Journal of Integer Sequences, Vol. 27 (2024),
Article 24.8.7

Counting Tournaments with a Specified Number of Circular Triads

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Abstract

We investigate the counting of tournaments with a specified number of circular triads (3-cycles). We derive the generating functions for the sequences of the counts, and give explicit expressions for the sequences for when the number of circular triads is from 0 through 14.

1 Introduction and historical background

A *tournament*, with n 'players', is a directed complete graph with n nodes, and $\binom{n}{2}$ directed edges. A *circular triad* (also known as a *cyclic triple* or *3-cycle*) occurs when we have 3 nodes A, B, C, such that $A \to B \to C \to A$. We are interested in counting the number of tournaments with n nodes with d circular triads, which we term $g_d(n)$. We consider the sequences $(g_d(n))_{n\geq 1}$, indexed by d.

Counting the number d of circular triads for a specific tournament of size n was first conducted in Kendall and Babington Smith [9], with the objective of analyzing paired comparisons data, as in Bradley and Terry [2], and Thurstone [18]. Kendall and Babington Smith [9] computed the number of tournaments of size n with exactly d circular triads for $n \leq 7$ and $d \leq 14$. Alway [1] extended these results to $n = 8, 9$, and 10 for $d \leq 40$, and Knezek et al. [11] further extended these results to $n \leq 15$.

Landau [12] introduced a one dimensional measure of the structure of a society where non-transitive domination relations exist between members. Davis [5] created a method to count the number of isomorphic tournament graphs for all n. David [3] derived the joint distribution of score sequences of a subset of players in a simple round-robin tournament under the null hypothesis of equal player strength.

Moon [15] collected together many results on tournaments, and provided an exhaustive collection of non-isomorphic tournaments for $1 \leq n \leq 6$. David [4] introduced different measures of transitivity for tournaments. Kendall and Gibbons [10] suggested approximating the distribution of d for a specific n with a χ^2 distribution, which is particularly useful for hypothesis testing. McShane [14] introduced a mathematical model for intransitive relations, and investigated algorithms for counting tournaments with specific numbers of circular triads. Harary and Moser [7] measured the transitivity of tournaments, and consider some properties of strongly connected tournaments. Stockmeyer [17] developed algorithms for counting the number of distinct score sequences for various classes of tournaments.

Our investigation is motivated by Kadane [8], who derived an expression for $g_d(n)$ using equivalence classes, and gave specific formulae for the cases $d = 0, 1, 2, 3, 4, 5$. In OEIS [16] these are given by A000142, A090672, A357242, A357257, A357248, A357266, respectively.

Following the principles outlined in Flajolet and Sedgewick [6], we derive the exponential generating function of the sequences $(g_d(n))_{n\geq 1}$. In addition, we use the generating function formula to obtain explicit expressions for $(g_d(n))_{n\geq 1}$ for $d = 6, \ldots, 14$.

The remainder of the paper is organized as follows. Section 2 gives some definitions, and summarizes some important ideas related to exponential generating functions and counting. Section 3 examines the decomposition of tournaments into strongly connected subtournaments, and defines generating functions that we term atomic generators. Section 4 examines how tournaments are built from strong subtournaments, and gives a general expression for the exponential generating function for $(g_d(n))_{n\geq 1}$. Section 5 uses the generating function to reproduce and clarify Kadane's explicit expression for $(g_d(n))_{n\geq 1}$. There are some concluding remarks in Section 6.

2 Exponential generating functions and notation

We will use the following terms and symbols interchangeably to mean that A dominates B: A wins against B, A beats B, A is superior to B, $A > B$, $A \to B$. In a similar way, A loses to B, A is inferior to B, $A < B$, $A \leftarrow B$, will refer to the opposite situation. Ties are not considered.

When S_1 and S_2 are disjoint sets of players, the expressions S_1 beats S_2, S_1 is superior to S_2, and $S_1 > S_2$ all imply the following:

$$S_1 \to S_2 \Leftrightarrow \{A \to B | \forall A \in S_1, \forall B \in S_2\}.$$

A similar definition is used for the reverse situation.

The exponential generating function (EGF) of sequence $(g(n))_{n\geq 0}$ is defined as

$$A(x) = \sum_{n=0}^{\infty} \frac{g(n)}{n!} x^n. \tag{1}$$

A key example of an EGF that plays an important role in this paper is the EGF of the sequence $(g(n))_{n\geq a}$ where $g(n) = n!\binom{n-a+b}{b}$. Theorem 1 describes the EGF.

