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Domination Polynomials of the Grid, the Cylinder, the Torus, and the King Graph

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Abstract

We present an algorithm to compute the domination polynomial of the $m \times n$ grid, cylinder, and torus graphs and the king graph. The time complexity of the algorithm is $O(m^2 n^2 \lambda^{2m})$ for the torus and $O(m^3 n^2 \lambda^m)$ for the other graphs, where $\lambda = 1 + \sqrt{2}$. The space complexity is $O(mn\lambda^m)$ for all of these graphs. We use this algorithm to compute domination polynomials for graphs up to size 24×24 and the total number of dominating sets for even larger graphs. This allows us to give precise estimates of the asymptotic growth rates of the number of dominating sets. We also extend several sequences in the *On-Line Encyclopedia of Integer Sequences*.

1 Introduction

A dominating set in a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that every node in V is either an element of S or has a neighbor in S.

Domination is one of the most widely studied topics in graph theory. According to Haynes, Hedetniemi, and Henning [18], more than 4000 papers on the subject were published by the year 2020. Domination problems originated in the 19th century in chess [20, 19]. A placement of chess pieces on a chessboard is called dominating if each free square of the chessboard is under attack by at least one piece. Figure 1 shows 9 kings dominating the 8×8 chessboard.

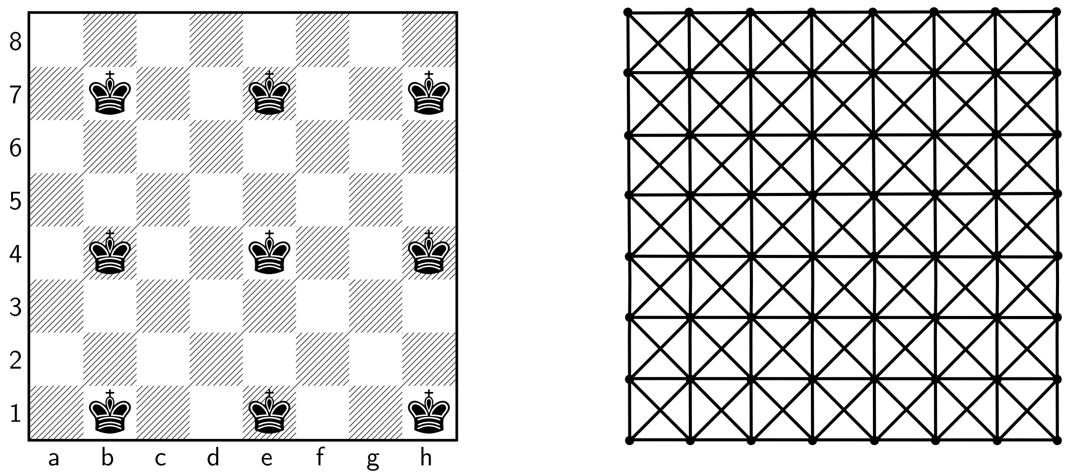

Figure 1: Nine kings are required to dominate the chessboard, or equivalently the 8×8 king graph.

The link from chess to graph theory is given by graphs like the king graph (Figure 1). In this graph, the vertices represent the squares of the board, and each edge represents a legal move of a king. Obviously a dominating placement of kings on the board corresponds to a dominating set of the king graph. Graphs for other chess pieces can be defined analogously.

In this contribution, we study domination in some related families of graphs: the $m \times n$ grid, cylinder, and torus graph (Figure 2). If P_n denotes the path graph of n vertices, the grid graph $G_{m\times n}$ is the Cartesian graph product $G_{m\times n} = P_m \square P_n$. The cylinder graph is $G_{\overline{m}\times n} = C_m \square P_n$, where C_m is the cycle graph: we use the overbar $\overline{m}$ to indicate the cyclic index. The $m \times n$ torus graph is $G_{\overline{m}\times\overline{n}} = C_m \square C_n$. We also study domination in the $m \times n$ king graph $K_{m\times n} = P_m \boxtimes P_n$, the strong graph product of two path graphs P_m and P_n.

Our goal is to compute the *domination polynomial* of these graphs. The domination polynomial of a graph G is the generating function of its dominating sets with respect to their size, i.e.,

$$D_G(z) = \sum_{S \subseteq V} z^{|S|}\,, \tag{1}$$

where the sum runs over all dominating sets in G. Like other graph polynomials, the domination polynomial encodes many interesting properties of a graph [2, 10, 11]. For example,

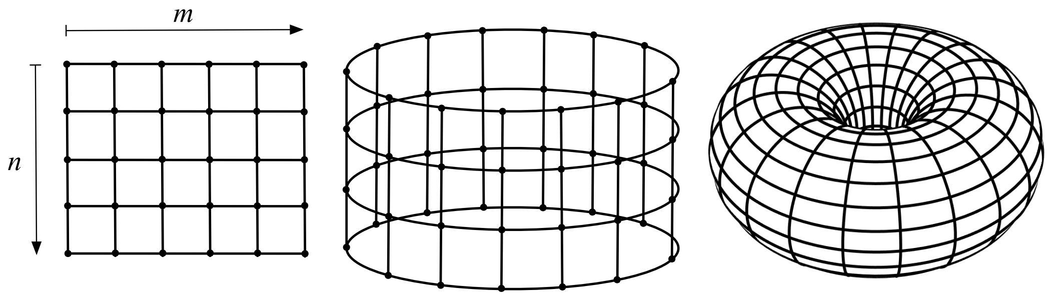

Figure 2: Examples of an $m \times n$ grid, cylinder and torus graph.

the lowest power of z appearing in $D_G(z)$ is the size of the smallest dominating set in G, which is known as the *domination number* $\gamma(G)$. A considerable fraction of the 4000 papers mentioned above are studies of $\gamma(G)$ for various graphs.

A closed form for the domination polynomial is known only for a few families of simple graphs like complete graphs, path and cycle graphs, wheel graphs, star graphs, and friendship graphs [6, 5, 7, 4]. Recently, rook graphs on $m \times n$ chessboards, i.e., Cartesian products of two complete graphs $K_m \square K_n$, were added to this list [23]. As far as we know, the rook is the first chess piece for which this has been achieved.

The domination polynomial of course can be computed numerically. However, this approach is challenging because of its computational complexity. In theoretical computer science, the decision problem Dominating Set asks, given a graph G and an integer k, whether G has a dominating set of size at most k, i.e., whether $\gamma(G) \leq k$. This problem is NP-complete, which can be shown by reduction from the Vertex Cover problem [16]. Unless P = NP, this means that no polynomial-time algorithm exists to compute $\gamma(G)$. Indeed, the fastest known algorithm to find the dominating set of minimum size for general graphs $G = (V, E)$ has time complexity $O(1.4969^{|V|})$ [27]. Because determining the domination number is NP-hard, calculating the domination polynomial is (at least) as hard.

The grid graph is planar and bipartite, and both of these properties often allow polynomial time algorithms for problems that are NP-complete for general graphs [24]. But not in this case: Dominating Set is NP-complete even on subgraphs of the $m \times n$ grid [13]. This suggests that we should not expect to find a polynomial-time algorithm that computes $\gamma(G)$ for the grid graph or its relatives, let alone one that computes the entire polynomial $D_G(z)$. But we can try to reduce the exponential running time as much as possible. This is the main goal of this work.

The paper is organized as follows. We begin by proving in Section 2 that the domination polynomial of the $m \times n$ grid, cylinder, and torus can be expressed in terms of the nth power of an $a \times a$ matrix A, the "transfer matrix", where $a = O(\lambda^m)$ with $\lambda = 1 + \sqrt{2} = 2.4142\ldots$

As we show in Section 3, this gives rise to an algorithm for the domination polynomial with time complexity $O(m^3n^2\lambda^m)$ for the grid and cylinder, and a closely related algorithm with time complexity $O(m^3n^2\lambda^{2m})$ for the torus. We also explain how this algorithm can be adapted to compute the domination polynomial of the king graph, with the same time and space complexity. Since the running times of these algorithms are exponential in the width m as opposed to the number of vertices $|V| = mn$, they represent a considerable improvement over the algorithm of [27].

In Sections 4 and 5 we discuss some numerical results obtained by our algorithm. In particular, we estimate the asymptotic growth rate of the total number of dominating sets for all these graphs. We give Conclusions in Section 6 and provide some combinatorial proofs and domination polynomials in the appendices.

2 The transfer matrix

The idea of the transfer matrix approach is to compute the domination polynomial of a grid or cylinder row by row. We do this by defining legal transitions from one row to the next, identifying which vertices in each row are in the dominating set S. Consider the following definitions, where we borrow some wording from domination problems in chess.

Definition 1. Given a graph $G = (V, E)$ and a set $S \subseteq V$, we say a vertex $v \in V$ is *occupied* if $v \in S$, *covered* if $v \notin S$ but some neighbor of v is in S, and *uncovered* if $v \notin S$ and no neighbor of v is in S.

Clearly every vertex is either occupied, covered, or uncovered. As we construct S, some vertices in the current row may be uncovered, because they will become covered by a neighboring occupied vertex in the next row. This gives us the following definition.

Definition 2. If G is the grid graph $G_{m\times n}$ or the cylinder graph $G_{\overline{m}\times n}$, we say S is *almost dominating* if every vertex in the subgraph $G_{m\times(n-1)}$ (resp., $G_{\overline{m}\times(n-1)}$) consisting of the first $n-1$ rows are occupied or covered.

We label vertices according to their state, namely • (occupied), • (covered), and ∘ (uncovered). Given an almost dominating set S, we define its *signature* σ as the string of length m over the alphabet $\{\circ, \bullet, \bullet\}$ that identifies the states of the vertices in the nth row. However, not all such strings can occur: since the neighbors of an occupied vertex are covered, the symbols • and ∘ cannot be adjacent.

Definition 3. A *signature* of length m is a string σ of length m over the alphabet $\{\circ, \bullet, \bullet\}$ which does not contain either of the substrings (∘,•) and (•,∘). A *cyclic signature* is one where this substring constraint also applies to the pair (σ_1, σ_m).

Signatures apply to the grid, and cyclic signatures apply to the cylinder.

The time and space complexity of our algorithms depend on the number of signatures or cyclic signatures. The number $a(m)$ of signatures is given by

$$
\begin{aligned}
a(m) &= \frac{1}{2}\left(1-\sqrt{2}\right)^{m+1} + \frac{1}{2}\left(1+\sqrt{2}\right)^{m+1} \\
&= 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, \ldots
\end{aligned} \tag{2}
$$

This sequence has two entries in the *On-Line Encyclopedia of Integer Sequences* (OEIS), A001333 and A078057, differing only in their offset. Since cyclic signatures are constrained at one more pair, the total number $\overline{a}(m)$ of cyclic signatures is smaller than $a(m)$ for $m \geq 3$, although with the same asymptotic growth rate:

$$
\begin{aligned}
\overline{a}(m) &= 1 + \left(1-\sqrt{2}\right)^{m} + \left(1+\sqrt{2}\right)^{m} \\
&= 3, 7, 15, 35, 83, 199, 479, 1155, 2787, 6727, \ldots
\end{aligned} \tag{3}
$$

This is sequence A124696 in the OEIS. We derive the formulas for $a(m)$ and $\overline{a}(m)$ in the Appendix.

