sum version*

$$\sum_{j\ge J,n\ge 1} b_{(2n-1)\cdot 2^j} = \sum_{p\ge 1, p\not\equiv 0 \pmod{2^J}} b_p. \tag{100}$$

# 6 The case of a double-indexed sequence

**Proposition 15.** *Consider a double-indexed sequence $(a_{p,q})$. We have*

$$\prod_{k\ge 0,m\ge 1}\prod_{j\ge 0,n\ge 1}\left(\frac{a_{(2n-1)\cdot 2^j,(2m-1)\cdot 2^k}a_{(2n)\cdot 2^j,(2m)\cdot 2^k}}{a_{(2n)\cdot 2^j,(2m-1)\cdot 2^k}a_{(2n-1)\cdot 2^j,(2m)\cdot 2^k}}\right)^{2^{j+k}} = \prod_{p\ge 1,q\ge 1} a_{p,q}. \tag{101}$$

*The sum version is, for an arbitrary sequence $(b_{p,q})$,*

$$\sum_{p,q\ge 1} b_{p,q} = \sum_{j,k\ge 0}\sum_{m,n\ge 1} 2^{j+k}(b_{(2n-1)\cdot 2^j,(2m-1)\cdot 2^k} + b_{(2n)\cdot 2^j,(2m)\cdot 2^k} \tag{102}$$
$$- b_{(2n)\cdot 2^j,(2m-1)\cdot 2^k} - b_{(2n-1)\cdot 2^j,(2m)\cdot 2^k}).$$

*Proof.* For a fixed value of $q$ we have

$$\prod_{j\ge 0,n\ge 1}\left(\frac{a_{(2n-1)\cdot 2^j,q}}{a_{(2n)\cdot 2^j,q}}\right)^{2^j} = \prod_{p\ge 1} a_{p,q}. \tag{103}$$

Denote

$$b_q = \prod_{j\ge 0,n\ge 1}\left(\frac{a_{(2n-1)\cdot 2^j,q}}{a_{(2n)\cdot 2^j,q}}\right)^{2^j} \tag{104}$$

so that

$$\prod_{q\ge 1} b_q = \prod_{p\ge 1,q\ge 1} a_{p,q}. \tag{105}$$

Using

$$\prod_{q\ge 1} b_q = \prod_{k\ge 0,m\ge 1}\left(\frac{b_{(2m-1)\cdot 2^k}}{b_{(2m)\cdot 2^k}}\right)^{2^k}, \tag{106}$$

we deduce

$$\prod_{p\ge 1} a_{p,q} = \prod_{k\ge 0,m\ge 1}\left(\frac{\prod_{j\ge 0,n\ge 1}\left(\frac{a_{(2n-1)\cdot 2^j,(2m-1)\cdot 2^k}}{a_{(2n)\cdot 2^j,(2m-1)\cdot 2^k}}\right)^{2^j}}{\prod_{j\ge 0,n\ge 1}\left(\frac{a_{(2n-1)\cdot 2^j,(2m)\cdot 2^k}}{a_{(2n)\cdot 2^j,(2m)\cdot 2^k}}\right)^{2^j}}\right)^{2^k}$$
$$= \prod_{k\ge 0,m\ge 1}\prod_{j\ge 0,n\ge 1}\left(\frac{a_{(2n-1)\cdot 2^j,(2m-1)\cdot 2^k}a_{(2n)\cdot 2^j,(2m)\cdot 2^k}}{a_{(2n)\cdot 2^j,(2m-1)\cdot 2^k}a_{(2n-1)\cdot 2^j,(2m)\cdot 2^k}}\right)^{2^{j+k}}. \tag{107}$$

□

**Corollary 16.** *The two Lambert series*

$$f(q) = \sum_{n\geq 1} n \frac{q^n}{1-q^n} \text{ and } g(q) = \sum_{n\geq 1} n \frac{q^n}{1+q^n} \tag{108}$$

*are related by the formula*

$$f(q) = \sum_{j,k\geq 0} 2^{2j+k} (g(q^{2^{j+k}}) - 4g(q^{2^{j+k+1}})). \tag{109}$$

*Proof.* Writing the function $f$ as the double sum

$$f(q) = \sum_{m,n} n q^{nm}, \tag{110}$$

the dissection formula (102) yields

$$\begin{aligned} f(q) = \sum_{j,k\geq 0} \sum_{n,m\geq 1} & 2^{j+k} ((2n-1)2^j q^{(2n-1)\cdot 2^j \cdot (2m-1)\cdot 2^k} - (2n-1)2^j q^{(2n-1)\cdot 2^j \cdot (2m)\cdot 2^k} \\ & + (2n)2^j q^{(2n)\cdot 2^j \cdot (2m)\cdot 2^k} - (2n)2^j q^{(2n)\cdot 2^j \cdot (2m-1)\cdot 2^k}). \end{aligned} \tag{111}$$

Each of the four sums over $n, m \geq 1$ is computed as follows:

$$\sum_{n,m\geq 1} (2n-1) q^{(2n-1)\cdot 2^j \cdot (2m-1)\cdot 2^k} = \sum_{n\geq 1} (2n-1) \frac{q^{(2n-1)2^{j+k}}}{1-q^{(2n-1)2^{j+k+1}}}, \tag{112}$$

$$\sum_{n,m\geq 1} (2n-1) q^{(2n-1)\cdot 2^j \cdot (2m)\cdot 2^k} = \sum_{n\geq 1} (2n-1) \frac{q^{(2n-1)2^{j+k+1}}}{1-q^{(2n-1)2^{j+k+1}}}, \tag{113}$$

$$\sum_{n,m\geq 1} (2n) q^{(2n)\cdot 2^j \cdot (2m)\cdot 2^k} = \sum_{n\geq 1} 2n \frac{q^{n\cdot 2^{j+k+2}}}{1-q^{n\cdot 2^{j+k+2}}}, \tag{114}$$

$$\sum_{n,m\geq 1} (2n) q^{(2n)\cdot 2^j \cdot (2m-1)\cdot 2^k} = \sum_{n\geq 1} 2n \frac{q^{n\cdot 2^{j+k+1}}}{1-q^{n\cdot 2^{j+k+2}}}. \tag{115}$$

With the notation $q_{j,k} = q^{2^{j+k}}$, this yields

$$\begin{aligned} f(q) &= \sum_{j,k\geq 0} 2^{2j+k} \sum_{n\geq 1} ((2n-1) \frac{q_{j,k}^{2n-1} - q_{j,k}^{2(2n-1)}}{1-q_{j,k}^{2(2n-1)}} + (2n) \frac{q_{j,k}^{4n} - q_{j,k}^{2n}}{1-q_{j,k}^{4n}}) \\ &= \sum_{j,k\geq 0} 2^{2j+k} \sum_{n\geq 1} ((2n-1) \frac{q_{j,k}^{2n-1}}{1+q_{j,k}^{2n-1}} - (2n) \frac{q_{j,k}^{2n}}{1+q_{j,k}^{2n}}). \end{aligned} \tag{116}$$

This inner sum coincides with the difference between the odd part $h_o$ and the even part $h_e$ of the function $h(q) = \sum_{n\geq 1} n \frac{q_{j,k}^n}{1+q_{j,k}^n}$, also equal to $h(q) - 2h_e(q)$, i.e., to

$$\sum_{n\geq 1}(n\frac{q_{j,k}^n}{1+q_{j,k}^n} - 4n\frac{q_{j,k}^{2n}}{1+q_{j,k}^{2n}}), \tag{117}$$

so that finally

$$\begin{aligned} f(q) &= \sum_{j,k\geq 0} 2^{2j+k} \sum_{n\geq 1}(n\frac{q^{n\cdot 2^{j+k}}}{1+q^{n\cdot 2^{j+k}}} - 4n\frac{q^{n\cdot 2^{j+k+1}}}{1+q^{n\cdot 2^{j+k+1}}}) \\ &= \sum_{j,k\geq 0} 2^{2j+k}(g(q^{2^{j+k}}) - 4g(q^{2^{j+k+1}})). \end{aligned} \tag{118}$$

□

We notice that the previous result can be extended in a straightforward way to the following identity; we skip the details.

