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Domination Polynomial of the Rook Graph

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Abstract

A placement of chess pieces on a chessboard is called *dominating* if each free square of the chessboard is under attack by at least one piece. In this contribution we compute the number of dominating arrangements of k rooks on an $n \times m$ chessboard. To this end we derive an expression for the corresponding generating function, the domination polynomial of the $n \times m$ rook graph.

1 Introduction

A placement of chess pieces on a chessboard is called *dominating* if each free square of the chessboard is under attack by at least one piece. Chess domination problems have been studied at least since 1862, when Jaenisch [11] posed the problem to find the minimum number of queens needed to dominate the 8×8 board. This number is known as the domination number $\gamma_{♕}$. The minimum number of knights needed to dominate the 8×8 board is called $\gamma_{♘}$. We can easily show that $\gamma_{♕} \leq 5$ and that $\gamma_{♘} \leq 12$ by presenting

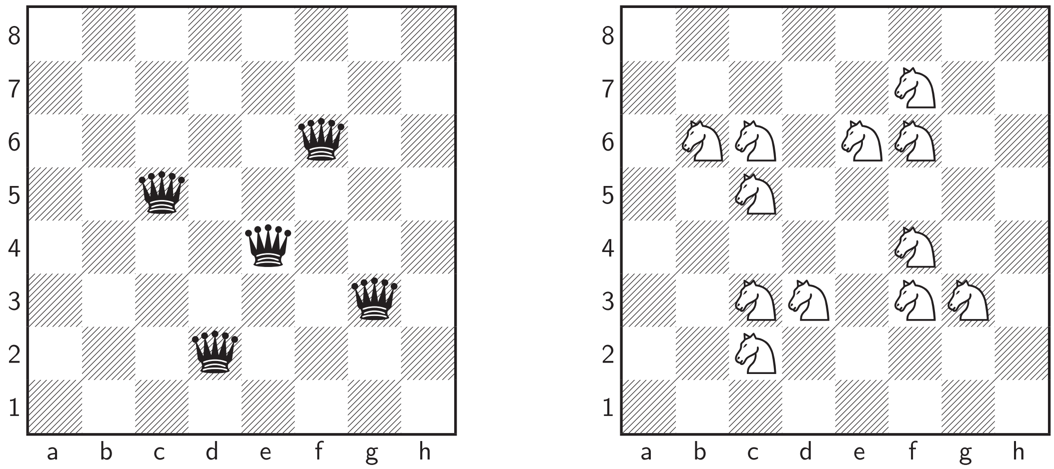

Figure 1: 5 queens or 12 knights can dominate the 8×8 board.

dominating placements of 5 queens and 12 knights (Fig. 1). Proving that both inequalities are actually equations is more challenging [16].

The domination number for the $n \times n$ chessboard defines the domination sequence. For queens, this sequence is $\gamma_{♛}(n) = 1, 1, 1, 2, 3, 3, 4, 5, 5, \ldots$, which is sequence A075458 in the On-Line Encyclopedia of Integer Sequences (OEIS) [15]. For knights, the sequence reads $\gamma_{♘}(n) = 1, 4, 4, 4, 5, 8, 10, 12, 14, \ldots$, which is sequence A006075 in the OEIS. Both sequences are hard to compute even for moderate values of n [7, 12, 13, 14]. The sequence $\gamma_{♛}(n)$ is only known for $n \leq 25$, and $\gamma_{♘}(n)$ for $n \leq 21$.

Domination on chessboards is a rich and active topic of research [6, 10]. The field got an additional boost when it was extended to domination in graphs [9]. A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

Chess domination problems can be recast in graph theory terms by defining an appropriate graph. Take, as an example, queens on an $n \times m$ board. In the queen graph $Q_{n,m}$ each vertex represents a square on the chessboard. Two vertices v and u share an edge if and only if a queen can legally move from v to u. The graphs $N_{n,m}$ for knights and $R_{n,m}$ for rooks are defined correspondingly.

A dominating placement of k queens corresponds to dominating set of cardinality k in the graph $G = Q_{n,m}$. The problem of computing $\gamma_{♛}(n)$ corresponds to finding the minimum cardinality of dominating sets in $Q_{n,n}$.