Theorem 1. *The sequence* $(g(n))_{n\geq a}$ *where* $g(n) = n!\binom{n-a+b}{b}$ *has exponential generating function*

$$A(x) = x^a(1-x)^{-(b+1)}.$$

Proof. First substitute the expression for $g(n)$ into 1 to get

$$A(x) = \sum_{n=a}^{\infty} \frac{n!}{n!}\binom{n-a+b}{b} x^n = x^a \sum_{n-a=0}^{\infty} \binom{n-a+b}{b} x^{n-a} = x^a \sum_{k=0}^{\infty} \binom{k+b}{b} x^k.$$

Thus to prove the result, we merely need to show that

$$\sum_{k=0}^{\infty} \binom{k+b}{b} x^k = (1-x)^{-(b+1)},$$

which follows from the generalized binomial theorem,

$$(1+X)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} X^k,$$

where, for $\alpha \in \mathbb{R}$, we define

$$\binom{\alpha}{k} = \frac{\alpha\,(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Set $\alpha = -(b+1)$, $X = -x$, and the result follows (after a little algebra). □

The two operations of addition and multiplication of EGFs are intimately connected to the combinatorial operations involved in counting the number of ways a structure can be built. These are summarized by the following propositions.

Proposition 2 simply states that if a count is a sum of counts then the EGF for the count is a sum of EGFs.

Proposition 2. *For $n \in \mathbb{N}$, let c_n be the number of ways to build a tournament with a specific structure on n players. Suppose that the structure can be decomposed into two disjoint subsets, such that a_n and b_n are the numbers of ways to build the structure in each of these two cases, and thus $c_n = a_n + b_n$. If $A(x), B(x)$, and $C(x)$ are the exponential generating functions of a_n,b_n and c_n, respectively, then*

$$C(x) = A(x) + B(x).$$

Proposition 3 states that if we can break a tournament into sub tournaments, and make choices on each sub tournament, then we obtain the EGF of the tournament by multiplying the corresponding EGFs for the sub tournaments.

Proposition 3. *For $n \in \mathbb{N}$, let a_n and b_n be the numbers of ways to build an $\alpha-$structure and a $\beta-$structure on a tournament on n players, respectively. Let c_n be the number of ways to partition $\{1, 2, \ldots, n\}$ into two sets S_1 and S_2, and then place an α-structure on the tournament of players in set S_1, and a β-structure on the tournament of players in set S_2. If $A(x), B(x)$, and $C(x)$ are the exponential generating functions of a_n,b_n and c_n, respectively, then*

$$C(x) = A(x)B(x).$$

Further information on the relationship between generating functions and combinatorial operations can be found in Flajolet and Sedgewick [6].

Tournaments that are structurally simple, or have a small n can be easily represented using a graph. For example a circular triad structure on 3 players has a simple graphical structure, shown in Figure 1. Here the directed arrow from A to B in the figure implies that A beats B.

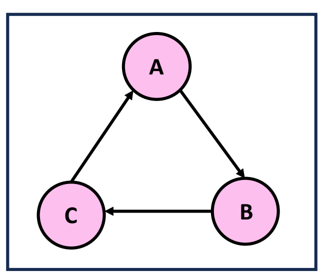

Figure 1: A circular triad.

A transitive chain of players can also be represented easily, as shown in Figure 2. Note that by convention, if a connection is omitted, it is assumed that players beat all players placed below them in the diagram. In Figure 2, player 1 beats players 2 through n, player 2 beats players 3 through n and so on. To aid visual display, some connecting arrows are kept in the diagram, but some authors, such as Moon [15], eliminate these also.

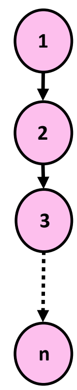

Figure 2: Chain.

Very complicated structures cannot be easily represented graphically, and require special notation to track. One tool to do so is to use the *adjacency matrix*. The adjacency matrix is a matrix of 0s and 1s such that the ij^{th} entry is 1 if player i dominates player j and is 0 if the converse is true. The simple circular triad structure on 3 players thus is represented by the 3×3 matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Although there is a 1-1 relationship between adjacency matrices and tournaments, for the purpose of counting circular triads, it is more efficient to work with summaries of the adjacency matrices.