Slightly abusing notation, we write $G_{m\times n}(z)$ for the domination polynomial of the grid $G_{m\times n}$ and $G_{\overline{m}\times n}(z)$ for the domination polynomial of the cylinder $G_{\overline{m}\times n}$. We also write $G^{\sigma}_{m\times n}(z)$ and $G^{\sigma}_{\overline{m}\times n}(z)$ for the generating functions of almost dominating sets on $G_{m\times n}$ (resp., $G_{\overline{m}\times n}$) with signature σ. The connection between these dominating polynomials and almost-dominating polynomials is then given by the following lemma.

Lemma 4. *Let $\circ(\sigma)$ denote the number of uncovered vertices in σ. Then*

$$
G_{m\times n}(z) = \sum_{\sigma\,:\,\circ(\sigma)=0} G^{\sigma}_{m\times n}(z)\,, \qquad G_{\overline{m}\times n}(z) = \sum_{\sigma\,:\,\circ(\sigma)=0} G^{\sigma}_{\overline{m}\times n}(z)\,, \tag{4}
$$

where the sum for the grid (resp., the cylinder) runs over all signatures (resp., cyclic signatures).

Proof. The dominating sets of $G_{m\times n}$ and $G_{\overline{m}\times n}$ consist of the almost dominating sets which are in fact dominating, i.e., where there are no uncovered vertices in the nth row. □

Now we have all the ingredients to implement the idea of constructing dominating sets row by row. Consider an almost dominating set in an $m\times n$ grid or cylinder with signature σ, and consider adding an $(n+1)$st row with signature τ. Only certain pairs σ, τ are compatible. Wherever σ has an uncovered vertex, its neighbor in τ must be occupied. Similarly, wherever σ is occupied, its neighbor in τ is occupied or covered by definition. Finally, a vertex in τ cannot be covered unless it has an occupied neighbor, either above it in σ or to either side in τ. Thus the new signature τ must be compatible with the previous signature σ according to the following definition.

Definition 5. A (cyclic) signature $\tau = (\tau_1, \ldots, \tau_m)$ is *compatible* with a (cyclic) signature $\sigma = (\sigma_1, \ldots, \sigma_m)$ if, for all $i = 1, \ldots, m$,

$$\begin{aligned} \sigma_i = \circ &\implies \tau_i = \bullet\,, \\ \sigma_i = \bullet &\implies \tau_i \in \{{\color{gray}\bullet}, \bullet\}\,, \\ \tau_i = {\color{gray}\bullet} &\implies (\sigma_i = \bullet) \text{ or } (\tau_{i-1} = \bullet) \text{ or } (\tau_{i+1} = \bullet)\,. \end{aligned} \tag{5}$$

In the last equation, we compute the indices $i \pm 1 \bmod m$ for cyclic signatures, and ignore τ_0 and τ_{m+1} in the non-cyclic case.

Finally, we define the transfer matrices A and $\overline{A}$, whose rows and columns are indexed by (cyclic) signatures.

Definition 6. Let $\bullet(\sigma)$ denote the number of occupied vertices in a (cyclic) signature σ. For a given integer $m \geq 0$, the transfer matrix $A = (A_{\tau,\sigma})$ is defined as

$$A_{\tau,\sigma} = \begin{cases} z^{\bullet(\tau)}, & \text{if } \tau \text{ is compatible with } \sigma; \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

where τ and σ range over all signatures of length m. The transfer matrix $\overline{A} = (\overline{A}_{\tau,\sigma})$ is defined similarly with τ and σ ranging over cyclic signatures of length m.

	$\circ\circ$	$\circ{\color{gray}\bullet}$	$\bullet\bullet$	$\bullet{\color{gray}\bullet}$	${\color{gray}\bullet}\circ$	${\color{gray}\bullet}\bullet$	${\color{gray}\bullet}{\color{gray}\bullet}$
$\circ\circ$	0	0	0	0	0	0	1
$\circ{\color{gray}\bullet}$	0	0	0	0	0	1	0
$\bullet\bullet$	z^2	z^2	z^2	z^2	z^2	z^2	z^2
$\bullet{\color{gray}\bullet}$	0	z	z	z	0	z	z
${\color{gray}\bullet}\circ$	0	0	0	1	0	0	0
${\color{gray}\bullet}\bullet$	0	0	z	z	z	z	z
${\color{gray}\bullet}{\color{gray}\bullet}$	0	0	1	0	0	0	0

Table 1: The transfer matrix A for domination on grids of width $m = 2$. Rows and columns are indexed by the new and old signatures τ and σ respectively.

Since the width m of the graph is usually clear from context, we suppress the dependence of A on m in our notation for the most part. Tables 1 and 2 show the transfer matrices for grids of width $m = 2$ and cylinders of width $m = 3$.

The next two theorems are our key results.

Theorem 7. *Let $\sigma_{\color{gray}\bullet}$ be the (cyclic) signature which is covered everywhere, $\sigma_{\color{gray}\bullet} = ({\color{gray}\bullet}, {\color{gray}\bullet}, \ldots, {\color{gray}\bullet})$. Then the domination polynomials $G_{m\times n}(z)$ and $G_{\overline{m}\times n}(z)$ can be computed as*

$$G_{m\times n}(z) = \sum_{\sigma:\circ(\sigma)=0} (A^n)_{\sigma,\sigma_{\color{gray}\bullet}}\,, \tag{7a}$$

	$\circ\circ\circ$	$\circ\circ{\color{gray}\bullet}$	$\circ{\color{gray}\bullet}\circ$	$\circ{\color{gray}\bullet}{\color{gray}\bullet}$	$\bullet\bullet\bullet$	$\bullet\bullet{\color{gray}\bullet}$	$\bullet{\color{gray}\bullet}\bullet$	$\bullet{\color{gray}\bullet}{\color{gray}\bullet}$	${\color{gray}\bullet}\circ\circ$	${\color{gray}\bullet}\circ{\color{gray}\bullet}$	${\color{gray}\bullet}\bullet\bullet$	${\color{gray}\bullet}\bullet{\color{gray}\bullet}$	${\color{gray}\bullet}{\color{gray}\bullet}\circ$	${\color{gray}\bullet}{\color{gray}\bullet}\bullet$	${\color{gray}\bullet}{\color{gray}\bullet}{\color{gray}\bullet}$
$\circ\circ\circ$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\circ\circ{\color{gray}\bullet}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$\circ{\color{gray}\bullet}\circ$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$\circ{\color{gray}\bullet}{\color{gray}\bullet}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\bullet\bullet\bullet$	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3	z^3
$\bullet\bullet{\color{gray}\bullet}$	0	z^2	0	z^2	z^2	z^2	z^2	z^2	0	z^2	z^2	z^2	0	z^2	z^2
$\bullet{\color{gray}\bullet}\bullet$	0	0	z^2	z^2	z^2	z^2	z^2	z^2	0	0	z^2	z^2	z^2	z^2	z^2
$\bullet{\color{gray}\bullet}{\color{gray}\bullet}$	0	0	0	z	z	z	z	z	0	0	z	z	0	z	z
${\color{gray}\bullet}\circ\circ$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
${\color{gray}\bullet}\circ{\color{gray}\bullet}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
${\color{gray}\bullet}\bullet\bullet$	0	0	0	0	z^2	z^2	z^2	z^2	z^2	z^2	z^2	z^2	z^2	z^2	z^2
${\color{gray}\bullet}\bullet{\color{gray}\bullet}$	0	0	0	0	z	z	z	z	0	z	z	z	0	z	z
${\color{gray}\bullet}{\color{gray}\bullet}\circ$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
${\color{gray}\bullet}{\color{gray}\bullet}\bullet$	0	0	0	0	z	z	z	z	0	0	z	z	z	z	z
${\color{gray}\bullet}{\color{gray}\bullet}{\color{gray}\bullet}$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

Table 2: The transfer matrix $\overline{A}$ for domination on the cylinder of width $m = 3$. Rows and columns are indexed by the new and old signatures τ and σ respectively.

$$G_{\overline{m}\times n}(z) = \sum_{\sigma:\circ(\sigma)=0} (\overline{A}^n)_{\sigma,\sigma_{\color{gray}\bullet}} \,. \tag{7b}$$

Proof. Consider $G^\tau_{m\times n}$, the generating function of almost dominating sets S in the $m \times n$ grid with signature τ. Now τ is compatible with multiple signatures σ on row $n-1$. For each such σ, placing τ on the nth row increases S by $\bullet(\tau)$ and thus multiplies $G^\sigma_{m\times(n-1)}$ by a factor $z^{\bullet(\tau)}$. Hence we can write

$$G^\tau_{m\times n}(z) = \sum_{\sigma\,:\,\tau \text{ compatible}} z^{\bullet(\tau)} G^\sigma_{m\times(n-1)}(z) = \sum_\sigma A_{\tau,\sigma} G^\sigma_{m\times(n-1)}(z) \,, \tag{8}$$

and applying this reasoning recursively gives

$$G^\tau_{m\times n}(z) = \sum_\sigma (A^{n-1})_{\tau,\sigma} G^\sigma_{m\times 1}(z) \,. \tag{9}$$

Now $G^\sigma_{m\times 1}$ is a valid generating function for an almost dominating set, i.e., for signatures σ with weights $z^{\bullet(\sigma)}$, with the additional property that σ does not contain any ${\color{gray}\bullet}$s. If you look again at Definition 5, this is equivalent to saying that σ is compatible with $\sigma_{\color{gray}\bullet}$. This gives

$$G^\sigma_{m\times 1}(z) = A_{\sigma,\sigma_{\color{gray}\bullet}} \,. \tag{10}$$

Combining this with (9) and (4) completes the proof of (7a). The proof of (7b) is similar. □

Domination on the torus is like domination on the cylinder, except that occupied vertices in the nth row can cover vertices in the 1st row and vice versa. As the following theorem shows, this corresponds to taking the trace of the nth power of the transfer matrix.

Theorem 8. *Let $G_{\overline{m}\times\overline{n}}(z)$ denote the domination polynomial of the $m \times n$ torus. Then*

$$G_{\overline{m}\times\overline{n}}(z) = \operatorname{Tr} \overline{A}^n . \qquad (11)$$

Proof. On the torus, in addition to requiring that the cyclic signature σ on the tth row is compatible to the signature in the $(t-1)$st row for all $1 < t \leq n$, we also need the signature on the 1st row to be compatible with the one on the nth row. We can find all such configurations by adding a 0th row to the graph with signature σ, applying the transfer matrix n times (note that the first application of $\overline{A}$ requires that the signature on the 1st row is compatible with σ) and picking out the entries of $\overline{A}^n$ where the signature on the nth row is also σ. Thus the generating function for dominating sets on the $m \times n$ torus with signature σ on the nth row is

$$G^{\sigma}_{\overline{m}\times\overline{n}}(z) = (\overline{A}^n)_{\sigma,\sigma} , \qquad (12)$$

and summing over all σ gives (11). □

Theorems 7 and 8 reduce the problem of computing the domination polynomials for the $m \times n$ grid, cylinder, or torus to computing the nth power of the transfer matrix A or $\overline{A}$. There are several ways to do this efficiently. We can compute the nth power of a matrix A by squaring it $\lfloor \log_2 n \rfloor$ times to obtain $A^1, A^2, A^4, \ldots, A^{2^{\lfloor \log_2 n \rfloor}}$, and multiplying whichever of these powers correspond to 1s in the binary expansion of n. Squaring an $N \times N$ matrix can be done in time $O(N^\omega)$, where $\omega = 3$ for the naive schoolbook method or $\omega = 2.371552$ with the fastest known algorithm [28]. In our case $N = a(m)$ or $\overline{a}(m)$, and in both cases $N = O(\lambda^m)$ with $\lambda = 1+\sqrt{2} = 2.4142\ldots$ This gives a time complexity of essentially $O(\lambda^{\omega m})$. In addition, squaring a matrix whose entries are polynomials of degree nm requires us to multiply such polynomials, which takes $O((nm)^2)$ time using the simplest method of adding all the cross-terms. However, rather than analyzing the running time of this repeated-squaring approach in detail, we present a faster algorithm in the next section.