Let $d_G(k)$ denote the number of dominating sets in G of cardinality k. The domination polynomial $D_G(x)$ is defined as the generating function of $d_G(k)$,

$$D_G(x) = \sum_{k=\gamma_G}^{|V|} d_G(k) x^k \,. \tag{1}$$

Like other graph polynomials, the domination polynomial encodes many interesting properties of a graph [1, 3].

In this contribution we will compute the domination polynomial of the rook graph $R_{n,m}$, which is the cartesian product of the complete graphs K_n and K_m. Rook domination is considerably easier to analyze than the domination of queens and knights. For example, the domination sequence is given by the simple formula

$$\gamma_{R_{n,m}} = \min(n, m) \,. \tag{2}$$

This follows from the fact that for domination, each row *or* each column must contain a rook.

Despite the simplicity of rook domination, very little is known about the domination polynomial. Notable exceptions are its unimodality [5] and its lowest degree coefficient [17, problem 34b]:

$$d_{R_{n,m}}(\gamma_{R_{n,m}}) = \begin{cases} \max(n,m)^{\min(n,m)}, & \text{if } n \neq m; \\ 2n^n - n!, & \text{if } n = m. \end{cases} \tag{3}$$

Proof. The case $n \neq m$ is obvious. For the square case we can place the n rooks to cover every row (n^n possibilities) or every column (another n^n possibilities). Adding these two numbers double counts the configurations that cover both all columns and all rows. Hence we need to subtract the number of those configurations, which is $n!$. □

To compute all the other coefficients, we will deploy the machinery of generating functions. But before doing this, we will derive a recursive equation that links $d_{R_{n,m}}(k)$ to enumerations in smaller boards.

2 Recursion

A dominating arrangement of rooks does not necessarily have a rook in every column *and* every row of the board. Think of n rooks in the first row, leaving all other $n-1$ rows empty. Arrangements of k rooks that contain at least one rook in every column *and* every row are a subset of all dominating configurations, and their number $E_{n,m}(k)$ is less than $d_{R_{n,m}}(k)$. We need $E_{n,m}(k)$ to compute $d_{R_{n,m}}(k)$:

Theorem 1. *Let $E_{n,m}(k)$ denote the number of placements of k indistinguishable rooks on an $n \times m$ chessboard such that each row and each column contain at least one rook. Then*

$$d_{R_{n,m}}(k) = \binom{nm}{k} - \sum_{r=1}^{n} \sum_{c=1}^{m} \binom{n}{r} \binom{m}{c} E_{n-r,m-c}(k) \,. \tag{4}$$

Proof. The first term in (4) is the number of all possible rook arrangements. Hence, we need to prove that the second term equals the number of non-dominating arrangements.

In a non-dominating placement, at least one square is not attacked by any rook. This means, that the row and column of that square is void of rooks. So we need to have one or more empty rows and one or more empty columns. The second term in (4) sums over all combinations of $r = 1, \ldots, n$ empty rows and $c = 1, \ldots, m$ empty columns. In order to avoid overcounting, each of the remaining $n - r$ rows and $m - c$ columns must contain at least one rook. The number of the corresponding arrangements is given by $E_{n-r,m-c}(k)$. □

Theorem 1 allows us to compute $d_{R_{n,m}}(k)$ only if we know how to compute $E_{n,m}(k)$, which seems to be as difficult as the original task. The square case $E_{n,n}(k)$ can be found in the OEIS as A055599, but we need $E_{n,m}(k)$ for general n and m. Luckily, we can compute $E_{n,m}(k)$ by recursion:

Theorem 2. *With base case $E_{0,m}(k) = E_{n,0}(k) = 0$, the numbers $E_{n,m}(k)$ can be computed by recursion over n and m:*

$$E_{n,m}(k) = \binom{nm}{k} - \sum_{r=0}^{n} \sum_{c=0}^{m} \binom{n}{r} \binom{m}{c} E_{n-r,m-c}(k)(1 - \delta_{0,r}\delta_{0,c}) \,, \tag{5}$$

where $\delta_{i,,j}$ is the Kronecker delta.