Landau [12] devised a summary of the adjacency matrix which involves finding the outdegrees for each vertex. The outdegree for a given vertex A is simply the number of vertices that are dominated by A, and thus are the set of row sums of the adjacency matrix. When the outdegrees are ordered from smallest to largest, the set of integers formed is known as the *score sequence*. For example, the circular triad in Figure 1 has score sequence $(1, 1, 1)$.

Independently, Kendall and Babington Smith [9] proposed the same summary in 1940, and call the score sequence the *alpha-vector* associated with a tournament.

Note that there is not a 1-1 relationship between score sequences and tournaments. For example, the score sequence $(1,1,2,3,3)$ is associated with two different, nonisomorphic tournaments. (See Moon [15, p. 92].)

David [3] proposed a slight simplification of the score sequence which is strikingly similar to MacMahon [13], whereby repetitions of a score are represented with powers. Thus in David's notation, the score sequence $(1,1,1)$ becomes 1^3. For another example, a tournament on 5 players represented by the score sequence $(1,1,2,3,3)$ becomes $3^2 2^1 1^2$. Kadane [8] makes a further modification to David's notation, counting the number of players that dominate $n-1$ players, the number that dominate $n-2$ players, and so on down to the number that dominate 0 players. This, of course, is simply the powers in David's notation, including "missing" powers. So for example, the score sequence $(1,1,2,3,3)$, which is $3^2 2^1 1^2$ in David's notation can be rewritten as $4^0 3^2 2^1 1^2 0^0$, and then extracting the powers this becomes $(0,2,1,2,0)$ in Kadane notation. Note that in the Kadane notation, a transitive chain is $(1,1,1,\ldots,1)$, and the circular triad on 3 vertices is $(0,3,0)$.

In this paper we use the traditional score sequence notation. This is due primarily to the following theorems, stated in Moon [15]. Note that we present these theorems without proofs, as the proofs can be found in the cited literature. For the definition of reducible, mentioned in Theorem 5, see the next section.

Theorem 4. *[15, p. 61] A sequence $s_1 \leq s_2 \leq \ldots \leq s_n$ is a valid score sequence if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}$$

for $1 \leq k \leq n$, with equality when $k = n$.

Theorem 5. *[15, p. 2] A tournament with score sequence $s_1 \leq s_2 \leq \ldots \leq s_n$ is reducible if and only if*

$$\sum_{i=1}^{k} s_i = \binom{k}{2}$$

for some positive integer $k < n$.

Theorem 6. *[15, p. 9] The number of circular triads in a tournament with score sequence $s_1 \leq s_2 \leq \ldots \leq s_n$ is*

$$\binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2}.$$

3 Atomic tournaments, generators and multipliers

A key concept in the theory of tournaments is that of an irreducible tournament. Moon [15] stated that a tournament is *reducible* if it is possible to partition its nodes into two nonempty sets S_1, S_2 such that $S_1 > S_2$. A tournament that is not reducible is termed *irreducible*. Moon [15] also states that every tournament $\mathbb{A}$ has a unique decomposition into irreducible subtournaments $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_k$, for some k, where $\mathbb{A}_1 > \mathbb{A}_2 > \cdots > \mathbb{A}_k$.

Interestingly, the fundamental concept of a unique decomposition of a tournament has been recognized independently by many authors, with the result that there is a variety of terminology associated with the concept. Harary and Moser [7] named an irreducible tournament a *strong tournament*, or *strongly connected tournament*. Kadane [8] refered to irreducible tournaments as *simple tournaments*, and reducible tournaments as *compound tournaments*. We add to this plethora of terminology by referring to an irreducible tournament by the term *atomic tournament*, or *atom* for short. We do not take this step lightly. We believe that the term *atom* is both brief, and emphasizes the essential nature of the decomposition of a tournament into its constituent atoms. We follow Kadane [8] and use the term *compound tournaments*, or *compound* to refer to reducible tournaments. Note that the simplest atom is a single node tournament. A chain can be defined to be a compound structure made up of the single atom elements.

There are no atoms on $n = 2$ nodes. For $n = 3$, there is just one type of atom, which is the basic circular triad, with score sequence $(1, 1, 1)$, shown in Figure 1. For $n = 4$ there is also just one type of atom, with score sequence $(1, 1, 2, 2)$, shown in Figure 3, and for $n = 5$, there are 3 different atomic score sequences,with score sequences $(1, 1, 2, 3, 3)$ $(d = 3)$, $(1, 2, 2, 2, 3)$ $(d = 4)$, and $(2, 2, 2, 2, 2)$ $(d = 5)$. For figures for $n = 5$, see Moon [15].