**Corollary 17.** *For $\mu \geq 1$, the two Lambert series*

$$f_\mu(q) = \sum_{n\geq 1} n^\mu \frac{q^n}{1-q^n} \text{ and } g_\mu(q) = \sum_{n\geq 1} n^\mu \frac{q^n}{1+q^n} \tag{119}$$

*are related by the formula*

$$f_\mu(q) = \sum_{j,k\geq 0} 2^{2j+k}(g_\mu(q^{2^{j+k}}) - 2^{\mu+1}g_\mu(q^{2^{j+k+1}})). \tag{120}$$

# 7 A generating functional approach

## 7.1 The base-2 case

### 7.1.1 Two proofs of Teixeira's identity

The series version of (8) applied to the case $a_m = t^m$ is

$$\sum_{m\geq 1} t^m = \sum_{k\geq 0, n\geq 1} 2^k t^{(2n-1)2^k} - 2^k t^{(2n)2^k}. \tag{121}$$

Summing over $n$ in the right hand-side and using the simplification

$$\sum_{k\geq 0} 2^k \frac{t^{2^k}(1-t^{2^k})}{1-t^{2^{k+1}}} = \sum_{k\geq 0} 2^k \frac{t^{2^k}}{1+t^{2^k}} \tag{122}$$

produces

$$\frac{t}{1-t} = \sum_{k\geq 0} 2^k \frac{t^{2^k}}{1+t^{2^k}}. \tag{123}$$

This identity is well-known and appears as Problem 10 of Chapter II in Comtet's book [7, page 118], where the result is attributed to F. G. Teixeira [19].

### First proof

A simple proof of (123) follows: define

$$\varphi(t) = \frac{t}{1-t}, \tag{124}$$

notice that this function satisfies

$$\varphi(t) + \varphi(-t) = 2\varphi(t^2), \tag{125}$$

and iterate this identity (see Milgram [11, Appendix A] for convergence details).

### Second proof

Another proof appeals to the theory of Lambert series. First, recall the definition of the Dirichlet eta function

$$\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s). \tag{126}$$

Then consider the following result provided by J. M. and P. B. Borwein [3, Problem 11, p. 300]:

**Proposition 18.** *Given two sequences* $(a_n)$ *and* $(b_n)$*,*

$$\eta(s) \sum_{n\geq 1} \frac{a_n}{n^s} = \sum_{n\geq 1} \frac{b_n}{n^s} \tag{127}$$

*if and only if*

$$\sum_{n\geq 1} a_n \frac{x^n}{1+x^n} = \sum_{n\geq 1} b_n x^n. \tag{128}$$

Let us apply this result to the sequence $(a_n)$ defined by

$$a_n = \begin{cases} 2^l, & \text{if } n = 2^l; \\ 0, & \text{otherwise.} \end{cases}$$

to obtain, on the left-hand side of (128),

$$\sum_{n\geq 1} a_n \frac{x^n}{1+x^n} = \sum_{l\geq 1} 2^l \frac{x^{2^l}}{1+x^{2^l}} \tag{129}$$

and, on the left-hand side of (127),

$$\eta(s)\sum_{n\geq 1}\frac{a_n}{n^s} = \eta(s)\sum_{l\geq 0}\frac{1}{2^{l(s-1)}} = \zeta(s). \tag{130}$$

We deduce from (127) that

$$\sum_{n\geq 1}\frac{b_n}{n^s} = \zeta(s), \tag{131}$$

so that $b_n = 1, n \geq 1$ and, as a consequence, we obtain Teixeira's identity (123).

A simple interpretation of Teixeira's identity (123) follows:

$$\frac{t}{1-t} = \sum_{n\geq 1} t^n \tag{132}$$

is the generating function of the sequence $(1, 1, \ldots)$, each integer $n$ being counted with a unit weight. The first term ($k = 0$) in the right-hand side of Teixeira's identity (123) is

$$\frac{t}{1+t} = \sum_{n\geq 1}(-1)^n t^n \tag{133}$$

so that all even integers are weighted by $+1$ but the odd ones are weighted by $(-1)$. The extra terms

$$\sum_{k\geq 1} 2^k \frac{t^{k^j}}{1+t^{2^k}} \tag{134}$$

will correct these negative weights so that the final sum has all positive unit weights. More precisely:

$$\begin{aligned}\frac{t}{1+t} &= t - t^2 + t^3 - t^4 + t^5 - t^6 + t^7 - t^8 + t^9 - t^{10} + t^{11} - t^{12}\\ &\quad + t^{13} - t^{14} + t^{15} - t^{16} + t^{17} - t^{18} + t^{19} - t^{20} + O(t^{21}),\end{aligned} \tag{135}$$

$$\begin{aligned}\frac{t}{1+t} + 2\frac{t^2}{1+t^2} &= t + t^2 + t^3 - 3t^4 + t^5 + t^6 + t^7 - 3t^8 + t^9 + t^{10} + t^{11} - 3t^{12}\\ &\quad + t^{13} + t^{14} + t^{15} - 3t^{16} + t^{17} + t^{18} + t^{19} - 3t^{20} + O(t^{21}),\end{aligned} \tag{136}$$

$$\frac{t}{1+t} + 2\frac{t^2}{1+t^2} + 4\frac{t^4}{1+t^4} = t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 - 7t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{15} - 7t^{16} + t^{17} + t^{18} + t^{19} + t^{20} + O(t^{21}), \quad (137)$$

so that all powers of $t$ in

$$\sum_{j=0}^{J} 2^j \frac{t^{2^j}}{1+t^{2^j}} \quad (138)$$

have unit weight except, those $t$ with power $k2^{J+1}, k \geq 1$ that have weight $1 - 2^{J+1}$. As $J \to \infty$, only terms with unit weight remain.