To conclude this section, we briefly discuss another transfer matrix approach to domination on the grid. Oh [25] proposed what he called the "state matrix recursion method" for the domination polynomial of the grid. Although the phrase "transfer matrix" does not appear in [25], it is essentially a transfer matrix method, and Oh's Theorems 1 and 2 provide expressions for $G_{m\times n}(z)$ and $G_{\overline{m}\times n}(z)$ in the same spirit as ours. However, Oh focuses on the edges of the graph rather than the vertices. In a graph with a dominating set, there are four types of edges, whose endpoints are labeled (•, •), (•, •), (•, •) and (•, •). Since each of the m vertical edges connecting one row to the next can be in one of these four states, Oh's transfer matrix is 4^m-dimensional rather than $O(\lambda^m)$-dimensional, making it less efficient than our transfer matrix to compute domination polynomials. On the other hand, Oh's transfer matrix can be computed by a surprisingly simple recurrence, giving it an elegant mathematical form.

3 The algorithm

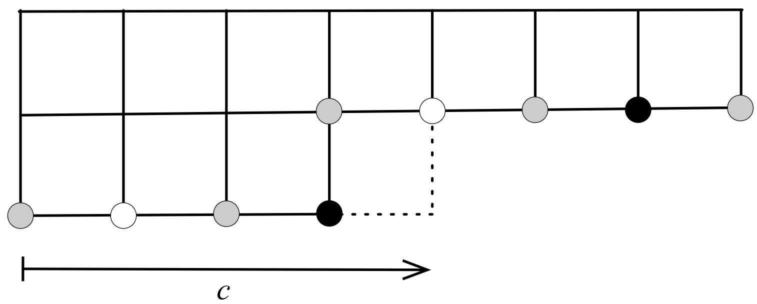

Figure 3: The algorithm adds a new row one vertex at a time from left to right. Here we illustrate a step where we add a vertex in column $c = 5$ at the dashed lines. The current signature is $\sigma =$ ●○●●'○●●● where ' marks the "kink." Whether the new vertex is unoccupied or occupied produces one of two new signatures, $\sigma_0 =$ ●○●●●'●●● or $\sigma_1 =$ ●○●●●'●●●. However, in this example $\sigma[c] = \circ$ (the uncovered vertex above the new vertex) so the new vertex must be occupied and σ_0 is invalid.

The repeated-squaring approach for computing the nth power of the transfer matrix, described in the previous section, fails to take advantage of A's (and $\overline{A}$'s) structure. First of all, these matrices are quite sparse, since most pairs of signatures are not compatible. Secondly, their nonzero entries are powers of z; so, rather than multiplying arbitrary polynomials, we can multiply by A or $\overline{A}$ by shifting the coefficients of each polynomial and adding the results. Thirdly, and most importantly, in this section we will show how to add a row, and thus multiply by A, using a series of even simpler operations. This will reduce the running time from $O(\lambda^{\omega m})$ to essentially $O(\lambda^m)$.

From a bird's eye perspective, the transfer matrix method turns a two-dimensional problem into a sequence of n one-dimensional problems. This idea can be applied again. By filling the new row one vertex at a time, from left to right, we can subdivide the one-dimensional problem of adding a row into a sequence of m zero-dimensional problems.

At each step, the nth row is filled up to column $c - 1$. The corresponding signature contains a "kink" at column c, where it hops up to the $(n-1)$st row; see Figure 3. When we add a vertex in the cth column, this signature can be mapped to two possible signatures, depending on whether this new vertex is occupied or not.

Across the kink at c, the substrings $(\circ,\bullet)$ and $(\bullet,\circ)$ are no longer forbidden, so the number of signatures is a bit larger than $a(m)$ or $\overline{a}(m)$. And in the case of the non-cyclic signatures, the number depends on c. However, we show in Appendix A that for each $1 \le c \le m$ the number of signatures grows as $O(\lambda^m)$.

Each step of this new transfer matrix algorithm is the addition of a new vertex in a partially filled or empty row. This gives a subroutine **extend**(σ, c), which we show for the grid in Figure 4. This subroutine interprets σ as a signature where the current row is filled

```
subroutine extend (σ, c)
σ₁ := σ
σ₁[c] := ●
if c > 1 and σ₁[c − 1] = ○ then
    σ₁[c − 1] := ●(grey)
end if
if σ[c] = ○ then                ▷ new vertex needs to cover σ[c]
    σ₀ := invalid
else
    σ₀ := σ
    if σ[c] = ● then
        σ₀[c] := ●(grey)
    else if c > 0 and σ₀[c − 1] = ● then
        σ₀[c] := ●(grey)
    else
        σ₀[c] := ○
    end if
end if
return σ₀, σ₁
```

Figure 4: Adding a new vertex at column c in the current row.

up to the $(c-1)$st column, and returns up to two signatures σ_0, σ_1 where σ_0 (resp., σ_1) results from σ by adding an unoccupied (resp., occupied) vertex in column c.

The **extend** subroutine takes care of the compatibility between σ, σ_0, and σ_1. If the vertex to the left of the new vertex is uncovered in σ, in σ_1 it becomes covered by the new occupied vertex. It also marks the new vertex as covered in σ_0 if the vertex to its left or above it is occupied in σ. Finally, if $\sigma[c] = \circ$, i.e., if the vertex immediately above the new vertex is uncovered as in Figure 3, then the new vertex must be occupied. In that case σ_0 is defined as *invalid*, and does not need to be pursued further by the algorithm.

We use the subroutine **extend** in an algorithm that loops over the n rows and m columns of the grid (Figure 5). This algorithm builds the rows of the grid one vertex at a time while maintaining a list of configurations, i.e., pairs (σ, G^σ) where σ is a signature and G^σ is the corresponding generating function. Whenever we add a new vertex we multiply by z if that vertex is occupied, adding G^σ to G^{σ_0} and adding zG^σ to G^{σ_1}. Note that σ_0 or σ_1 might already be in the list of new signatures, since adding the new vertex hides the vertex above it. That is, $\sigma_0 = \sigma_0'$ if σ and σ' differ only in column c, and similarly for σ_1 and σ_1'. Each loop where c ranges from 1 to m thus adds a new row and effectively applies the transfer matrix. This continues until we complete the nth row and obtain the dominating polynomial for the entire grid.

In order to carry out these computations for large grids, it turns out that memory, not time, is the limiting resource. Thus to reach grids as large as possible, we need to

```
L_old := (σ_•, 1)   ▷ zeroth row configuration, σ_• = (•, •, . . . , •)
for r = 1, . . . , n do
    L_new := empty list
    for c = 1, . . . , m do
        while L_old not empty do
            take (σ, G^σ) out of L_old
            σ_0, σ_1 := extend(σ, c)
            if σ_0 = invalid then ignore σ_0
            else if (σ_0, ·) ∉ L_new then add (σ_0, G^σ) to L_new
            else replace (σ_0, G) in L_new with (σ_0, G + G^σ)
            end if
            if (σ_1, ·) ∉ L_new then add (σ_1, zG^σ) to L_new
            else replace (σ_1, G) in L_new with (σ_1, G + zG^σ)
            end if
        end while
        L_old := L_new
    end for
end for
G_{n,m}(z) := Σ_{σ∈L_old, ○(σ)=0} G^σ(z)
```

Figure 5: The algorithm to compute the domination polynomial of the grid $G_{m\times n}$. Completing each row, i.e., completing the inner loop over the m columns, has the effect of multiplying by the transfer matrix A. As a programming detail, the assignment $L_{\text{old}} := L_{\text{new}}$ is by reference (i.e., by moving a pointer) to avoid copying data from one location in memory to another.

think carefully about how to represent and store both signatures σ and their polynomials G^σ as efficiently as possible. To some readers the rest of this section will seem like mere implementation details. But these details play an essential role. While both the time and space requirements of our algorithm are exponential, they reduce the exponent, and without them we would have no hope of obtaining the results we present in the next section.

First, to represent the signatures σ, we treat the three symbols ○, •, and • as ternary digits, and interpret each σ as an integer between 0 and $3^m - 1$. Since $3^{40} \leq 2^{64}$, the signatures fit into 64-bit integers as long as $m \leq 40$.

We store the polynomials G^σ as vectors of integer coefficients. However, since these coefficients grow exponentially in mn, they quickly get too large to store as fixed-width integers with 32, 64, or 128 bits. One could use variable-length integers to deal with this problem, but this would add a factor nm to both the time and the space complexity.

Instead, we stick with fixed-width integers and use modular arithmetic. For some integer b, there is a set of prime moduli $p_i < 2^b$ such that $\prod_i p_i \geq 2^{mn}$. We then carry out our calculations mod p_i using b-bit integers, and use the Chinese remainder theorem [14] to recover the coefficients. Even for our largest computations, integers of length $b = 16$ suffice. This approach trades space (the length of the integers) for time (one run for each prime modulus). But the runs for different moduli can be done in parallel, and we do them on separate processors. The final computation using the Chinese remainder theorem has to be done with variable-length integers to produce the coefficients of G^σ, but this takes time and space which is polynomial in their length mn.

To contain lists of configurations of exponential size, an efficient data structure is mandatory. There is no point in using tables of size 3^m (the number of possible ternary sequences) when only $O(\lambda^m)$ signatures actually appear. We use an ordered associative container like `set` or `map` from the standard C++ library where signatures are ordered according to their ternary value. These data structures guarantee logarithmic complexity for search, insert and delete operations, so for lists of size exponential in m they work in $O(m)$ time [14].

Since the maximum degree of G^σ is mn, our integers have fixed width b, and there are $O(\lambda^m)$ different signatures σ, the total space complexity of the algorithm is $O(mn\lambda^m)$. The time complexity of our algorithm is $O(m^3n^2\lambda^m)$. The factor λ^m comes from the size of the lists. One factor of mn comes from the loops in Figure 5 that add the mn vertices one at a time. Another factor of mn comes from copying the polynomials G^σ, shifting them (i.e., multiplying them by z) and adding them together. The last factor of m comes from the logarithmic complexity of the list operations. Ignoring polynomial factors, then, our time and space complexity is $O(\lambda^m)$.

What we have explained so far is the algorithm for the grid. The cylinder requires only a small change in the subroutine **extend**. When adding the last vertex of a row at $c = m$, the subroutine has to ensure compatibility with the vertex at $c = 1$ to make the full signature cyclic. Other than that, no changes are required, and the space and time complexity is the same as for the grid. In particular the algorithm shown in Figure 5 stays the same.

The time complexity increases, however, when we adapt our algorithm to the torus. Here we have to run the algorithm of Figure 5 for each cyclic signature σ in the zeroth row, instead of just starting with $\sigma_\bullet$. Hence we need an additional outer loop of length $\overline{a}(m) = O(\lambda^m)$, resulting in an overall time complexity of $O(m^3n^2\lambda^{2m})$, or $O(\lambda^{2m})$ ignoring polynomial factors. The space complexity remains the same.