Proof. The proof is almost identical to the proof of Theorem 1, except that here the sums over r and c start at 0. This is because even with all rows being covered ($r = 0$), a configuration does not count if a single column is not covered ($c > 0$). And vice versa. The only case that needs to be excluded is $c = r = 0$. This is the reason for the factor $(1 - \delta_{0,r}\delta_{0,c})$. □

Theorems 1 and 2 are sufficient to compute $d_{R_{n,m}}(k)$ numerically. A literal implementation of (4) and (5) in a simple Python script computes $d_{R_{10,10}}(k)$ in a few seconds. As a sanity check for an implementation one can compare the numerical results to (3) and to the following "high density" formula:

Corollary 3. *For $k > nm - n - m - \min(n, m) + 2$,*

$$d_{R_{n,m}}(k) = \binom{nm}{k} - nm\binom{(n-1)(m-1)}{k} \,. \tag{6}$$

Proof. An unattacked square implies that its row and its column are void of rooks. One empty row and one empty column contain $m + n - 1$ squares. If k is larger than $nm - (n + m - 1) = (n - 1)(m - 1)$, we have too many rooks on the board to clear a column and a row and all $\binom{nm}{k}$ placements are dominating. The second binomial in (6) is zero in this case, as it should be.

If we want two unattacked squares we need to clear one row, one column and another row or column (whichever is shorter). This means $n + m + \min(n, m) - 2$ empty squares. For

$k > nm - (n + m + \min(n, m) - 2)$ we have again too many rooks on the board to achieve this. Hence we are left with a single unattacked square (x, y), which can be anywhere on the board (factor nm). The k rooks can be placed arbitrarily on the $nm - n - m + 1$ squares other than row x and column y, which explains the second binomial in (6). □

3 The domination polynomial

Theorem 1 tells us that we can compute the generating function for $d_{R_{n,m}}(k)$ once we know the generating function for $E_{n,m}(k)$. So let us have a closer look on the latter.

In the rook graph $R_{n,m}$, the vertices represent the squares on the board. There is another graph $K_{n,m}$, in which the *edges* represent the squares. Think of square (x, y) as connecting row x with column y. Hence, the vertices in $K_{n,m}$ are the rows and the columns, and because each row is connected to each column by the square in their intersection, $K_{n,m}$ is the complete bipartite graph.

The set of squares with rooks correspond to a subset of edges of $K_{n,m}$, and each row and each column contain a rook if and only if the corresponding edges are an *edge cover*, i.e. a set F of edges such that each vertex of $K_{n,m}$ is adjacent to at least one $f \in F$. Hence, $E_{n,m}(k)$ denotes the number edge coverings of cardinality k of the complete bipartite graph $K_{n,m}$. The corresponding generating function, the *edge cover polynomial* of $K_{n,m}$, is given by [2, Corollary 5]:

$$\sum_{k=0}^{nm} E_{n,m}(k)x^k = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} ((1 + x)^k - 1)^n . \tag{7}$$

Theorem 4. *The domination polynomial of the $n \times m$ rook graph can be written as*

$$D_{R_{n,m}}(x) = \left((1 + x)^n - 1\right)^m - (-1)^m \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} \left((1 + x)^k - 1\right)^n . \tag{8}$$

Proof. Multiplication of (4) by x^k and summation over $k = 0, \ldots, nm$ yields

$$D_{R_{n,m}}(x) = \sum_{k=0}^{nm} \binom{nm}{k} x^k - \sum_{r=1}^{n} \sum_{c=1}^{m} \binom{n}{r} \binom{m}{c} \sum_{k=0}^{nm} E_{n-r,m-c}(k)x^k . \tag{9}$$

The first term is $(1+x)^{nm}$. In the second term, the sum over k is the edge covering polynomial (7). Inserting these terms and changing the summation indices $r \mapsto n - r$ and $c \mapsto m - c$ provide us with

$$D_{R_{n,m}}(x) = (1 + x)^{nm} - \sum_{r=0}^{n-1} \sum_{c=0}^{m-1} \binom{n}{r} \binom{m}{c} \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} \left((1 + x)^k - 1\right)^r . \tag{10}$$

Using the identity

$$\sum_{r=0}^{n-1} \binom{n}{r} A^r = (1+A)^n - A^n \tag{11}$$

with $A = (1+x)^k - 1$, we can compute the sum over r to obtain

$$D_{R_{n,m}}(x) = \big((1+x)^n - 1\big)^m + \sum_{c=0}^{m-1} \binom{m}{c} \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} \big((1+x)^k - 1\big)^n . \tag{12}$$