#### 7.1.2 A generalization of Teixeira's identity

Now consider, for an arbitrary real number $q$ such that $|q| < 1$, the functions

$$\varphi(z) = \frac{z}{1-z}, \ \chi(z) = \frac{z - (q-1)z^2}{1-z^2}; \quad (139)$$

they satisfy the identity

$$\varphi(z) - \chi(z) = q\varphi(z^2). \quad (140)$$

We deduce

$$\varphi(z) = \chi(z) + q\varphi(z^2) = \chi(z) + q\chi(z^2) + q^2\varphi(z^4) = \cdots \quad (141)$$

so that

$$\frac{z}{1-z} = \sum_{k\geq 0} q^k \chi(z^{2^k}) = \sum_{k\geq 0} q^k \frac{z^{2^k} - (q-1)z^{2^{k+1}}}{1 - z^{2^{k+1}}} = \sum_{k\geq 0, n\geq 1} q^k (z^{(2n-1)\cdot 2^k} - (q-1)z^{(2n)\cdot 2^k}), \quad (142)$$

which is the $b_n = z^n$ version of the generalized identity

$$\sum_{m\geq 1} b_m = \sum_{j\geq 0, n\geq 1} q^j (b_{(2n-1)\cdot 2^j} - (q-1)b_{(2n)\cdot 2^j}). \quad (143)$$

### 7.2 The arbitrary base-$b$ case

Consider the functions

$$\varphi(z) = \frac{z}{1-z} \text{ and } \chi(z) = \frac{z + z^2 + \cdots + z^{b-1} - (q-1)z^b}{1-z^b}; \quad (144)$$

they satisfy the identity

$$\varphi(z) - \chi(z) = q\varphi(z^b), \quad (145)$$

and we deduce by iterating

$$\varphi(z) = \chi(z) + q\chi(z^b) + q^2\chi(z^{b^2}) + \cdots$$
$$= \sum_{j\geq 0} q^j \chi(z^{b^j}), \tag{146}$$

so that

$$\frac{z}{1-z} = \sum_{j\geq 0} q^j \frac{z^{b^j}}{1-z^{b^{j+1}}} + q^j \frac{z^{2\cdot b^j}}{1-z^{b^{j+1}}} + \cdots + q^j \frac{z^{(b-1)\cdot b^j}}{1-z^{b^{j+1}}} - (q-1)q^j \frac{z^{b\cdot b^j}}{1-z^{b^{j+1}}}$$
$$= \sum_{j\geq 0, n\geq 1} q^j z^{(bn-1)\cdot b^j} + q^j z^{(bn-2)\cdot b^j} + \cdots + q^j z^{(bn-(b-1))\cdot b^j} - (q-1)q^j z^{(bn)\cdot b^j}. \tag{147}$$

# 8 A q-calculus application

This section produces a $q$-calculus application of our main identity.

**Proposition 19.** *With the* $q$-Pochhammer *symbol*

$$(a;q)_\infty = \prod_{k\geq 0}(1-aq^k), \tag{148}$$

*we have the identities, for* $q \in \mathbb{C}$ *such that* $|q| < 1$*,*

$$\prod_{j\geq 0}\left(\frac{(q^{2^j};q^{2^{j+1}})_\infty}{(q^{2^{j+1}};q^{2^{j+1}})_\infty}\right)^{2^j} = (q;q)_\infty, \tag{149}$$

*and more generally, for every complex number* $p$*, we have*

$$\prod_{j\geq 0}\left(\frac{(q^{2^j};q^{2^{j+1}})_\infty}{(q^{2^{j+1}};q^{2^{j+1}})_\infty^{p-1}}\right)^{p^j} = (q;q)_\infty \tag{150}$$

*and*

$$\prod_{j\geq 0}\left(\frac{(q^{3^j};q^{3^{j+1}})_\infty (q^{2\cdot 3^j};q^{3^{j+1}})_\infty}{(q^{3^{j+1}};q^{3^{j+1}})_\infty^{p-1}}\right)^{p^j} = (q;q)_\infty. \tag{151}$$

*The special case* $p = 1$ *produces respectively*

$$\prod_{j\geq 0}(q^{2^j};q^{2^{j+1}})_\infty = (q;q)_\infty \tag{152}$$

*and*

$$\prod_{j\geq 0}(q^{3^j};q^{3^{j+1}})_\infty (q^{2\cdot 3^j};q^{3^{j+1}})_\infty = (q;q)_\infty. \tag{153}$$

*Proof.* In the identity

$$\prod_{n\geq 1, j\geq 0} \left( \frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}} \right)^{2^j} = \prod_{n\geq 1} a_n, \tag{154}$$

take $a_n = 1 - q^n$ so that the right-hand side is $(q;q)_\infty$. Moreover,

$$\prod_{n\geq 1, j\geq 0} (1 - q^{(2n-1)\cdot 2^j}) = \prod_{n\geq 0, j\geq 0} (1 - q^{(2n+1)\cdot 2^j}) = \prod_{j\geq 0} (q^{2^j}; q^{2^{j+1}})_\infty, \tag{155}$$

and

$$\prod_{n\geq 1, j\geq 0} (1 - q^{(2n)\cdot 2^j}) = \prod_{n\geq 0, j\geq 0} (1 - q^{(2n+2)\cdot 2^j}) = \prod_{j\geq 0} (q^{2^{j+1}}; q^{2^{j+1}})_\infty, \tag{156}$$

so that

$$\prod_{j\geq 0} \left( \frac{(q^{2^j}; q^{2^{j+1}})_\infty}{(q^{2^{j+1}}; q^{2^{j+1}})_\infty} \right)^{2^j} = (q;q)_\infty. \tag{157}$$

In the more general case

$$\prod_{n\geq 1, j\geq 0} \left( \frac{a_{(2n-1)\cdot 2^j}}{a^{p-1}_{(2n)\cdot 2^j}} \right)^{p^j} = \prod_{n\geq 1} a_n, \tag{158}$$

we deduce

$$\prod_{j\geq 0} \left( \frac{(q^{2^j}; q^{2^{j+1}})_\infty}{(q^{2^{j+1}}; q^{2^{j+1}})^{p-1}_\infty} \right)^{p^j} = (q;q)_\infty. \tag{159}$$

For example, $p = 3$ produces

$$\prod_{j\geq 0} \left( \frac{(q^{2^j}; q^{2^{j+1}})_\infty}{(q^{2^{j+1}}; q^{2^{j+1}})^{2}_\infty} \right)^{3^j} = (q;q)_\infty. \tag{160}$$

In the base-3 case, using

$$\prod_{n\geq 1, j\geq 0} \left( \frac{a_{(3n-1)\cdot 3^j} a_{(3n-2)\cdot 3^j}}{a^{p-1}_{(3n)\cdot 3^j}} \right)^{p^j} = \prod_{n\geq 1} a_n, \tag{161}$$

we have

$$\prod_{n\geq 1, j\geq 0} (1 - q^{(3n-1)\cdot 3^j}) = \prod_{n\geq 0, j\geq 0} (1 - q^{(3n+2)\cdot 3^j}) = \prod_{j\geq 0} (q^{2\cdot 3^j}; q^{3^{j+1}})_\infty, \tag{162}$$

and

$$\prod_{n\geq 1, j\geq 0} (1 - q^{(3n-2)\cdot 3^j}) = \prod_{n\geq 0, j\geq 0} (1 - q^{(3n+1)\cdot 3^j}) = \prod_{j\geq 0} (q^{3^j}; q^{3^{j+1}})_\infty, \tag{163}$$

and

$$\prod_{n\geq 1, j\geq 0}(1-q^{(3n).3^j}) = \prod_{n\geq 0, j\geq 0}(1-q^{(3n+3)\cdot 3^j}) = \prod_{j\geq 0}(q^{3^{j+1}};q^{3^{j+1}})_\infty, \tag{164}$$

so that

$$\prod_{j\geq 0}(\frac{(q^{3^j};q^{3^{j+1}})_\infty (q^{2\cdot 3^j};q^{3^{j+1}})_\infty}{(q^{3^{j+1}};q^{3^{j+1}})_\infty^{p-1}})^{p^j} = (q;q)_\infty. \tag{165}$$