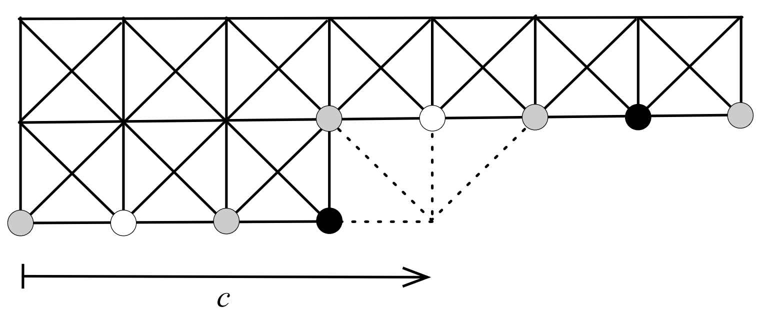

Figure 6: In the king graph, the new vertex at c has to ensure compatibility with 4 neighbors.

For the king graph $K_{m\times n}$, we just need to modify the **extend** subroutine, since it must consider all four neighbors of the new vertex to ensure compatibility (Figure 6). Because of the neighbor in the north-west, the number of signatures is now $a(m+1)$. The main algorithm in Figure 5 stays the same, and the time and space complexity are the same as for the grid and the cylinder.

To push our computations further, we take advantage of symmetries. We can identify each signature of a full row ($c = n$) with its mirror image, which roughly halves the number of signatures. For cyclic signatures we have also translational symmetry, which reduces the number of cyclic signatures by a factor of approximately $1/m$. See Appendix A for the precise factors.

To give the reader an idea about the actual computational resources needed, consider the computationally largest task we solved. Using 16-bit integers for the coefficients, computing $G_{\overline{24}\times 24}(z) \bmod p_i$ for each prime modulus took 125 hours of wall-clock time and required 481 GB of memory. Finally, we needed 36 parallel runs for different moduli p_i to recover the coefficients using the Chinese remainder theorem.

4 Results

	$m \leq$	$n \leq$	$m+n \leq$
$G_{m\times n}(z)$			44
$G_{\overline{m}\times n}(z)$	24	24	
$G_{\overline{m}\times \overline{n}}(z)$	17	17	
$K_{m\times n}(z)$			44
$G_{n\times n}(1)$		24	
$G_{m\times n}(1)$	22	100	
$G_{\overline{m}\times n}(1)$	22	100	
$G_{\overline{n}\times n}(1)$		26	
$K_{m\times n}(1)$	22	100	

Table 3: Sizes of graphs for which we have computed domination polynomials or the total number of dominating sets.

Our algorithm allowed us to compute domination polynomials and the total number of dominating sets for the graphs listed in Table 3. We show the complete domination polynomials for examples of size $m = n \leq 8$ in Appendix B. The complete data is available from the author's website [1].

The varying sizes for which we can carry out these computations are due to two facts. First, the number of cyclic signatures $\overline{a}(m)$ is less than the number of signatures $a(m)$, making computations for the cylinder somewhat easier than those for the grid. Secondly, as discussed above, the computation time for the torus has an extra factor of λ^m due to the need to sum over all starting signatures.

To compute the total number of dominating sets, we used a modified version of our algorithm, in which we do not store the full domination polynomial with each signature, but only its value at $z = 1$. This saves us a factor of mn in time and space complexity and allows us to solve larger systems. In particular, we calculated the number of dominating sets for $m \leq 22$ and $n \leq 100$ for the grid, the cylinder, and the king graph. This allowed us to compute precise numerical estimates for the growth rate of these graphs (see Section 5).

n \ m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8
2	1	2	2	2	3	4	4	4	5	6	6	6	7	8	8	8	9	10	10	10	11	12	12	12
3	1	2	3	3	4	5	6	6	7	8	9	9	10	11	12	12	13	14	15	15	16	17	18	18
4	2	3	4	4	6	6	7	8	10	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	2	3	4	5	7	8	9	10	12	12	14	15	17	17	19	20	21	22	24	24	26	27	29	29
6	2	4	5	6	8	9	11	12	14	15	16	18	20	20	22	24	25	26	28	30	31	32	34	35
7	3	4	6	7	9	10	12	14	16	17	19	20	22	24	25	27	29	30	32	34	36	37	39	40
8	3	5	7	8	10	12	14	16	18	19	21	23	25	27	29	30	32	34	36	38	40	42	44	46
9	3	5	7	9	11	13	15	18	20	21	24	26	28	30	32	34	36	38	41	42	44	46	49	51
10	4	6	8	10	12	14	17	20	22	24	26	28	31	33	36	38	40	42	45	47	49	51	54	56
11	4	6	9	11	13	16	18	21	24	26	28	31	34	36	39	41	44	46	49	52	54	56	59	62
12	4	7	10	12	14	17	20	23	26	28	31	34	37	39	42	45	48	50	53	56	59	61	64	67
13	5	7	10	13	15	18	21	25	28	30	33	36	40	42	45	48	51	54	57	60	63	66	69	72
14	5	8	11	14	16	20	23	27	30	32	36	39	42	45	48	52	55	58	61	64	68	70	74	77
15	5	8	12	15	17	21	24	28	32	34	38	41	45	48	51	55	58	62	65	68	72	75	79	82
16	6	9	13	16	18	22	26	30	34	36	40	44	48	51	54	58	62	66	69	72	76	80	84	87
17	6	9	13	17	19	24	27	32	36	38	43	46	51	54	57	62	65	70	73	76	81	84	89	92
18	6	10	14	18	20	25	29	34	38	40	45	49	54	57	60	65	69	73	77	80	85	89	93	97
19	7	10	15	19	21	26	30	36	40	42	47	51	57	60	63	68	72	77	81	84	89	93	98	102
20	7	11	16	20	22	28	32	38	42	44	50	54	60	63	66	72	76	81	85	88	94	98	103	107
21	7	11	16	21	23	29	33	39	44	46	52	56	62	66	69	75	79	85	89	92	98	102	108	112
22	8	12	17	22	24	30	35	41	46	48	54	59	65	69	72	78	83	89	93	96	102	107	113	117
23	8	12	18	23	25	32	36	43	48	50	57	61	68	72	75	82	86	92	97	100	107	111	117	122
24	8	13	19	24	26	33	38	45	50	52	59	64	71	75	78	85	90	96	101	104	111	116	122	127

Table 4: Domination numbers $\gamma(G_{\overline{m}\times n})$ of the cylinder graph. The array $\gamma(G_{\overline{m}\times n})$ is sequence A375603 in the OEIS, the sequence $\gamma(G_{\overline{n}\times n})$ is sequence A375601

.

From the domination polynomials we can get other parameters like the domination number γ (the minimum cardinality of a dominating set) and the number of these minimum dominating sets. There are, however, more efficient algorithms to compute γ without computing the full domination polynomial. For example, Alanko et al. [3] computed $\gamma(G_{m\times n})$ for $m, n \leq 29$. And in the same year, Gonçalves et al. [17] proved the general formula

$$\gamma(G_{m\times n}) = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4 \qquad (n, m \geq 16)\,. \tag{13}$$

The sequence $\gamma(G_{n\times n})$ is sequence A104519 in the OEIS.

For the cylinder, we did not find any results for the domination number in the literature. Therefore we present Table 4 obtained with our algorithm, giving $\gamma(G_{\overline{m}\times n})$ for all $m, n \leq 24$.

For the torus, Shao et al. [26] computed $\gamma(G_{\overline{n}\times\overline{n}})$ for $n \leq 24$. The corresponding sequence A094087 in the OEIS lists values up to $n = 27$. Crevals and Östergård [15] found formulae for $\gamma(G_{\overline{m}\times\overline{n}})$ for $m < 20$ and arbitrary n.

Finally, for the king graph, no computation is necessary to find γ. Arshad, Hayat, and Jamil [8] showed

$$\gamma(K_{m\times n}) = \left\lceil \frac{m}{3} \right\rceil \left\lceil \frac{n}{3} \right\rceil . \tag{14}$$

The sequence $\gamma(K_{m\times n})$ is sequence A075561 in the OEIS.

	OEIS sequence	OEIS # elements	this work # elements
$N_\gamma(G_{m\times n})$	A350820	276	946
$N_\gamma(G_{n\times n})$	A347632	12	22
$N_\gamma(G_{\overline{m}\times n})$	A375566		300
$N_\gamma(G_{\overline{n}\times n})$	A375569		24
$N_\gamma(G_{\overline{n}\times\overline{n}})$	A347557	8	17
$N_\gamma(K_{m\times n})$	A350815	276	946
$N_\gamma(K_{n\times n})$	A347554	12	22

Table 5: The number of minimum dominating sets, OEIS vs. our results.

Much less is known about the number N_γ of dominating sets of minimum size γ in these graphs. As often, the OEIS is the only source of knowledge for these sequences. Table 5 shows the OEIS results in comparison to our data. Note that the OEIS stores 2-dimensional sequences in linear order read by antidiagonals. Hence if one knows a 2-dimensional sequence $A_{m,n}$ for all $m + n \leq k$, the linear sequence contains $k(k-1)/2$ elements.

Table 6 shows our results for N_γ on $n\times n$ grids, cylinders, tori, and king graphs for various n. Interestingly, all these sequences are non monotonic. This is most easily understood for the king graph: whenever n is divisible by 3, the board can be tiled by $(n/3)^2$ king's neighborhoods of size 3×3, and the unique minimum dominating set has one king in the center of each tile. For other values of n, there are many more arrangements of kings to cope with the interactions between them, including "defects" where the same vertex is covered by more than one king.

As for N_γ, the OEIS is the only source of knowledge for the total number of dominating sets. Table 7 compares the OEIS entries and our results, and Table 8 shows the total number on square grids, i.e., $G_{n\times n}(1)$, for all $n \leq 24$. Results on cylinders, tori, and king graphs are available from the author's website [1].

n	grid $G_{n\times n}$	cylinder $G_{\overline{n}\times n}$	torus $G_{\overline{n}\times\overline{n}}$	king $K_{n\times n}$
2	6	6	6	4
3	10	34	48	1
4	2	16	40	256
5	22	320	10	79
6	288	36	18	1
7	2	56	686	243856
8	52	5565	129224	3600
9	32	20196	36	1
10	4	32210	10	581571283
11	32	88	6292	281585
12	21600	121428	162	1
13	18	388284	2704	2722291223553
14	540360	224	56	32581328
15	34528	1489960	10	1
16	100406	12800	916736	21706368614058886
17	70266144	251464	29327728	5112264019
18	1380216154	2304		1
19	1682689266	36784		268740319616196074546
20	77900162	73062090		1028516654620
21	233645826	29787744		1
22	200997249200	738959760		4839916638142874877046813
23		73600		
24		884736		

Table 6: The number of minimum dominating sets N_γ in various $n \times n$ graphs.