In order to compute the sum over c, we change the order of summation,

$$\sum_{c=0}^{m-1} \sum_{k=0}^{c} \cdots = \sum_{k=0}^{m-1} \sum_{c=k}^{m-1} \cdots , \tag{13}$$

to get

$$D_{R_{n,m}}(x) = \big((1+x)^n - 1\big)^m + \sum_{k=0}^{m-1} \big((1+x)^k - 1\big)^n \sum_{c=k}^{m-1} (-1)^{c-k} \binom{m}{c} \binom{c}{k} . \tag{14}$$

If the sum over c would run from k to m, it would evaluate to 0, see [8, Eq. (5.24)]. Hence

$$\sum_{c=k}^{m-1} (-1)^{c-k} \binom{m}{c} \binom{c}{k} = -(-1)^{m-k} \binom{m}{k} , \tag{15}$$

which yields (8). □

Of course (10), (12) and (14) are also valid representations of the domination polynomial. It is a matter of taste to choose (8) as "the" domination polynomial. Our choice was guided by the observation that the "single sum" form of (8) is the most efficient for computations with Mathematica. With (8), the computation of $D_{R_{50,50}}(x)$ took about 2 minutes on a laptop.

A blemish of (8) is that it does not display the symmetry $D_{R_{n,m}}(x) = D_{R_{m,n}}(x)$. But of course there is a variant that does:

Corollary 5. *The domination polynomial of the $n \times m$ rook graph can also be written as*

$$D_{R_{n,m}}(x) = \big((1+x)^n - 1\big)^m + \big((1+x)^m - 1\big)^n - (-1)^{n+m} \sum_{\ell=0}^{n} \sum_{k=0}^{m} \binom{n}{\ell} \binom{m}{k} (-1)^{k+\ell} (1+x)^{k\ell} . \tag{16}$$

Proof. Binomial expansion of $\big((1+x)^k - 1\big)^n$ in (8) provides us with

$$D_{R_{n,m}}(x) = \big((1+x)^n - 1\big)^m - (-1)^{m+n} \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{\ell} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (1+x)^{k\ell} . \tag{17}$$

The sum over k can be computed according to (11):

$$D_{R_{n,m}}(x) = \left((1+x)^n - 1\right)^m + \left((1+x)^m - 1\right)^n - (-1)^n \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} \left((1+x)^\ell - 1\right)^m . \quad (18)$$