□

This result can be extended to the more general case of the $q$-Pochhammer symbol (148) by considering the sequence

$$a_n = 1 - aq^{n-1}, n \geq 1. \tag{166}$$

**Proposition 20.** *The following multisection formula holds:*

$$\prod_{j\geq 0}\left(\frac{(aq^{2^j-1};q^{2^{j+1}})_\infty}{(aq^{2^{j+1}-1};q^{2^{j+1}})_\infty}\right)^{2^j} = (a;q)_\infty. \tag{167}$$

*Proof.* We have

$$\begin{aligned}\prod_{j\geq 0, n\geq 1}\left(\frac{a_{(2n-1)\cdot 2^j}}{a_{(2n)\cdot 2^j}}\right)^{2^j} &= \prod_{j\geq 0, n\geq 1}(\frac{1-aq^{(2n-1)\cdot 2^j-1}}{1-aq^{(2n)\cdot 2^j-1}})^{2^j}\\ &= \prod_{j\geq 0}\left(\frac{\prod_{n\geq 0}(1-aq^{(2n+1)\cdot 2^j-1})}{\prod_{n\geq 0}(1-aq^{(2n+2)\cdot 2^j-1})}\right)^{2^j}\\ &= \prod_{j\geq 0}\left(\frac{\prod_{n\geq 0}(1-aq^{2^j-1}q^{n\cdot 2^{j+1}})}{\prod_{n\geq 0}(1-aq^{2^{j+1}-1}q^{n\cdot 2^{j+1}})}\right)^{2^j}\\ &= \prod_{j\geq 0}\left(\frac{(aq^{2^j-1};q^{2^{j+1}})_\infty}{(aq^{2^{j+1}-1};q^{2^{j+1}})_\infty}\right)^{2^j}.\end{aligned} \tag{168}$$

□

*Remark* 21. A combinatorial interpretation of identity (152) follows: rewrite (152) as

$$\frac{1}{(q;q)_\infty} = \prod_{j\geq 0}\frac{1}{(q^{2^j};q^{2^{j+1}})_\infty}. \tag{169}$$

The left-hand side is the generating function for the number of partitions $p(n)$ of the integer $n$. More generally, the infinite product

$$\prod_{i\in I}\frac{1}{1-q^i} \tag{170}$$

is the generating function for the number of partitions $p_I(n)$ of $n$ with parts belonging to the set $I$. Hence

$$\frac{1}{(q^{2^j};q^{2^{j+1}})_\infty} = \prod_{i\in I_j} \frac{1}{1-q^i} \tag{171}$$

is the generating function for the number of partitions of $n$ with parts in the set $I_j = \{2^j + \mathbb{N}\cdot 2^{j+1}, j\ge 0\}$. It is easy to check that the set of subsets $I_j, j\ge 0$ forms a partition of $\mathbb{N}$, i.e.,

$$\bigcup_{j\ge 0} I_j = \mathbb{N} \text{ and } I_j \cap I_k = \varnothing, j\ne k. \tag{172}$$

In fact, the partition $\cup_{j\ge 0} I_j = \mathbb{N}$ has an elementary interpretation in base 2: since $I_j = \{n : \nu_2(n) = j\}$, this partition decomposes the set of integers $n$ according to their 2-adic valuation.

# 9 Applications

We begin with a simple application of example (29). Let

$$a(m) = 1 + x^2/m^2, \tag{173}$$

so that

$$\prod_{j\ge 0,n\ge 1} \left(\frac{a((2n-1)\cdot 2^j)}{(a(2n)\cdot 2^j)^{q-1}}\right)^{q^j} = \prod_{m=1}^{\infty} (1+x^2/m^2) = \sinh(\pi x)/x\pi, \tag{174}$$

which, after substitution in the series equivalent case (30), becomes

$$\sum_{j\ge 0,n\ge 1} q^j(\ln\left(1+\frac{x^2}{(2n-1)^2 2^{2j}}\right) - (q-1)\ln\left(1+\frac{x^2}{2^{2j+2}n^2}\right)) = \ln\left(\frac{\sinh(\pi x)}{x\pi}\right). \tag{175}$$

## 9.1 By differentiation

It is interesting to differentiate (175) once with respect to $x$, and, since there is no overall $q$ dependence, letting $q=1$ solves the double summation

$$\sum_{j\ge 0,n\ge 1} \frac{1}{n(n-1)2^{2+2j} + x^2 + 4^j} = \frac{\pi x\coth(\pi x) - 1}{2x^2}. \tag{176}$$

However, since the inner sum is known [10, 22], that is,

$$\sum_{n=1}^{\infty} \frac{1}{n(n-1)2^{2+2j} + x^2 + 4^j} = \frac{\pi \tanh(\pi x/2^{1+j})}{(2^{j+2}x)}, \tag{177}$$

we eventually reproduce the (telescoping-based) identity listed by Hansen [9, Eq. (43.6.4)]

$$\sum_{j=0}^{\infty} \frac{\tanh(\frac{x}{2^{1+j}})}{2^j} = 2\frac{x\coth(x)-1}{x}. \tag{178}$$

Continuing, by differentiating (175) with respect to $q$, we obtain

$$\sum_{j\geq 0, n\geq 1} j\ln\left(1+\frac{x}{2^{2j}(2n-1)^2}\right) = \sum_{j\geq 0, n\geq 1} \ln\left(1+\frac{x}{n^2 2^{2j+2}}\right) \tag{179}$$

with $q=1$. Again, differentiating (179) with respect to $x$ yields the transformation

$$\sum_{j=1}^{\infty} \frac{j}{2^j}\tanh\left(\frac{x}{2^{1+j}}\right) = \frac{2}{x}\sum_{j=0}^{\infty}\left(\frac{x}{2^{1+j}}\coth\left(\frac{x}{2^{1+j}}\right)-1\right). \tag{180}$$

## 9.2 By power series expansion

In the following, we make extensive use of the identity listed by Dieckmann [8]

$$\prod_{k=1}^{\infty}\left(1+\left(\frac{x}{k+b}\right)^n\right) = \frac{\Gamma(1+b)^n}{b^n+x^n}\prod_{k=1}^{n}\frac{1}{\Gamma(b-xe^{\imath\pi(2k+1)/n})}, \tag{181}$$

valid for $x\in\mathbb{C}$. See also Prudnikov et al. [15, Eq. (20]. Consider the case

$$a(m) = 1 - x^k/m^k, \qquad m>1, \tag{182}$$

and recall, for example, the simple identity

$$\ln\left(1-\frac{x^k}{((3n-2)3^j)^k}\right) = -\sum_{p=1}^{\infty}\frac{1}{p}\left(\frac{x^k}{((3n-2)3^j)^k}\right)^p, \qquad |x|<3. \tag{183}$$

From (42) and (183) we have, after the (convergent) sums are reordered,

$$\begin{aligned}&\sum_{j\geq 1, n\geq 1}(\ln(a((3n-1)\cdot 3^j))+\ln(a((3n-2)\cdot 3^j))) \\ &= \sum_{p\geq 1, j\geq 1}\frac{(x/3^j)^{kp}}{p}\sum_{n=1}^{\infty}\left(\left(\frac{1}{(3n-1)^k}\right)^p+\left(\frac{1}{(3n-2)^k}\right)^p\right), \quad |x|<3.\end{aligned} \tag{184}$$