5 Growth rates

The length of the integers in Table 8 demonstrates visually that the total number of dominating sets $G_{n\times n}(1)$ grows exponentially in the area, i.e., as μ^{n^2} for some μ. In fact, it follows from supermultiplicativity and Fekete's Lemma that

$$\lim_{m,n\to\infty} G_{m\times n}(1)^{\frac{1}{mn}} = \sup G_{m\times n}(1)^{\frac{1}{mn}}\ , \tag{15}$$

Since $1 \leq G_{m\times n}(1) \leq 2^{mn}$, the supremum is finite and the limit

$$\mu = \lim_{m,n\to\infty} G_{m\times n}(1)^{\frac{1}{mn}} \tag{16}$$

exists. By the same argument, for any fixed m the limit

$$\mu_m = \lim_{n\to\infty} G_{m\times n}(1)^{\frac{1}{mn}} \tag{17}$$

	OEIS		this work
	sequence	# elements	# elements
$G_{m\times n}(1)$	A218354	198	946
$G_{n\times n}(1)$	A133515	15	24
$G_{\overline{m}\times n}(1)$	A286514	91	325
$G_{\overline{n}\times n}(1)$	A286914	12	26
$G_{\overline{n}\times \overline{n}}(1)$	A303334	8	17
$K_{m\times n}(1)$	A218663	240	946
$K_{n\times n}(1)$	A133791	18	22

Table 7: The total number of dominating sets, OEIS vs. our results.

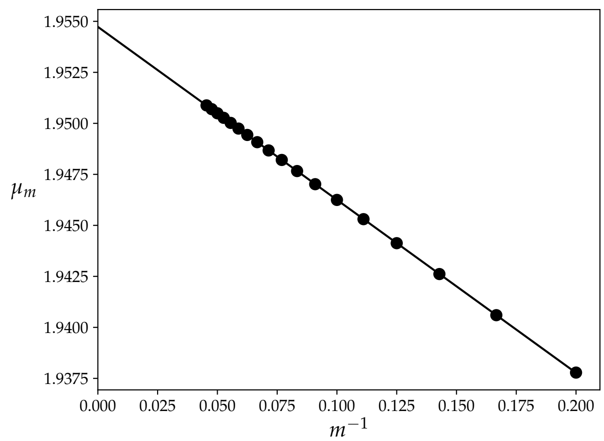

Figure 7: Growth constants μ_m (17) for the grid versus m^{-1}.

exists, and that $\lim_{m\to\infty} \mu_m = \mu$. The same arguments apply to growth rates on the cylinder, torus, and king graph.

Thus, in order to estimate μ numerically, we compute μ_m for some finite values of m and then extrapolate to $m = \infty$. Numerically, we find that the sequence on the right-hand side of

$$\mu_m = \lim_{n\to\infty} \left(\frac{G_{m\times n}(1)}{G_{m\times (n-1)}(1)} \right)^{1/m} \tag{18}$$

converges very quickly: the first 50 decimals no longer change for $n > 30$. A plot of these estimates of μ_m as a function of m^{-1} (Figure 7) suggests that

$$\mu_m \simeq \mu + \frac{\mu^{(1)}}{m} \tag{19}$$

for some negative constant $\mu^{(1)}$.

We could use a linear fit in Figure 7 to estimate μ. But we proceed more carefully, and

1	1
2	11
3	291
4	28661
5	10982565
6	16031828359
7	89373230342147
8	1904212088591018521
9	155026375803222057878889
10	48225130114674924906540348115
11	573224778112724865207700531151 40403
12	2603512578122720760266605183563782799220 77
13	451832336702919293832395537362709344159899302 3433
14	299624906403253780837722041979448648614417149864627538623
15	759209253151473516433210266443037972912262378020875269294775 29215
16	73506889909106192805189860657770727246194962991718803056889525904074284073
17	2719428083163928491947440976457856629839655399596870592597482049076952052830 51026009
18	384423636853239186664840025696090954157887218683165855547606128468077265794742105185 9726699115
19	2076469530351946351039085360915858373667866820688645831018712080835730547861970482231521855946857 72535619
20	428572220521461056343733491916678199495581039073176218002011483464349149322917216079120873741409921393210951745 22717
21	337991241584940716779247214383300275523266782209433800102898023186917008826635091624034957925052629905976346840894055942303943 97
22	1018520678138754342906778088132072422420279415507113849867222415897092226717289124647571996234666411585723180626543154377473388117877845 59599
23	1172781831506875248624491128214157705060276306376593017925817288859622363961093172623234050354080607835750132458893872967965324411437914582111375760429975
24	515997482734451043575475632820633274089783784635404143072896060610383534212287009080497165540248921189521456048180811887786119449831207521812488578297758085704156 70777

Table 8: The total number of dominating sets in the $n \times n$ grid, i.e., $G_{n\times n}(1)$.

take higher order terms into account. We assume that

$$\mu_m = \mu + \sum_{k=1}^{\infty} \frac{\mu^{(k)}}{m^k} \tag{20}$$

and then use Bulirsch-Stoer extrapolation [12], a reliable, rapidly converging method based on rational interpolation. See [22, Sec. 4.3] for a detailed description of this method applied in a similar situation. As result we get

$$\mu = 1.9547511954080(8)\,. \tag{21}$$

For the growth rate of the cylinder we get

$$\overline{\mu} = 1.9547511954085(3)\,, \tag{22}$$

which equals the growth rate for the grid within the error bars. Based on the assumption that the vertical boundaries of the grid have a decaying effect as $m \to \infty$, we conjecture that these two growth rates are in fact equal.

Our data for the torus is not sufficient to compute its growth constant with the same accuracy, but we do not have to! If you look at (7b) and (11), you see that in the limit $n \to \infty$, both right-hand sides are dominated by the largest eigenvalue of the matrix $\overline{A}$ for $z = 1$, which equals the growth rate $\overline{\mu}_m$. Hence the growth rate for the torus equals that for the cylinder (22).

Let η denote the growth rate of the king graph. With the same methods, we estimate

$$\eta = 1.997064386596(3)\,. \tag{23}$$

This value fits right between the bounds proved by Baumann et al. [9],

$$1.9969 \leq \eta \leq 1.9972\,, \tag{24}$$

and we conjecture that the first ten decimal digits of (23) are correct.

6 Conclusions

We have presented a transfer matrix algorithm for computing dominating polynomials, and in particular counting dominating sets and minimum dominating sets, on the grid, cylinder, and torus graphs, and on the king graph. While our algorithm takes exponential time and requires exponential space, we are able to significantly reduce the exponent by breaking the induction over rows into an induction over single vertices.

Along with a careful use of representations and data structures, including representing large integers using the Chinese remainder theorem, this reduces the running time (ignoring polynomial factors) to $O(\lambda^m)$ for the grid, cylinder, and king graph, and $O(\lambda^{2m})$ for the torus, where $\lambda = 1 + \sqrt{2} = 2.4142...$ We use this algorithm to count dominating sets on these graphs, where the number of rows n and columns m range up to 24. This allows us to extend several OEIS sequences considerably, and to obtain high-precision estimates of the growth rate μ, where the number of dominating sets on $m \times n$ graphs grows asymptotically as μ^{mn}. We believe that similar techniques can be applied to many other periodic graphs based on low-dimensional lattices, and to other kinds of sets of interest in graph theory.

7 Acknowledgments

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A The number of signatures

Let $a_\circ(m)$, $a_{\color{gray}\bullet}(m)$ and $a_\bullet(m)$ denote the number of signatures of length m that end with $\circ$, ${\color{gray}\bullet}$, and $\bullet$ respectively. Obviously $a(m) = a_\circ(m) + a_{\color{gray}\bullet}(m) + a_\bullet(m)$. We also have

$$\begin{aligned}
a_\circ(m) &= a_\circ(m-1) + a_{\color{gray}\bullet}(m-1)\,, \\
a_{\color{gray}\bullet}(m) &= a_\circ(m-1) + a_{\color{gray}\bullet}(m-1) + a_\bullet(m-1) = a(m-1)\,, \\
a_\bullet(m) &= a_{\color{gray}\bullet}(m-1) + a_\bullet(m-1)\,.
\end{aligned}$$

Adding these three equations yields

$$a(m) = 2a(m-1) + a_{\color{gray}\bullet}(m-1)\,,$$

and inserting the equation for $a_{\color{gray}\bullet}(m-1)$ provides us with the Pell-type recurrence

$$a(m) = 2a(m-1) + a(m-2)\,. \tag{25}$$

The characteristic polynomial of the recurrence is $P(\lambda) = \lambda^2 - 2\lambda - 1$ with zeroes $1 \pm \sqrt{2}$. Hence the solution of (25) is

$$a(m) = A_-\left(1 - \sqrt{2}\right)^m + A_+\left(1 + \sqrt{2}\right)^m\,, \tag{26a}$$

where the coefficients A_+ and A_- are fixed by the base cases $a(0)$ and $a(1)$,

$$A_- = \frac{\left(1 + \sqrt{2}\right)a(0) - a(1)}{2\sqrt{2}} \qquad A_+ = \frac{a(0) - \left(1 - \sqrt{2}\right)a(1)}{2\sqrt{2}} \tag{26b}$$

In our case, $a(1) = 3$ and $a(2) = 7$, which implies $a(0) = 1$ and yields (2).

As we discussed, we can identify a signature with its mirror image. Taking into account this reflection symmetry, the resulting number of signatures is

$$\hat{a}(m) = \frac{1}{2}\Big(a(m) + a(\lfloor \tfrac{m+1}{2} \rfloor)\Big)\,, \tag{27}$$

which is sequence A030270. This formula is easily understood. Reflection symmetry gives us a factor of 1/2 for all non-symmetric signatures. If we apply the factor 1/2 to all signatures, we need to add back the number of symmetric signatures, which are completely specified by their first half.

For signatures with a kink between $c - 1$ and c, the recurrence reads

$$a_c(m) = \begin{cases} 3a_c(m-1), & \text{if } m = c; \\ 2a_c(m-1) + a_c(m-2), & \text{otherwise.} \end{cases} \tag{28}$$

Obviously, $a_c(m)$ follows (2) for $m < c$ and for $m > c$. Hence, $a_c(m)$ is also solved by (26a), but with change of A_- and A_+ as m passes c. The asymptotic scaling $O(\lambda^m)$ persists.

For cyclic signatures, the derivation of (3) is a bit more involved. Knopfmacher et al. [21] used Chebyshev polynomials to derive the generating function for $\overline{a}(m)$. Here we give a more elementary derivation.