A binomial expansion of $\left((1+x)^\ell - 1\right)^m$ yields (16). □

$$\begin{aligned}
D_{R_{1,1}}(x) &= x \\
D_{R_{2,2}}(x) &= 6\,x^2 + 4\,x^3 + x^4 \\
D_{R_{3,3}}(x) &= 48\,x^3 + 117x^4 + 126\,x^5 + 84\,x^6 + 36\,x^7 + 9\,x^8 + x^9 \\
D_{R_{4,4}}(x) &= 488\,x^4 + 2\,640\,x^5 + 6\,712\,x^6 + 10\,864\,x^7 + 12\,726\,x^8 + 11\,424\,x^9 + 8\,008\,x^{10} + \\
&\quad 4\,368\,x^{11} + 1\,820\,x^{12} + 560\,x^{13} + 120\,x^{14} + 16\,x^{15} + x^{16} \\
D_{R_{5,5}}(x) &= 6\,130\,x^5 + 58\,300\,x^6 + 269\,500\,x^7 + 808\,325\,x^8 + 1\,778\,875\,x^9 + 3\,075\,160\,x^{10} + \\
&\quad 4\,349\,400\,x^{11} + 5\,154\,900\,x^{12} + 5\,186\,300\,x^{13} + 4\,454\,400\,x^{14} + 3\,268\,360\,x^{15} + \\
&\quad 2\,042\,950\,x^{16} + 1\,081\,575\,x^{17} + 480\,700\,x^{18} + 177\,100\,x^{19} + 53\,130\,x^{20} + \\
&\quad 12\,650\,x^{21} + 2\,300\,x^{22} + 300\,x^{23} + 25\,x^{24} + x^{25} \\
D_{R_{6,6}}(x) &= 92\,592\,x^6 + 1\,356\,480\,x^7 + 9\,859\,140\,x^8 + 47\,187\,180\,x^9 + 167\,284\,836\,x^{10} + \\
&\quad 469\,268\,496\,x^{11} + 1\,086\,623\,400\,x^{12} + 2\,137\,381\,200\,x^{13} + 3\,642\,777\,000\,x^{14} + \\
&\quad 5\,453\,014\,080\,x^{15} + 7\,235\,196\,885\,x^{16} + 8\,558\,765\,100\,x^{17} + 9\,057\,864\,300\,x^{18} + \\
&\quad 8\,591\,124\,600\,x^{19} + 7\,305\,959\,610\,x^{20} + 5\,567\,447\,160\,x^{21} + 3\,796\,214\,400\,x^{22} + \\
&\quad 2\,310\,778\,800\,x^{23} + 1\,251\,676\,800\,x^{24} + 600\,805\,260\,x^{25} + 254\,186\,856\,x^{26} + \\
&\quad 94\,143\,280\,x^{27} + 30\,260\,340\,x^{28} + 8\,347\,680\,x^{29} + 1\,947\,792\,x^{30} + 376\,992\,x^{31} + \\
&\quad 58\,905\,x^{32} + 7\,140\,x^{33} + 630\,x^{34} + 36\,x^{35} + x^{36} \\
D_{R_{7,7}}(x) &= 1\,642\,046\,x^7 + 34\,112\,526\,x^8 + 355\,943\,644\,x^9 + 2\,472\,314\,110\,x^{10} + \\
&\quad 12\,823\,222\,482\,x^{11} + 52\,933\,543\,012\,x^{12} + 181\,178\,358\,774\,x^{13} + \\
&\quad 529\,116\,154\,896\,x^{14} + 1\,346\,298\,997\,554\,x^{15} + 3\,031\,523\,389\,181\,x^{16} + \\
&\quad 6\,112\,557\,579\,744\,x^{17} + 11\,134\,728\,203\,116\,x^{18} + 18\,446\,369\,091\,724\,x^{19} + \\
&\quad 27\,928\,246\,211\,796\,x^{20} + 38\,781\,291\,222\,674\,x^{21} + 49\,515\,597\,595\,786\,x^{22} + \\
&\quad 58\,230\,726\,508\,164\,x^{23} + 63\,144\,145\,569\,911\,x^{24} + 63\,175\,905\,655\,695\,x^{25} + \\
&\quad 58\,330\,909\,718\,550\,x^{26} + 49\,695\,284\,721\,096\,x^{27} + 39\,048\,436\,087\,654\,x^{28} + \\
&\quad 28\,277\,118\,318\,876\,x^{29} + 18\,851\,589\,456\,070\,x^{30} + 11\,554\,240\,013\,008\,x^{31} + \\
&\quad 6\,499\,267\,511\,814\,x^{32} + 3\,348\,108\,643\,131\,x^{33} + 1\,575\,580\,671\,714\,x^{34} + \\
&\quad 675\,248\,870\,772\,x^{35} + 262\,596\,783\,715\,x^{36} + 92\,263\,734\,836\,x^{37} + \\
&\quad 29\,135\,916\,264\,x^{38} + 8\,217\,822\,536\,x^{39} + 2\,054\,455\,634\,x^{40} + 450\,978\,066\,x^{41} + \\
&\quad 85\,900\,584\,x^{42} + 13\,983\,816\,x^{43} + 1\,906\,884\,x^{44} + 211\,876\,x^{45} + 18\,424\,x^{46} + \\
&\quad 1\,176\,x^{47} + 49\,x^{48} + x^{49}
\end{aligned}$$

Table 1: Domination polynomials of the $n \times n$ rook graph.