The sum indexed by $j$ is trivially evaluated, and the sum indexed by $n$ is recognizable as a series representation of a special case of the Hurwitz Zeta function $\zeta(s,a)\equiv\sum_{n=0}^{\infty}1/(n+a)^s$, leading to the identity

$$\sum_{p=1}^{\infty}\frac{x^{kp}}{(3^{kp}-1)p}(\zeta(kp,\frac{2}{3})+\zeta(kp,\frac{1}{3})) = \ln(-x^k\prod_{j=0}^{k-1}\Gamma(-xe^{2\pi\imath j/k})), \quad |x|<1, \tag{185}$$

where the right-hand side arises from (181), and we have replaced $x := 3x$. Because $k$ and $p$ are both integers and $k > 1$, (185) can also be rewritten using [14, Eq. (25.11.12)] as

$$\sum_{p=1}^{\infty} \frac{(\psi(kp-1,\frac{2}{3})+\psi(kp-1,\frac{1}{3}))(-x)^{kp}}{(3^{kp}-1)\Gamma(kp+1)} = \frac{1}{k}\ln(-x^k \prod_{j=0}^{k-1}\Gamma(-xe^{2\imath\pi j/k})), \qquad |x|<1, \quad (186)$$

where $\psi(n,x)$ is the polygamma function. For the case (185) using $k=2$, we find

$$\sum_{p=1}^{\infty} \frac{x^{2p}(\psi(2p-1,\frac{2}{3})+\psi(2p-1,\frac{1}{3}))}{(3^{2p}-1)\Gamma(2p+1)} = \frac{1}{2}\ln\left(\frac{x\pi}{\sin(\pi x)}\right), \qquad |x|<1, \quad (187)$$

for $k=3$ we obtain

$$\sum_{p=1}^{\infty} \frac{(\psi(3p-1,\frac{2}{3})+\psi(3p-1,\frac{1}{3}))(-1)^p x^{3p}}{(3^{3p}-1)\Gamma(3p+1)} = \frac{1}{3}\ln(x^2\left|\Gamma(-\frac{\imath\sqrt{3}x}{2}+\frac{x}{2})\right|^2\Gamma(1-x)), \quad (188)$$

and by setting $x := \imath x$ and $k := 2k$ in (186) we discover

$$\sum_{p=1}^{\infty} \frac{(-1)^p x^{2kp}(\psi(2kp-1,\frac{2}{3})+\psi(2kp-1,\frac{1}{3}))}{\Gamma(2kp+1)(3^{2kp}-1)} = \frac{\ln(x^{2k}\prod_{j=0}^{2k-1}\Gamma(-\imath x e^{i\pi j/k}))}{2k}, \quad (189)$$

corresponding to the related case

$$a(m) = 1 + x^k/m^k, \qquad m > 1. \quad (190)$$

Similarly let $x := \imath x$ and $k := 2k+1$, to find

$$\sum_{p=1}^{\infty} \frac{(\psi((2k+1)p-1,\frac{2}{3})+\psi((2k+1)p-1,\frac{1}{3}))x^{(2k+1)p}e^{-\frac{\imath\pi p}{2}}(-1)^{kp}}{(3^{(2k+1)p}-1)\Gamma((2k+1)p+1)} \quad (191)$$

$$= \frac{\ln(-\imath(-1)^k x^{2k+1}\prod_{j=0}^{\infty}\Gamma(-\imath x e^{\frac{2\imath\pi j}{2k+1}}))}{2k+1}.$$

Choosing $k=1$ in (189), or $k=0$ in the real part of (191), gives

$$\sum_{p=1}^{\infty} \frac{(-1)^p x^{2p}(\psi(2p-1,\frac{2}{3})+\psi(2p-1,\frac{1}{3}))}{(3^{2p}-1)\Gamma(2p+1)} = \frac{1}{2}\ln\left(\frac{x\pi}{\sinh(\pi x)}\right). \quad (192)$$

## 9.3 Variation

We continue with another simple example that utilizes the methods applied above. Let

$$a(m) = 1/(1 + x^s/m^s). \quad (193)$$

Then, according to (8),

$$\prod_{j=0}^{\infty}\left(\prod_{n=1}^{\infty}(1+\frac{x^s}{(2^{j+1})^s n^s})/(1+\frac{x^s}{(2^{j+1})^s(n-\frac{1}{2})^s})\right)^{2^j}=\prod_{m=1}^{\infty}\frac{1}{1+\frac{x^s}{m^s}}, \tag{194}$$

a result that can be tested numerically for $s \geq 2$. After identifying $s$ in the above with $n$ in (181), we find the general identity

$$\prod_{j=0}^{\infty}\left(\frac{(1+(-\frac{2^j}{x})^n)}{\pi^{\frac{n}{2}}}\prod_{k=1}^{n}\frac{\Gamma(-\frac{1}{2}-\frac{x}{2^{j+1}}e^{\imath\pi(2k+1)/n})}{\Gamma(-\frac{x}{2^{j+1}}e^{\imath\pi(2k+1)/n})}\right)^{2^j}=\prod_{m=1}^{\infty}\frac{1}{1+x^n/m^n}. \tag{195}$$

If $n=2$ in (195), after some simplification involving [14, Eqs. (5.4.3) and (5.4.4)], we arrive at the original curious and provocative result (7). In the case that $n=4$ and $x \in \mathbb{R}$, after further simplification and the redefinition $x := 2\sqrt{2}x/\pi$, we find

$$\prod_{j=1}^{\infty}\left(\frac{2^{2j}}{2x^2}\left|\tan\left(\frac{(1+\imath)x}{2^j}\right)\right|^2\right)^{2^j}=\frac{16x^4}{(\cosh(2x)-\cos(2x))^2}. \tag{196}$$

### 9.4 A sum involving $\zeta(k)$

With the slight variation $a(m)=1+x^k/m^k$ applied to (30) using $q=1$ and analyzed as above, we find the identities

$$\sum_{j=1}^{\infty}\frac{(-1)^{j+1}x^{2jk}\zeta(2jk)}{j}=\ln(\frac{(-\imath)^k}{\pi^k x^k}\prod_{j=1}^{k}\sin((-1)^{\frac{2j-1}{2k}}\pi x)) \tag{197}$$

and

$$\sum_{j=1}^{\infty}\frac{(-1)^{j+1}x^{(2k+1)j}\zeta((2k+1)j)}{j}$$
$$=\ln(\frac{1}{\Gamma(x+1)}\prod_{j=1}^{2k}\frac{1}{\pi}\sin(\pi(-1)^{\frac{j(2+2k)}{2k+1}}x)\Gamma((-1)^{\frac{j(2+2k)}{2k+1}+1}x)), \tag{198}$$

where the right-hand sides of both are available from (181). In the case $k=1$, (198) reduces to

$$\sum_{k=1}^{\infty}\frac{x^k\zeta(3k)}{k}=\ln\left(\Gamma(1-x^{\frac{1}{3}})\Gamma(1+\frac{x^{\frac{1}{3}}(1+\imath\sqrt{3})}{2})\Gamma(1+\frac{x^{\frac{1}{3}}(1-\imath\sqrt{3})}{2})\right), \qquad x<1, \tag{199}$$

an identity known to Mathematica. See also H. M. Srivastava and J. Choi [18, section 3.4].