Let $\overline{a}_{\sigma_m,\sigma_1}(m)$ denote the number of cyclic signatures of length m with values σ_1 and σ_m at their 1st and mth position. Then

$$\begin{aligned}
\overline{a}_{\circ,\circ}(m) &= \overline{a}_{\circ,\circ}(m-1) + \overline{a}_{{\color{gray}\bullet},\circ}(m-1)\,,\\
\overline{a}_{\circ,{\color{gray}\bullet}}(m) &= \overline{a}_{\circ,{\color{gray}\bullet}}(m-1) + \overline{a}_{{\color{gray}\bullet},{\color{gray}\bullet}}(m-1)\,,\\
\overline{a}_{{\color{gray}\bullet},\circ}(m) &= \overline{a}_{\circ,\circ}(m-1) + \overline{a}_{{\color{gray}\bullet},\circ}(m-1)\,,\\
\overline{a}_{{\color{gray}\bullet},{\color{gray}\bullet}}(m) &= \overline{a}_{\circ,{\color{gray}\bullet}}(m-1) + \overline{a}_{{\color{gray}\bullet},{\color{gray}\bullet}}(m-1) + \overline{a}_{\bullet,{\color{gray}\bullet}}(m-1)\,,\\
\overline{a}_{{\color{gray}\bullet},\bullet}(m) &= \overline{a}_{{\color{gray}\bullet},\bullet}(m-1) + \overline{a}_{\bullet,\bullet}(m-1)\,,\\
\overline{a}_{\bullet,{\color{gray}\bullet}}(m) &= \overline{a}_{\circ,\circ}(m-1) + \overline{a}_{{\color{gray}\bullet},\circ}(m-1) + \overline{a}_{\circ,{\color{gray}\bullet}}(m-1) + \overline{a}_{{\color{gray}\bullet},{\color{gray}\bullet}}(m-1) + \overline{a}_{\bullet,{\color{gray}\bullet}}(m-1)\,,\\
\overline{a}_{\bullet,\bullet}(m) &= \overline{a}_{{\color{gray}\bullet},\bullet}(m-1) + \overline{a}_{\bullet,\bullet}(m-1)\,.
\end{aligned} \tag{29}$$

On the right-hand sides, $\overline{a}_{{\color{gray}\bullet},\bullet}$, $\overline{a}_{\bullet,{\color{gray}\bullet}}$ and $\overline{a}_{\bullet,\bullet}$ appear twice, and all other σ's appear three times. Hence, adding all these equations yields

$$\overline{a}(m) = 3\overline{a}(m-1) - \big[\overline{a}_{{\color{gray}\bullet},\bullet}(m-1) + \overline{a}_{\bullet,{\color{gray}\bullet}}(m-1) + \overline{a}_{\bullet,\bullet}(m-1)\big]\,. \tag{30}$$

When we apply the recurrence (29) to the terms in brackets, we notice that $\overline{a}_{{\color{gray}\bullet},\bullet}$ and $\overline{a}_{\bullet\bullet}$ appear twice, and all other σ's appear exactly once. This gives

$$[\cdots] = \overline{a}(m-2) + \big\{\overline{a}_{{\color{gray}\bullet},\bullet}(m-2) + \overline{a}_{\bullet\bullet}(m-2)\big\}\,. \tag{31}$$

If we apply (29) to the terms in the curly brackets, we get $\{\cdots\} = \overline{a}(m-3)$, and finally

$$\overline{a}(m) = 3\overline{a}(m-1) - \overline{a}(m-2) - \overline{a}(m-3)\,. \tag{32}$$

The characteristic polynomial of this recurrence is

$$P(\lambda) = \lambda^3 - 3\lambda^2 + \lambda + 1 = (\lambda - 1)(\lambda^2 - 2\lambda - 1)\,, \tag{33}$$

with zeroes 1, $1 - \sqrt{2}$, and $1 + \sqrt{2}$. Hence the solution of (32) is

$$\overline{a}(m) = C_1 + C_- \left(1 - \sqrt{2}\right)^m + C_+ \left(1 + \sqrt{2}\right)^m \,, \tag{34}$$

where C_1, C_-, and C_+ depend on the base case $\overline{a}(0)$, $\overline{a}(1)$, and $\overline{a}(2)$:

$$\begin{aligned} C_1 &= \frac{1}{2}\overline{a}(0) + \overline{a}(1) - \frac{1}{2}\overline{a}(2)\,, \\ C_- &= \frac{2+\sqrt{2}}{4\sqrt{2}}\overline{a}(0) - \frac{2+2\sqrt{2}}{4\sqrt{2}}\overline{a}(1) + \frac{1}{4}\overline{a}(2)\,, \\ C_+ &= -\frac{2-\sqrt{2}}{4\sqrt{2}}\overline{a}(0) + \frac{2-2\sqrt{2}}{4\sqrt{2}}\overline{a}(1) + \frac{1}{4}\overline{a}(2)\,. \end{aligned} \tag{35}$$

In our case we have $\overline{a}(1) = 3$, $\overline{a}(2) = 7$ and $\overline{a}(3) = 15$ which implies $\overline{a}(0) = 3$ and therefore $C_1 = C_- = C_+ = 1$, which gives (3).

If one takes into account circular and reflection symmetry, the number of signatures is approximately $\overline{a}(m)/2m$, as can be checked by dividing sequence A208716 by A124696.

B Domination polynomials

Tables 9, 10, 11 and 12 show the domination polynomials of the $n \times n$ grid, cylinder, torus and king graph for $n \leq 8$. The domination polynomials for larger and rectangular graphs can be downloaded from the author's website [1].

$$G_{1\times 1}(z) = z$$

$$G_{2\times 2}(z) = 6\,z^{2} + 4\,z^{3} + z^{4}$$

$$G_{3\times 3}(z) = 10\,z^{3} + 57\,z^{4} + 98\,z^{5} + 80\,z^{6} + 36\,z^{7} + 9\,z^{8} + z^{9}$$

$$\begin{aligned} G_{4\times 4}(z) = {} & 20\,z^{4} + 40\,z^{5} + 554\,z^{6} + 2\,484\,z^{7} + 5\,494\,z^{8} + 7\,268\,z^{9} + 6\,402\,z^{10} + \\ & 3\,964\,z^{11} + 1\,760\,z^{12} + 556\,z^{13} + 120\,z^{14} + 16\,z^{15} + z^{16} \end{aligned}$$

$$\begin{aligned} G_{5\times 5}(z) = {} & 22\,z^{7} + 1\,545\,z^{8} + 22\,594\,z^{9} + 140\,304\,z^{10} + 492\,506\,z^{11} + 1\,126\,091\,z^{12} + \\ & 1\,823\,057\,z^{13} + 2\,204\,694\,z^{14} + 2\,063\,202\,z^{15} + 1\,528\,544\,z^{16} + 908\,623\,z^{17} + \\ & 435\,832\,z^{18} + 168\,426\,z^{19} + 51\,953\,z^{20} + 12\,550\,z^{21} + 2\,296\,z^{22} + 300\,z^{23} + \\ & 25\,z^{24} + z^{25} \end{aligned}$$

$$\begin{aligned} G_{6\times 6}(z) = {} & 288\,z^{10} + 20\,896\,z^{11} + 478\,624\,z^{12} + 5\,119\,512\,z^{13} + 32\,070\,018\,z^{14} + \\ & 133\,299\,396\,z^{15} + 397\,278\,079\,z^{16} + 894\,777\,804\,z^{17} + 1\,581\,325\,412\,z^{18} + \\ & 2\,254\,665\,800\,z^{19} + 2\,648\,227\,540\,z^{20} + 2\,602\,834\,832\,z^{21} + 2\,165\,708\,332\,z^{22} + \\ & 1\,538\,223\,528\,z^{23} + 937\,732\,160\,z^{24} + 492\,091\,912\,z^{25} + 222\,401\,360\,z^{26} + \\ & 86\,397\,060\,z^{27} + 28\,715\,172\,z^{28} + 8\,101\,900\,z^{29} + 1\,917\,814\,z^{30} + 374\,360\,z^{31} + \\ & 58\,757\,z^{32} + 7\,136\,z^{33} + 630\,z^{34} + 36\,z^{35} + z^{36} \end{aligned}$$

$$\begin{aligned} G_{7\times 7}(z) = {} & 2\,z^{12} + 682\,z^{13} + 69\,818\,z^{14} + 2\,809\,634\,z^{15} + 58\,346\,490\,z^{16} + 722\,332\,499\,z^{17} + \\ & 5\,873\,091\,754\,z^{18} + 33\,720\,209\,068\,z^{19} + 144\,326\,231\,696\,z^{20} + 479\,699\,210\,510\,z^{21} + \\ & 1\,277\,484\,819\,726\,z^{22} + 2\,793\,279\,785\,490\,z^{23} + 5\,112\,738\,876\,944\,z^{24} + \\ & 7\,956\,389\,260\,884\,z^{25} + 10\,659\,803\,571\,300\,z^{26} + 12\,421\,321\,161\,300\,z^{27} + \\ & 12\,692\,372\,752\,380\,z^{28} + 11\,448\,278\,299\,084\,z^{29} + 9\,162\,679\,913\,216\,z^{30} + \\ & 6\,533\,166\,152\,352\,z^{31} + 4\,161\,998\,104\,421\,z^{32} + 2\,373\,420\,930\,490\,z^{33} + \\ & 1\,212\,661\,131\,156\,z^{34} + 555\,107\,862\,078\,z^{35} + 227\,428\,059\,844\,z^{36} + \\ & 83\,222\,666\,789\,z^{37} + 27\,112\,560\,820\,z^{38} + 7\,828\,049\,130\,z^{39} + 1\,990\,771\,673\,z^{40} + \\ & 442\,325\,654\,z^{41} + 84\,949\,536\,z^{42} + 13\,902\,582\,z^{43} + 1\,901\,827\,z^{44} + 211\,672\,z^{45} + \\ & 18\,420\,z^{46} + 1\,176\,z^{47} + 49\,z^{48} + z^{49} \end{aligned}$$

$$\begin{aligned} G_{8\times 8}(z) = {} & 52\,z^{16} + 15\,864\,z^{17} + 1\,722\,568\,z^{18} + 88\,226\,896\,z^{19} + 2\,530\,732\,136\,z^{20} + \\ & 45\,375\,987\,524\,z^{21} + 550\,599\,054\,884\,z^{22} + 4\,804\,379\,992\,724\,z^{23} + \\ & 31\,600\,623\,255\,338\,z^{24} + 162\,562\,260\,288\,736\,z^{25} + 673\,394\,654\,370\,166\,z^{26} + \\ & 2\,299\,264\,864\,482\,900\,z^{27} + 6\,594\,998\,844\,457\,680\,z^{28} + 16\,140\,569\,091\,024\,412\,z^{29} + \\ & 34\,145\,122\,808\,773\,410\,z^{30} + 63\,119\,173\,723\,897\,716\,z^{31} + 102\,895\,753\,969\,864\,066\,z^{32} + \\ & 149\,077\,597\,217\,535\,156\,z^{33} + 193\,230\,536\,934\,785\,376\,z^{34} + 225\,335\,102\,676\,614\,928\,z^{35} + \\ & 237\,544\,411\,406\,921\,016\,z^{36} + 227\,287\,805\,873\,540\,304\,z^{37} + 198\,057\,834\,976\,389\,932\,z^{38} + \\ & 157\,618\,769\,172\,704\,668\,z^{39} + 114\,817\,612\,849\,042\,346\,z^{40} + 76\,694\,678\,728\,213\,904\,z^{41} + \\ & 47\,038\,041\,070\,108\,638\,z^{42} + 26\,511\,846\,459\,068\,480\,z^{43} + 13\,738\,205\,846\,668\,894\,z^{44} + \\ & 6\,545\,243\,405\,852\,040\,z^{45} + 2\,865\,791\,004\,809\,792\,z^{46} + 1\,152\,143\,554\,074\,948\,z^{47} + \\ & 424\,740\,089\,888\,210\,z^{48} + 143\,310\,533\,096\,044\,z^{49} + 44\,147\,026\,143\,576\,z^{50} + \\ & 12\,377\,560\,349\,296\,z^{51} + 3\,146\,185\,878\,694\,z^{52} + 721\,528\,535\,044\,z^{53} + \\ & 148\,407\,392\,344\,z^{54} + 27\,176\,088\,292\,z^{55} + 4\,389\,826\,708\,z^{56} + 618\,261\,932\,z^{57} + \\ & 74\,786\,314\,z^{58} + 7\,615\,724\,z^{59} + 635\,108\,z^{60} + 41\,660\,z^{61} + 2\,016\,z^{62} + 64\,z^{63} + z^{64} \end{aligned}$$

Table 9: Domination polynomials of the grid graph $G_{n\times n}$.