$$
\begin{aligned}
D_{R_{8,8}}(x) \quad = \quad & 33\,514\,112\,x^{8} + 933\,879\,296\,x^{9} + 13\,161\,955\,968\,x^{10} + 124\,392\,729\,216\,x^{11} + \\
& 883\,565\,332\,160\,x^{12} + 5\,020\,456\,535\,808\,x^{13} + 23\,745\,692\,294\,080\,x^{14} + \\
& 96\,124\,772\,710\,912\,x^{15} + 339\,958\,097\,017\,896\,x^{16} + 1\,067\,094\,188\,274\,240\,x^{17} + \\
& 3\,009\,775\,897\,325\,792\,x^{18} + 7\,703\,325\,822\,650\,304\,x^{19} + \\
& 18\,031\,600\,637\,765\,680\,x^{20} + 38\,843\,543\,834\,346\,048\,x^{21} + \\
& 77\,392\,553\,377\,032\,096\,x^{22} + 143\,185\,055\,260\,371\,264\,x^{23} + \\
& 246\,761\,069\,109\,093\,336\,x^{24} + 397\,106\,882\,820\,897\,536\,x^{25} + \\
& 597\,898\,212\,185\,747\,424\,x^{26} + 843\,500\,295\,460\,142\,656\,x^{27} + \\
& 1\,116\,294\,749\,822\,105\,392\,x^{28} + 1\,387\,019\,957\,382\,904\,768\,x^{29} + \\
& 1\,619\,086\,454\,915\,331\,808\,x^{30} + 1\,776\,352\,520\,871\,483\,072\,x^{31} + \\
& 1\,832\,208\,846\,791\,514\,422\,x^{32} + 1\,776\,875\,996\,843\,390\,912\,x^{33} + \\
& 1\,620\,187\,226\,242\,379\,648\,x^{34} + 1\,388\,775\,090\,898\,717\,312\,x^{35} + \\
& 1\,118\,753\,489\,141\,190\,336\,x^{36} + 846\,631\,073\,977\,386\,432\,x^{37} + \\
& 601\,555\,988\,478\,702\,432\,x^{38} + 401\,038\,042\,815\,966\,528\,x^{39} + \\
& 250\,648\,973\,984\,891\,272\,x^{40} + 146\,721\,398\,729\,422\,272\,x^{41} + \\
& 80\,347\,442\,945\,600\,992\,x^{42} + 41\,107\,995\,982\,971\,456\,x^{43} + \\
& 19\,619\,725\,660\,610\,544\,x^{44} + 8\,719\,878\,112\,062\,656\,x^{45} + \\
& 3\,601\,688\,789\,838\,944\,x^{46} + 1\,379\,370\,175\,208\,256\,x^{47} + \\
& 488\,526\,937\,076\,444\,x^{48} + 159\,518\,999\,862\,656\,x^{49} + \\
& 47\,855\,699\,958\,816\,x^{50} + 13\,136\,858\,812\,224\,x^{51} + 3\,284\,214\,703\,056\,x^{52} + \\
& 743\,595\,781\,824\,x^{53} + 151\,473\,214\,816\,x^{54} + 27\,540\,584\,512\,x^{55} + \\
& 4\,426\,165\,368\,x^{56} + 621\,216\,192\,x^{57} + 74\,974\,368\,x^{58} + 7\,624\,512\,x^{59} + \\
& 635\,376\,x^{60} + 41\,664\,x^{61} + 2\,016\,x^{62} + 64\,x^{63} + x^{64}
\end{aligned}
$$

Table 2: Domination polynomial of the 8×8 rook graph.

Tables 1 and 2 show the domination polynomials $D_{R_{n,n}}(x)$ for $n = 1, \ldots, 8$. The coefficients of $D_{R_{n,n}}$ have become sequence A368831 in the OEIS. The *total number* of dominating sets,

$$D_{R_{n,m}}(1) = (2^n - 1)^m + (2^m - 1)^n - (-1)^{n+m} \sum_{\ell=0}^{n} \sum_{k=0}^{m} (-1)^{k+\ell} 2^{k\ell}\,, \tag{19}$$

is in the OEIS as A287274.

4 Conclusions

The connection between the domination polynomial of the rook graph $R_{n,m}$ and the edge cover polynomial of the complete bipartite graph $K_{n,m}$ allowed us to compute the former. Theorem 4 is our main result. As far as we know, the rook is the first chess piece for which the domination polynomial has been computed.

Evaluating the domination polynomial with a computer algebra system like Mathematica seems to be the fastest way to compute the numerical values of $d_{R_{n,m}}(k)$. These values have applications in cryptography [4], which was the initial motivation for this work.

The domination polynomial can also be used to study structural properties of the sequences $d_{R_{n,m}}(k)$, like unimodality (which has been proven recently using general arguments [5]), the maximum, or the asymptotics for large board sizes. We leave this for further studies.

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