## 9.5 An infinite product study

Consider the case

$$a(m) = 1 + e^{-\pi(2m+1)} \tag{200}$$

listed by Hansen [9, Eq. (89.21.4)] as

$$\prod_{m=1}^{\infty}(1 + e^{-\pi(2m+1)}) = \frac{2^{\frac{1}{4}}e^{-\frac{\pi}{24}}}{1 + e^{-\pi}}. \tag{201}$$

Apply (200) to (30) with $q = 1$ (see Example 4) to obtain the additive multisection

$$\sum_{j\geq 0,k\geq 1} \frac{e^{-k\pi}(-1)^k}{k\sinh(2k\pi 2^j)} = -2\ln\left(\frac{2^{\frac{1}{4}}e^{-\pi/24}}{1 + e^{-\pi}}\right) \tag{202}$$

after expanding the logarithmic terms in analogy to (183). The outer sum (over $j$) can be evaluated by applying the Hansen's listed identity [9, Eq. (25.1.1)] after setting $x := \imath x$ and evaluating the limit $n \to \infty$ in that identity, to yield

$$\sum_{j=0}^{\infty} \frac{1}{\sinh(2k\pi 2^j)} = \coth(2k\pi) - 1 + \operatorname{csch}(2k\pi), \tag{203}$$

in which case (202) reduces, after some simplification, to

$$\sum_{k=1}^{\infty} \frac{(-1)^k e^{-k\pi}\coth(k\pi)}{k} = \ln\left(\frac{(1 + e^{-\pi})e^{\pi/12}}{\sqrt{2}}\right). \tag{204}$$

Comparing (204) with a naive expansion of $\ln(1 + e^{-\pi(2m+1)})$ in (201), verifies the simple transformation

$$\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2k\pi}}{k\sinh(k\pi)} = \sum_{k=1}^{\infty} \frac{(-1)^k e^{-k\pi}\coth(k\pi)}{k} + \ln(1 + e^{-\pi}), \tag{205}$$

so that, comparing (204) and (205) identifies

$$\sum_{k=1}^{\infty} \frac{(-1)^k e^{-2k\pi}}{k\sinh(k\pi)} = \ln\left(\frac{(1 + e^{-\pi})^2 e^{\pi/12}}{\sqrt{2}}\right). \tag{206}$$

Similarly, letting $q = 2$ in (30) leads to

$$\sum_{j=0}^{\infty} 2^j \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{n=1}^{\infty}(-e^{-k(2(2n-1)2^j+1)\pi} + e^{-k(4n2^j+1)\pi}) = \ln\left(\frac{2^{\frac{1}{4}}e^{-\frac{\pi}{24}}}{1 + e^{-\pi}}\right). \tag{207}$$

The innermost sum (over $n$) can be written in terms of hyperbolic functions, eventually producing the double sum identity

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{j=0}^{\infty}\frac{2^j e^{-k\pi(2^j+1)}}{\cosh(k\pi 2^j)}=\ln\left(\frac{(1+e^{-\pi})^2 e^{\pi/12}}{\sqrt{2}}\right), \tag{208}$$

after transposing the two sums. Noting the equality of the right-hand sides of (206) and (208), suggests that the inner sum of (208) (over $j$) equates to the equivalent terms in the summand of (206), i.e.,

$$\sum_{j=0}^{\infty}\frac{2^j e^{-k\pi(2^j+1)}}{\cosh(k\pi 2^j)}=\frac{e^{-2k\pi}}{\sinh(k\pi)}, \tag{209}$$

a proof of which can be found in section 12.

## 9.6 Application of (70)

Consider the case (70) using $a_m = 1 + x^2/m^2$, giving

$$\prod_{j=0,n=1}^{\infty}\left(1+\frac{1}{(2n-1)^2 2^{2j}}\right)^{\frac{1}{j+1}}\left(1+\frac{1}{4n^2 2^{2j}}\right)^{\frac{1}{(j+1)(j+2)}}=\frac{\sinh(\pi)}{\pi}. \tag{210}$$

From (181) we have

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{(2n-1)^2(2^j)^2}\right)=\cosh\left(\frac{\pi}{2^{j+1}}\right), \tag{211}$$

and

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{4n^2(2^j)^2}\right)=\frac{2^{j+1}}{\pi}\sinh\left(\frac{\pi}{2^{j+1}}\right), \tag{212}$$

leading to the identity

$$\prod_{j=0}^{\infty}\left(\cosh(\pi x/2^{j+1})\right)^{\frac{1}{j+1}}\left(\frac{\sinh(\pi x/2^{j+1})2^{j+1}}{\pi x}\right)^{\frac{1}{(j+1)(j+2)}}=\frac{\sinh(\pi x)}{\pi x} \tag{213}$$

or its equivalent

$$\sum_{j=1}^{\infty}\frac{\ln(\cosh(\pi x/2^{j+1}))+\ln(\frac{2^{j+1}}{\pi x}\sinh(\pi x/2^{j+1}))/(j+2)}{j+1}=\frac{1}{2}\ln(\frac{2}{\pi x}\sinh(\frac{\pi x}{2})). \tag{214}$$

Differentiating (214) with respect to $x$ yields

$$\sum_{j=1}^{\infty}\frac{2^{-j-1}}{j+1}(\tanh(\pi x 2^{-j-1})+\frac{1}{j+2}\coth(\pi x 2^{-j-1}))=\frac{\coth(\frac{\pi x}{2})}{4}, \tag{215}$$

and further differentiating produces

$$\sum_{j=1}^{\infty} \frac{2^{-2j-2}}{j+1}(\operatorname{sech}^2(\pi x 2^{-j-1}) - \frac{1}{j+2}\operatorname{csch}^2(\pi x 2^{-j-1})) = -\frac{\operatorname{csch}(\frac{\pi x}{2})^2}{8}. \tag{216}$$

In (213), setting $x := \imath x$ gives

$$\prod_{j=1}^{\infty} \left(2\cos(\pi x 2^{-j-1})\right)^{\frac{1}{j+1}} \left(\frac{\sin(\pi x 2^{-j-1})}{2\pi x}\right)^{\frac{1}{(j+1)(j+2)}} = \sqrt{\frac{2}{\pi x}\sin(\frac{\pi x}{2})}, \qquad |x| < 4. \tag{217}$$

Further, employing $a_m = 1 + x^3/m^3$ with $x \in \mathbb{R}$, produces the identity

$$\prod_{j=0}^{\infty} \left(\frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{2} + 2^{-j-1}x)\left|\Gamma(\frac{1}{2} - 2^{-j-2}x(1-\imath\sqrt{3}))\right|^2}\right)^{\frac{1}{j+1}} \tag{218}$$

$$\times \left(\frac{1}{\Gamma(1 + 2^{-j-1}x)\left|\Gamma(1 - 2^{-j-2}x(1-\imath\sqrt{3}))\right|^2}\right)^{\frac{1}{(j+1)(j+2)}} = \frac{1}{\Gamma(x+1)\left|\Gamma(1 - \frac{x}{2}(1-\imath\sqrt{3}))\right|^2}.$$