$$
\begin{aligned}
G_{\overline{1}\times 1}(z) &= z\\
G_{\overline{2}\times 2}(z) &= 6\,z^2+4\,z^3+z^4\\
G_{\overline{3}\times 3}(z) &= 34\,z^3+99\,z^4+120\,z^5+84\,z^6+36\,z^7+9\,z^8+z^9\\
G_{\overline{4}\times 4}(z) &= 16\,z^4+248\,z^5+1\,560\,z^6+4\,752\,z^7+8\,308\,z^8+9\,376\,z^9+7\,404\,z^{10}+4\,264\,z^{11}+1\,812\,z^{12}+\\
&\quad 560\,z^{13}+120\,z^{14}+16\,z^{15}+z^{16}\\
G_{\overline{5}\times 5}(z) &= 320\,z^7+8\,525\,z^8+77\,240\,z^9+354\,768\,z^{10}+1\,000\,860\,z^{11}+1\,934\,895\,z^{12}+2\,744\,825\,z^{13}+\\
&\quad 2\,988\,230\,z^{14}+2\,571\,838\,z^{15}+1\,783\,400\,z^{16}+1\,007\,095\,z^{17}+464\,780\,z^{18}+174\,710\,z^{19}+\\
&\quad 52\,905\,z^{20}+12\,640\,z^{21}+2\,300\,z^{22}+300\,z^{23}+25\,z^{24}+z^{25}\\
G_{\overline{6}\times 6}(z) &= 36\,z^9+5\,304\,z^{10}+182\,640\,z^{11}+2\,674\,472\,z^{12}+20\,888\,976\,z^{13}+102\,474\,888\,z^{14}+349\,290\,996\,z^{15}+\\
&\quad 883\,272\,549\,z^{16}+1\,733\,585\,388\,z^{17}+2\,727\,960\,890\,z^{18}+3\,525\,246\,624\,z^{19}+3\,808\,843\,866\,z^{20}+\\
&\quad 3\,487\,178\,896\,z^{21}+2\,732\,164\,086\,z^{22}+1\,844\,521\,704\,z^{23}+1\,077\,669\,852\,z^{24}+545\,975\,556\,z^{25}+\\
&\quad 239\,780\,520\,z^{26}+91\,042\,704\,z^{27}+29\,727\,648\,z^{28}+8\,277\,408\,z^{29}+1\,941\,108\,z^{30}+376\,584\,z^{31}+\\
&\quad 58\,893\,z^{32}+7\,140\,z^{33}+630\,z^{34}+36\,z^{35}+z^{36}\\
G_{\overline{7}\times 7}(z) &= 56\,z^{12}+17\,878\,z^{13}+1\,155\,252\,z^{14}+31\,054\,898\,z^{15}+456\,455\,958\,z^{16}+4\,228\,396\,193\,z^{17}+\\
&\quad 27\,003\,670\,764\,z^{18}+126\,567\,019\,852\,z^{19}+455\,787\,743\,684\,z^{20}+1\,305\,495\,024\,212\,z^{21}+\\
&\quad 3\,054\,799\,279\,140\,z^{22}+5\,964\,099\,864\,170\,z^{23}+9\,880\,494\,881\,782\,z^{24}+14\,079\,356\,852\,554\,z^{25}+\\
&\quad 17\,447\,648\,954\,876\,z^{26}+18\,972\,152\,485\,706\,z^{27}+18\,232\,693\,610\,636\,z^{28}+15\,575\,358\,475\,348\,z^{29}+\\
&\quad 11\,880\,424\,274\,852\,z^{30}+8\,119\,023\,303\,202\,z^{31}+4\,982\,943\,200\,557\,z^{32}+2\,750\,423\,714\,766\,z^{33}+\\
&\quad 1\,366\,055\,406\,058\,z^{34}+610\,263\,826\,646\,z^{35}+244\,883\,991\,996\,z^{36}+88\,057\,328\,933\,z^{37}+\\
&\quad 28\,275\,236\,934\,z^{38}+8\,068\,294\,570\,z^{39}+2\,032\,827\,433\,z^{40}+448\,443\,744\,z^{41}+85\,669\,472\,z^{42}+\\
&\quad 13\,968\,430\,z^{43}+1\,906\,219\,z^{44}+211\,862\,z^{45}+18\,424\,z^{46}+1\,176\,z^{47}+49\,z^{48}+z^{49}\\
G_{\overline{8}\times 8}(z) &= 5\,556\,z^{16}+877\,312\,z^{17}+53\,209\,280\,z^{18}+1\,705\,112\,768\,z^{19}+33\,445\,432\,384\,z^{20}+\\
&\quad 439\,072\,279\,040\,z^{21}+4\,109\,617\,399\,080\,z^{22}+28\,780\,589\,281\,584\,z^{23}+156\,652\,617\,731\,416\,z^{24}+\\
&\quad 683\,114\,966\,762\,944\,z^{25}+2\,445\,690\,796\,232\,104\,z^{26}+7\,333\,807\,159\,180\,640\,z^{27}+18\,724\,721\,152\,985\,788\,z^{28}+\\
&\quad 41\,265\,837\,337\,782\,160\,z^{29}+79\,400\,630\,946\,848\,664\,z^{30}+134\,680\,399\,945\,312\,528\,z^{31}+\\
&\quad 203\,039\,926\,797\,499\,914\,z^{32}+273\,950\,585\,370\,935\,584\,z^{33}+332\,770\,579\,433\,142\,856\,z^{34}+\\
&\quad 365\,749\,751\,152\,851\,088\,z^{35}+365\,293\,505\,626\,221\,476\,z^{36}+332\,720\,567\,077\,905\,776\,z^{37}+\\
&\quad 277\,203\,692\,560\,942\,216\,z^{38}+211\,771\,844\,116\,575\,568\,z^{39}+148\,641\,968\,502\,148\,908\,z^{40}+\\
&\quad 96\,000\,555\,048\,304\,144\,z^{41}+57\,112\,559\,682\,929\,880\,z^{42}+31\,318\,418\,200\,248\,960\,z^{43}+\\
&\quad 15\,833\,769\,466\,628\,176\,z^{44}+7\,379\,242\,217\,245\,312\,z^{45}+3\,168\,290\,754\,707\,192\,z^{46}+\\
&\quad 1\,251\,914\,193\,916\,144\,z^{47}+454\,574\,372\,292\,346\,z^{48}+151\,368\,545\,763\,424\,z^{49}+\\
&\quad 46\,103\,561\,935\,240\,z^{50}+12\,802\,119\,434\,064\,z^{51}+3\,227\,917\,903\,348\,z^{52}+735\,359\,555\,024\,z^{53}+\\
&\quad 150\,440\,930\,640\,z^{54}+27\,431\,963\,344\,z^{55}+4\,416\,833\,096\,z^{56}+620\,587\,536\,z^{57}+74\,943\,232\,z^{58}+\\
&\quad 7\,623\,504\,z^{59}+635\,360\,z^{60}+41\,664\,z^{61}+2\,016\,z^{62}+64\,z^{63}+z^{64}
\end{aligned}
$$

Table 10: Domination polynomials of the cylinder graph $G_{\overline{n}\times n}$.

$$
\begin{aligned}
G_{\overline{1}\times\overline{1}}(z) &= z \\
G_{\overline{2}\times\overline{2}}(z) &= 6\,z^{2} + 4\,z^{3} + z^{4} \\
G_{\overline{3}\times\overline{3}}(z) &= 48\,z^{3} + 117\,z^{4} + 126\,z^{5} + 84\,z^{6} + 36\,z^{7} + 9\,z^{8} + z^{9} \\
G_{\overline{4}\times\overline{4}}(z) &= 40\,z^{4} + 560\,z^{5} + 2\,736\,z^{6} + 6\,800\,z^{7} + 10\,310\,z^{8} + 10\,560\,z^{9} + 7\,832\,z^{10} + 4\,352\,z^{11} + 1\,820\,z^{12} + \\
&\quad 560\,z^{13} + 120\,z^{14} + 16\,z^{15} + z^{16} \\
G_{\overline{5}\times\overline{5}}(z) &= 10\,z^{5} + 200\,z^{6} + 3\,050\,z^{7} + 31\,525\,z^{8} + 188\,700\,z^{9} + 677\,690\,z^{10} + 1\,610\,700\,z^{11} + 2\,740\,775\,z^{12} + \\
&\quad 3\,527\,075\,z^{13} + 3\,562\,700\,z^{14} + 2\,895\,610\,z^{15} + 1\,923\,600\,z^{16} + 1\,053\,175\,z^{17} + 475\,950\,z^{18} + \\
&\quad 176\,600\,z^{19} + 53\,105\,z^{20} + 12\,650\,z^{21} + 2\,300\,z^{22} + 300\,z^{23} + 25\,z^{24} + z^{25} \\
G_{\overline{6}\times\overline{6}}(z) &= 18\,z^{8} + 792\,z^{9} + 42\,480\,z^{10} + 901\,692\,z^{11} + 9\,417\,660\,z^{12} + 57\,622\,212\,z^{13} + 234\,273\,096\,z^{14} + \\
&\quad 686\,972\,304\,z^{15} + 1\,535\,339\,241\,z^{16} + 2\,718\,976\,500\,z^{17} + 3\,925\,148\,718\,z^{18} + 4\,717\,557\,288\,z^{19} + \\
&\quad 4\,795\,710\,066\,z^{20} + 4\,172\,271\,408\,z^{21} + 3\,133\,155\,636\,z^{22} + 2\,042\,728\,812\,z^{23} + 1\,160\,244\,930\,z^{24} + \\
&\quad 574\,802\,640\,z^{25} + 248\,126\,706\,z^{26} + 93\,014\,644\,z^{27} + 30\,098\,664\,z^{28} + 8\,330\,940\,z^{29} + 1\,946\,676\,z^{30} + \\
&\quad 376\,956\,z^{31} + 58\,905\,z^{32} + 7\,140\,z^{33} + 630\,z^{34} + 36\,z^{35} + z^{36} \\
G_{\overline{7}\times\overline{7}}(z) &= 686\,z^{12} + 205\,996\,z^{13} + 9\,203\,432\,z^{14} + 182\,205\,912\,z^{15} + 2\,082\,222\,660\,z^{16} + 15\,633\,666\,139\,z^{17} + \\
&\quad 83\,589\,101\,666\,z^{18} + 336\,543\,504\,122\,z^{19} + 1\,062\,883\,834\,964\,z^{20} + 2\,715\,977\,010\,936\,z^{21} + \\
&\quad 5\,751\,616\,552\,262\,z^{22} + 10\,287\,521\,966\,512\,z^{23} + 15\,778\,748\,654\,928\,z^{24} + 21\,007\,961\,215\,738\,z^{25} + \\
&\quad 24\,521\,234\,114\,524\,z^{26} + 25\,294\,410\,442\,980\,z^{27} + 23\,207\,364\,109\,062\,z^{28} + 19\,035\,405\,413\,402\,z^{29} + \\
&\quad 14\,013\,460\,448\,554\,z^{30} + 9\,286\,179\,999\,558\,z^{31} + 5\,549\,897\,026\,821\,z^{32} + 2\,994\,639\,956\,448\,z^{33} + \\
&\quad 1\,459\,111\,542\,322\,z^{34} + 641\,506\,327\,014\,z^{35} + 254\,073\,916\,530\,z^{36} + 90\,407\,322\,159\,z^{37} + \\
&\quad 28\,792\,214\,486\,z^{38} + 8\,164\,773\,470\,z^{39} + 2\,047\,811\,969\,z^{40} + 450\,329\,306\,z^{41} + 85\,854\,230\,z^{42} + \\
&\quad 13\,981\,660\,z^{43} + 1\,906\,835\,z^{44} + 211\,876\,z^{45} + 18\,424\,z^{46} + 1\,176\,z^{47} + 49\,z^{48} + z^{49} \\
G_{\overline{8}\times\overline{8}}(z) &= 129\,224\,z^{16} + 14\,681\,344\,z^{17} + 651\,801\,600\,z^{18} + 15\,758\,203\,520\,z^{19} + 240\,372\,029\,072\,z^{20} + \\
&\quad 2\,528\,654\,078\,528\,z^{21} + 19\,500\,205\,324\,032\,z^{22} + 115\,290\,942\,264\,448\,z^{23} + 540\,832\,229\,850\,464\,z^{24} + \\
&\quad 2\,068\,173\,372\,971\,840\,z^{25} + 6\,588\,920\,903\,240\,288\,z^{26} + 17\,801\,592\,852\,676\,672\,z^{27} + \\
&\quad 41\,390\,172\,398\,524\,272\,z^{28} + 83\,839\,998\,055\,557\,568\,z^{29} + 149\,484\,557\,713\,246\,144\,z^{30} + \\
&\quad 236\,656\,119\,110\,649\,024\,z^{31} + 335\,142\,837\,708\,961\,654\,z^{32} + 427\,236\,939\,021\,347\,072\,z^{33} + \\
&\quad 492\,905\,450\,386\,702\,720\,z^{34} + 517\,004\,156\,810\,313\,664\,z^{35} + 494\,919\,960\,091\,734\,336\,z^{36} + \\
&\quad 433\,802\,866\,482\,847\,616\,z^{37} + 349\,085\,443\,267\,295\,680\,z^{38} + 258\,463\,881\,482\,739\,136\,z^{39} + \\
&\quad 176\,377\,167\,134\,882\,296\,z^{40} + 111\,074\,953\,233\,247\,104\,z^{41} + 64\,609\,763\,870\,627\,264\,z^{42} + \\
&\quad 34\,728\,863\,089\,747\,456\,z^{43} + 17\,251\,322\,181\,046\,784\,z^{44} + 7\,916\,762\,958\,356\,992\,z^{45} + \\
&\quad 3\,353\,820\,958\,699\,552\,z^{46} + 1\,310\,034\,044\,881\,664\,z^{47} + 471\,036\,957\,313\,244\,z^{48} + \\
&\quad 155\,565\,089\,543\,040\,z^{49} + 47\,060\,663\,909\,504\,z^{50} + 12\,995\,994\,842\,880\,z^{51} + 3\,262\,480\,436\,912\,z^{52} + \\
&\quad 740\,719\,463\,168\,z^{53} + 151\,153\,208\,768\,z^{54} + 27\,511\,470\,912\,z^{55} + 4\,424\,085\,048\,z^{56} + 621\,106\,688\,z^{57} + \\
&\quad 74\,970\,592\,z^{58} + 7\,624\,448\,z^{59} + 635\,376\,z^{60} + 41\,664\,z^{61} + 2\,016\,z^{62} + 64\,z^{63} + z^{64}
\end{aligned}
$$