## 9.7 A recurring family

Consider the identity (67) using $b(m) = \exp(-zm)$, $\Re(z) > 0$. The sums over $n$ on both sides can be evaluated in closed form, leading to the transformation

$$\sum_{j=1}^{\infty} \frac{B_{2p+1}(j) e^{z2^j}}{(e^{z2^{j+1}} - 1)(2p+1)} = \sum_{j=1}^{\infty} \frac{j^{2p}}{e^{z2^{j+1}} - 1}, \tag{219}$$

since $B_{2p+1}(0) = 0$ for $p \geq 1$. We notice that, with $f(j) = B_{2p+1}(j)/(2p+1)$ and $g(j) = 1/(1 - e^{z2^j})$ and with the finite difference operator $\Delta f(j) = f(j+1) - f(j)$, we have

$$\Delta f(j) = j^{2p}, \;\; \Delta g(j) = \frac{e^{z2^j}}{e^{z2^{j+1}} - 1}, \tag{220}$$

so that (219) can be interpreted as a summation by parts identity [5]

$$\sum_{j\geq 1} f(j)\Delta g(j) = -\sum_{j \geq 1} g(j+1)\Delta f(j). \tag{221}$$

More interesting is the case $b(m) = \exp(-xm^2)$, $x > 0$ applied to (67), leading to

$$\sum_{j=1}^{\infty} \frac{B_{2p+1}(j)}{2p+1}\vartheta_2(0, e^{-2^{2+2j}}) = \sum_{j=1}^{\infty} j^{2p}(\vartheta_3(0, e^{-2^{2+2j}}) - 1), \tag{222}$$

by recognizing that

$$\sum_{n=1}^{\infty} e^{-4xn^2 2^{2j}} = \frac{1}{2}(\vartheta_3(0, e^{-x2^{2+2j}}) - 1) \tag{223}$$

and

$$\sum_{n=1}^{\infty} e^{-x(2n-1)^2 2^{2j}} = \frac{1}{2}\vartheta_2(0, e^{-x2^{2+2j}}) \tag{224}$$

coincide with the usual [14, Section 20.2] definitions of the Jacobi theta functions $\vartheta_2$ and $\vartheta_3$. Identity (222) can also be interpreted as a summation by parts identity since

$$\Delta\vartheta_3(0, q) = \vartheta_2(0, q), \tag{225}$$

a special case $z = 0$ of the more general but classic identity

$$\vartheta_3(z, q) = \vartheta_3(2z, q^4) + \vartheta_2(2z, q^4) \tag{226}$$

that can be found in [21, Example 1, p. 464].

Continuing with the choice $b(m) = \exp(-xm^2)$, the two identities (223) and (224) recur in many other identities presented in previous sections. Applied to (71) we find

$$\sum_{j=0}^{\infty}\left(\frac{\vartheta_2(0, e^{-x2^{2+2j}})}{j+1} + \frac{\vartheta_3(0, e^{-x2^{2+2j}}) - 1}{(j+1)(j+2)}\right) = \vartheta_3(0, e^{-x}) - 1, \tag{227}$$

and applied to (69) we obtain

$$\sum_{j=0}^{\infty}((-j^2 + j + 1)\vartheta_2(0, e^{-x2^{2+2j}}) + 2j(\vartheta_3(0, e^{-x2^{2+2j}}) - 1)) = \vartheta_3(0, e^{-x}) - 1. \tag{228}$$

# 10 Additional general identities

We list in this section two additional identities as a consequence of the dissection identity

$$\prod_{m\geq 1} a_m = \prod_{j\geq 0, n\geq 1} a_{(2n-1)\cdot 2^j}. \tag{229}$$

**Proposition 22.** *The function*

$$f(a, z) = \frac{\Gamma^2(a+1)}{\Gamma(a+1-\imath z)\Gamma(a+1+\imath z)} \tag{230}$$

*satisfies the identity*

$$f(a, z) = \prod_{j\geq 0} f\left(\frac{a}{2^{j+1}} - \frac{1}{2}, \frac{z}{2^{j+1}}\right). \tag{231}$$

*Proof.* The function $f(a,z)$ has the infinite product representation

$$f(a,z)=\prod_{m\geq 1}(1+(\frac{z}{m+a})^2). \tag{232}$$

We deduce from the identity (229) that

$$\begin{aligned} f(a,z) &= \prod_{j\geq 0,n\geq 1}(1+(\frac{z}{(2n-1)\cdot 2^j+a})^2) \\ &= \prod_{j\geq 0,n\geq 1}(1+(\frac{z}{n\cdot 2^{j+1}+a-2^j})^2) \\ &= \prod_{j\geq 0,n\geq 1}(1+(\frac{\frac{z}{2^{j+1}}}{n+(\frac{a}{2^{j+1}}-\frac{1}{2})})^2) \\ &= \prod_{j\geq 0} f(\frac{a}{2^{j+1}}-\frac{1}{2},\frac{z}{2^{j+1}}). \end{aligned} \tag{233}$$

□

Explicitly, (231) reads

$$\frac{\Gamma^2(a+1)}{\Gamma(a+1-\imath z)\Gamma(a+1+\imath z)}=\prod_{j\geq 0}\frac{\Gamma^2(\frac{a}{2^{j+1}}+\frac{1}{2})}{\Gamma(\frac{a}{2^{j+1}}+\frac{1}{2}-\imath\frac{z}{2^{j+1}})\Gamma(\frac{a}{2^{j+1}}+\frac{1}{2}+\imath\frac{z}{2^{j+1}})}. \tag{234}$$

If $z=x,\ x\in\mathbb{R}$, because $\Gamma(x)$ is its own complex conjugate, (234) can be rewritten

$$\frac{\Gamma^2(a+1)}{|\Gamma(a+1+\imath x)|^2}=\prod_{j\geq 0}\frac{\Gamma^2(\frac{a}{2^{j+1}}+\frac{1}{2})}{\left|\Gamma(\frac{a}{2^{j+1}}+\frac{1}{2}+\imath\frac{x}{2^{j+1}})\right|^2}. \tag{235}$$

With $a=0$, we have

$$f(0,z)=\frac{\sinh(\pi z)}{\pi z},\quad f(-\frac{1}{2},z)=\frac{1}{\cosh(\pi z)} \tag{236}$$

and we deduce

$$\frac{\sinh(\pi z)}{\pi z}=\prod_{j\geq 0}\cosh(\frac{\pi z}{2^{j+1}}), \tag{237}$$

reproducing the identity [9, Eq. (92.1.3)], again listed by Hansen. When written in sum form, (234) becomes

$$\begin{aligned} \sum_{j=0}^{\infty}(2\ln(\Gamma(\frac{a}{2^{j+1}}+\frac{1}{2}))-\ln(\Gamma(\frac{a}{2^{j+1}}+\frac{1}{2}-\frac{\imath z}{2^{j+1}}))-\ln(\Gamma(\frac{a}{2^{j+1}}+\frac{1}{2}+\frac{\imath z}{2^{j+1}}))) \\ =\ln\left(\frac{\Gamma(a+1)^2}{\Gamma(-\imath z+a+1)\Gamma(\imath z+a+1)}\right), \end{aligned} \tag{238}$$

so that, by either differentiating with respect to $a$ or $z$, and demanding that $z := x \in \mathbb{R}$, we respectively find two Euler sums:

$$\sum_{j=0}^{\infty} 2^{-j}(\psi(\frac{a}{2^{j+1}}+\frac{1}{2}) - \Re(\psi(\frac{a+\imath x}{2^{j+1}}+\frac{1}{2}))) \\ = 2\psi(a+1) - 2\Re(\psi(\imath x + a + 1)), \tag{239}$$

and

$$\sum_{j=0}^{\infty} 2^{-j}\Im(\psi(\frac{a+\imath x}{2^{j+1}}+\frac{1}{2})) = 2\Im(\psi(\imath x + a + 1)). \tag{240}$$