Table 11: Domination polynomials of the torus graph $G_{\overline{n}\times\overline{n}}$.

$$K_{1\times1}(z) = z$$

$$K_{2\times2}(z) = 4\,z^1 + 6\,z^2 + 4\,z^3 + z^4$$

$$K_{3\times3}(z) = z^1 + 10\,z^2 + 48\,z^3 + 106\,z^4 + 122\,z^5 + 84\,z^6 + 36\,z^7 + 9\,z^8 + z^9$$

$$\begin{aligned}K_{4\times4}(z) = {} & 256\,z^4 + 1\,536\,z^5 + 4\,480\,z^6 + 8\,320\,z^7 + 10\,896\,z^8 + 10\,560\,z^9 + 7\,744\,z^{10} + 4\,320\,z^{11} + 1\,816\,z^{12} + \\ & 560\,z^{13} + 120\,z^{14} + 16\,z^{15} + z^{16}\end{aligned}$$

$$\begin{aligned}K_{5\times5}(z) = {} & 79\,z^4 + 1\,593\,z^5 + 14\,672\,z^6 + 81\,524\,z^7 + 307\,244\,z^8 + 842\,506\,z^9 + 1\,764\,068\,z^{10} + 2\,918\,828\,z^{11} + \\ & 3\,909\,834\,z^{12} + 4\,311\,034\,z^{13} + 3\,955\,232\,z^{14} + 3\,038\,092\,z^{15} + 1\,957\,940\,z^{16} + 1\,056\,965\,z^{17} + \\ & 475\,304\,z^{18} + 176\,256\,z^{19} + 53\,046\,z^{20} + 12\,646\,z^{21} + \\ & 2\,300\,z^{22} + 300\,z^{23} + 25\,z^{24} + z^{25}\end{aligned}$$

$$\begin{aligned}K_{6\times6}(z) = {} & z^4 + 56\,z^5 + 1\,652\,z^6 + 31\,664\,z^7 + 404\,770\,z^8 + 3\,416\,472\,z^9 + 19\,840\,300\,z^{10} + 84\,209\,540\,z^{11} + \\ & 275\,031\,868\,z^{12} + 718\,655\,796\,z^{13} + 1\,546\,177\,306\,z^{14} + 2\,797\,874\,908\,z^{15} + 4\,326\,011\,372\,z^{16} + \\ & 5\,782\,863\,816\,z^{17} + 6\,741\,695\,574\,z^{18} + 6\,897\,654\,436\,z^{19} + 6\,220\,635\,186\,z^{20} + 4\,958\,580\,672\,z^{21} + \\ & 3\,498\,131\,846\,z^{22} + 2\,184\,049\,652\,z^{23} + 1\,205\,216\,450\,z^{24} + 586\,259\,808\,z^{25} + 250\,349\,560\,z^{26} + \\ & 93\,305\,796\,z^{27} + 30\,113\,038\,z^{28} + 8\,327\,600\,z^{29} + 1\,945\,800\,z^{30} + 376\,864\,z^{31} + 58\,901\,z^{32} + \\ & 7\,140\,z^{33} + 630\,z^{34} + 36\,z^{35} + z^{36}\end{aligned}$$

$$\begin{aligned}K_{7\times7}(z) = {} & 243\,856\,z^9 + 7\,483\,274\,z^{10} + 108\,525\,780\,z^{11} + 995\,661\,210\,z^{12} + 6\,526\,376\,452\,z^{13} + 32\,723\,647\,242\,z^{14} + \\ & 131\,188\,032\,404\,z^{15} + 433\,817\,785\,292\,z^{16} + 1\,211\,009\,331\,050\,z^{17} + 2\,904\,839\,371\,392\,z^{18} + \\ & 6\,071\,176\,663\,246\,z^{19} + 11\,178\,937\,768\,294\,z^{20} + 18\,295\,752\,974\,580\,z^{21} + \\ & 26\,804\,759\,801\,972\,z^{22} + 35\,356\,180\,710\,524\,z^{23} + 42\,178\,267\,079\,370\,z^{24} + \\ & 45\,670\,952\,317\,403\,z^{25} + 45\,011\,034\,604\,106\,z^{26} + 40\,458\,849\,573\,846\,z^{27} + \\ & 33\,215\,036\,685\,152\,z^{28} + 24\,925\,366\,211\,032\,z^{29} + 17\,102\,403\,546\,926\,z^{30} + \\ & 10\,726\,989\,678\,404\,z^{31} + 6\,145\,751\,104\,023\,z^{32} + 3\,212\,103\,217\,512\,z^{33} + \\ & 1\,528\,690\,222\,560\,z^{34} + 660\,843\,701\,416\,z^{35} + 258\,681\,402\,216\,z^{36} + 91\,330\,527\,514\,z^{37} + \\ & 28\,943\,075\,360\,z^{38} + 8\,183\,779\,088\,z^{39} + 2\,049\,421\,399\,z^{40} + 450\,371\,272\,z^{41} + 85\,843\,308\,z^{42} + \\ & 13\,979\,844\,z^{43} + 1\,906\,704\,z^{44} + 211\,872\,z^{45} + 18\,424\,z^{46} + 1\,176\,z^{47} + 49\,z^{48} + z^{49}\end{aligned}$$

$$\begin{aligned}K_{8\times8}(z) = {} & 3\,600\,z^9 + 260\,234\,z^{10} + 9\,161\,844\,z^{11} + 205\,624\,178\,z^{12} + 3\,259\,026\,956\,z^{13} + 38\,509\,091\,104\,z^{14} + \\ & 351\,743\,132\,940\,z^{15} + 2\,555\,393\,428\,502\,z^{16} + 15\,128\,696\,395\,436\,z^{17} + 74\,541\,297\,707\,306\,z^{18} + \\ & 311\,267\,686\,259\,112\,z^{19} + 1\,118\,844\,024\,839\,124\,z^{20} + 3\,507\,981\,273\,108\,664\,z^{21} + \\ & 9\,702\,498\,525\,018\,636\,z^{22} + 23\,899\,882\,018\,866\,672\,z^{23} + 52\,858\,603\,217\,834\,524\,z^{24} + \\ & 105\,690\,774\,510\,597\,180\,z^{25} + 192\,179\,344\,747\,568\,048\,z^{26} + 319\,368\,084\,410\,733\,612\,z^{27} + \\ & 487\,117\,660\,190\,269\,044\,z^{28} + 684\,379\,499\,046\,113\,744\,z^{29} + 888\,386\,977\,466\,277\,426\,z^{30} + \\ & 1\,068\,217\,222\,601\,672\,912\,z^{31} + 1\,192\,321\,377\,072\,934\,280\,z^{32} + 1\,237\,548\,909\,927\,735\,548\,z^{33} + \\ & 1\,196\,127\,084\,749\,768\,650\,z^{34} + 1\,077\,740\,592\,175\,963\,352\,z^{35} + 905\,994\,491\,238\,380\,692\,z^{36} + \\ & 710\,965\,651\,477\,267\,076\,z^{37} + 520\,969\,168\,389\,552\,836\,z^{38} + 356\,483\,920\,242\,132\,856\,z^{39} + \\ & 227\,748\,014\,955\,114\,180\,z^{40} + 135\,792\,828\,381\,540\,616\,z^{41} + 75\,513\,548\,059\,989\,048\,z^{42} + \\ & 39\,130\,117\,374\,538\,132\,z^{43} + 18\,872\,828\,052\,876\,618\,z^{44} + 8\,460\,284\,139\,138\,604\,z^{45} + \\ & 3\,518\,912\,510\,054\,954\,z^{46} + 1\,355\,245\,912\,038\,020\,z^{47} + 482\,129\,585\,758\,940\,z^{48} + \\ & 157\,983\,537\,865\,980\,z^{49} + 47\,524\,258\,972\,966\,z^{50} + 13\,073\,010\,514\,020\,z^{51} + \\ & 3\,273\,341\,812\,692\,z^{52} + 741\,978\,339\,844\,z^{53} + 151\,266\,210\,264\,z^{54} + 27\,518\,246\,208\,z^{55} + \\ & 4\,424\,188\,406\,z^{56} + 621\,078\,384\,z^{57} + 74\,967\,272\,z^{58} + 7\,624\,272\,z^{59} + 635\,372\,z^{60} + 41\,664\,z^{61} + \\ & 2\,016\,z^{62} + 64\,z^{63} + z^{64}\end{aligned}$$

Table 12: Domination polynomials of the king graph $K_{n\times n}$.

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