In the limit $x \to 0$, from (240) we also find

$$\sum_{j\geq 0} \frac{\psi'(\frac{a}{2^{j+1}}+\frac{1}{2})}{2^{2j+1}} = 2\psi'(1+a). \tag{241}$$

**Proposition 23.** *The function*

$$g(a,b,z) = \frac{b}{(b-z)} \frac{\Gamma(a-\sqrt{a^2-b})\Gamma(a+\sqrt{a^2-b})}{\Gamma(a-\sqrt{a^2-b+z})\Gamma(a+\sqrt{a^2-b+z})} \tag{242}$$

*satisfies the identity*

$$g(a,b,z) = \prod_{j\geq 0} g(\frac{a}{2^j}-1, 1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}}, \frac{z}{2^{2j+2}}). \tag{243}$$

*Proof.* The function $g(a,b,z)$ has the infinite product representation

$$g(a,b,z) = \prod_{m\geq 1}(1-\frac{z}{m^2+2am+b}). \tag{244}$$

We deduce

$$\begin{aligned} g(a,b,z) &= \prod_{n\geq 1, j\geq 0}(1-\frac{z}{(2n-1)^2 2^{2j}+2a(2n-1)2^j+b}) \\ &= \prod_{n\geq 1, j\geq 0}(1-\frac{z}{2^{2j+2}n^2-2^{2j+2}n+2^{2j}+2^{j+2}an+b-2^{j+1}a}) \\ &= \prod_{n\geq 1, j\geq 0}(1-\frac{z}{2^{2j+2}n^2+n(-2^{2j+2}+2^{j+2}a)+(2^{2j}+b-2^{j+1}a)}) \\ &= \prod_{n\geq 1, j\geq 0}(1-\frac{\frac{z}{2^{2j+2}}}{n^2+n(\frac{a}{2^j}-1)+(1+\frac{b}{2^{2j+2}}-\frac{a}{2^{j+1}})}) \\ &= \prod_{j\geq 0} g(\frac{a}{2^j}-1, 1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}}, \frac{z}{2^{2j+2}}). \end{aligned} \tag{245}$$

□

Explicitly, (243) reads

$$\frac{b}{(b-z)}\frac{\Gamma(a-\sqrt{a^2-b})\Gamma(a+\sqrt{a^2-b})}{\Gamma(a-\sqrt{a^2-b+z})\Gamma(a+\sqrt{a^2-b+z})}$$
$$=\prod_{j\geq 0}\frac{1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}}}{(1-\frac{a}{2^{j+1}}+\frac{b-z}{2^{2j+2}})}\frac{\Gamma(\frac{a}{2^j}-1-\sqrt{(\frac{a}{2^j}-1)^2-(1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}})})}{\Gamma(\frac{a}{2^j}-1-\sqrt{(\frac{a}{2^j}-1)^2-(1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}})+\frac{z}{2^{2j+2}}})}$$
$$\times\prod_{j\geq 0}\frac{\Gamma(\frac{a}{2^j}-1+\sqrt{(\frac{a}{2^j}-1)^2-(1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}})})}{\Gamma(\frac{a}{2^j}-1+\sqrt{(\frac{a}{2^j}-1)^2-(1-\frac{a}{2^{j+1}}+\frac{b}{2^{2j+2}})+\frac{z}{2^{2j+2}}})}. \tag{246}$$

# 11 Conclusion

The equality of the multisets $D_b = \{(bn)\cdot b^j, n \geq 1, j \geq 0\}$ and $E_b = \{m^{(\nu_b(m))}, m \in \mathbb{N}\}$ allowed us to express identities of the multisection type for a variety of special functions. The versatility of this approach is due to the fact that it exploits the equivalence of two summation domains (multisets) regardless of the values of the entities summed over these domains. We conclude this study by suggesting several directions of research.

The first direction would consider multidimensional versions of this approach, i.e., multidimensional multisets allowing summation over lattices in $\mathbb{R}^q$. This new context would allow the study of multivariate summations of interesting special functions such as the multidimensional Riemann theta functions. We refer the reader to [2] for an excellent reference in this domain.

A second possibility would require further development to extend the results reported here to finite series and products. Since matrix multiplication is essentially a multiple sum of products, it may be possible to reduce the arithmetic load inherent in large (or possibly infinite) matrix multiplication (in specialized cases) into a single sum, which is effectively what has been done here. Let us notice that similar results for finite sums associated with the Hurwitz-Lerch zeta function [16] were recently obtained; they are however derived using a completely different method, namely contour integration in the complex plane.

In a similar vein, it is possible that the multisection of a slowly converging, or even divergent series, could result in a more rapidly converging series, particularly if one of the double series resulting from the reordering is analytically summable, as has been demonstrated here. For a review of techniques for acceleration of slowly converging series, see [20].

Finally, the principle that underlies the results presented in this article can be enunciated as follows: for an arbitrary function $f$ such that the following series converge,

$$\sum_{n,j\geq 1} f(n\cdot\alpha(j)) = \sum_{m\geq 1}\beta(m)f(m), \tag{247}$$

where $\beta(m)$ is the number of integers of the form $\alpha(j)$ that divide $m$. For example,

- with $\alpha(j) = 2^{j+1}$, the function $\beta(m)$ enumerates the number of distinct powers of 2 that divide $m$, which is also the 2-adic valuation $\nu_2(m)$;
- with $\alpha(j) = j$, the function $\beta(m)$ coincides with the number-of-divisors function $\sigma_0(m)$;
- with $\alpha(j) = j^2$ then $\beta(m)$ is the number of square integers that divide $m$ (see OEIS entry A046951).

# 12 Appendix: Proof of identity (209)

Identity (209) claims that

$$\sum_{j\geq 0} 2^j \frac{e^{-k\pi(2^j+1)}}{\cosh(k\pi 2^j)} = \frac{e^{-2k\pi}}{\sinh(k\pi)}. \tag{248}$$

Denoting $q = e^{-k\pi}$, this is

$$\sum_{j\geq 0} 2^{j+1} \frac{q^{2^j+1}}{q^{2^j} + q^{-2^j}} = \frac{2q^2}{q^{-1} - q} \tag{249}$$

or

$$\sum_{j\geq 0} 2^{j+1} \frac{1}{q^{-2^{j+1}} + 1} = \frac{2q^2}{1 - q^2}. \tag{250}$$

Substituting $q := \frac{1}{q}$ produces

$$\sum_{j\geq 0} 2^{j+1} \frac{1}{q^{2^{j+1}} + 1} = \frac{2}{q^2 - 1}. \tag{251}$$

We know from (123) that

$$\frac{q}{1-q} = \sum_{j\geq 0} 2^j \frac{q^{2^j}}{q^{2^j} + 1}. \tag{252}$$

Substituting $q := \frac{1}{q}$ produces

$$\frac{1}{q-1} = \sum_{j\geq 0} 2^j \frac{1}{q^{2^j} + 1}, \tag{253}$$

so that

$$\sum_{j\geq 0} 2^{j+1} \frac{1}{q^{2^{j+1}} + 1} = \frac{1}{q-1} - \frac{1}{1+q} = \frac{2}{q^2 - 1}, \tag{254}$$

which is (251).

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