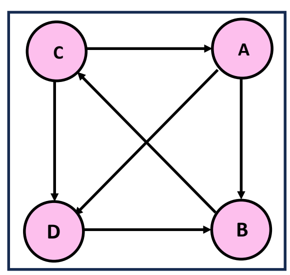

Figure 3: Atomic two circular triads tournament.

We make a distinction between *trivial atoms*, which are a single node, with score sequence (1), and *non trivial atoms*, such as the basic circular triad with score sequence $(1, 1, 1)$.

An example of a compound tournament is shown in Figure 4. The tournament can be

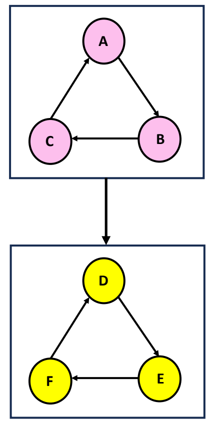

Figure 4: Compound two circular triads tournament.

decomposed into two sub tournaments, with one circular triad in each part. Note the score sequence for the compound in Figure 4 is $(1,1,1,4,4,4)$. The score sequences for the two atoms are both $(1,1,1)$. To obtain the score sequence for the compound, we increment one of these score sequences by 3 to get $(4,4,4)$, and then append these to get $(1,1,1,4,4,4)$.

For specific n, d, we count the total number of non-trivial atoms of n players with exactly d circular triads, which we call $N(n, d)$. Table 1 gives these counts for small n, d. The entry

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$d=1$	0	0	2	0	0	0	0	0
$d=2$	0	0	0	24	0	0	0	0
$d=3$	0	0	0	0	240	0	0	0
$d=4$	0	0	0	0	280	2880	0	0
$d=5$	0	0	0	0	24	3360	40320	0
$d=6$	0	0	0	0	0	8640	58880	645120

Table 1: Atomic tournament counts.

in Table 1 for $d = 4, n = 6$ means there are $N(6, 4) = 2880$ distinct ways to arrange the 6 players in an atom with 4 circular triads (score sequence $(1,1,2,3,4,4)$). Note that a particular combination of d, n is in general associated with more than one score sequence. For example, $d = 6, n = 7$ is associated with 3 different score sequences, $(1,2,2,2,4,5,5)$, $(1,1,2,4,4,4,5)$, and $(1,1,3,3,3,5,5)$. Despite these differences, we heuristically refer to all of the atoms with $d = 6, n = 7$ as 6_7, and in general use the notation d_n to refer to all atoms with n players and d circular triads. Extending this concept a little further, we use the term *isotopes of* d for atoms with the same d (*atomic number*) but different n (*atomic mass*). Note that Appendix A contains some details on how the $N(n, d)$ are obtained.

Definition 7. An *atomic generator* $A_d(x)$ is the exponential generating function of the sequence $(N(n, d))_{n\geq 1}$ for a given d.

As an example, for $d = 1$, there is just one non-zero value in the sequence, $(N(n, 1))_{n\geq 1}$, which is $N(3, 1) = 2$, so the EGF is

$$A_1(x) = \frac{2}{3!}x^3 = \frac{1}{3}x^3.$$

A second example is the EGF of $(N(n, 5))_{n\geq 1}$, the EGF of the sequence $0, 0, 0, 0, 0, 24, 3360, 40320, 0, 0, \ldots$ which is

$$A_5(x) = \frac{24}{5!}x^5 + \frac{3360}{6!}x^6 + \frac{40320}{7!}x^7 = \frac{1}{5}x^5 + \frac{14}{3}x^6 + 8x^7.$$

The atomic generators play a critical role in the construction of our main results in Section 4.

Of course, since each of these sequences of $N(n, d)$ are finite, the atomic generators are finite polynomials. We can describe the $A_d(x)$ in the following theorem.

Theorem 8.

$$A_d(x) = \sum_{k=L(d)}^{d+2} \frac{N(k, d)}{k!}x^k,$$

where $I_0(d)$ *is the smallest positive odd integer solution to* $x^3 - x - 24d \geq 0$*,* $I_e(d)$ *the smallest positive even integer solution to* $x^3 - 4x - 24d \geq 0$*, and* $L(d) = \min(I_0(d), I_e(d))$*.*

Proof. The highest degree power in the atomic generator is always $d + 2$, and corresponds to the atoms associated with score sequences of the form $(1, 1, 2, 3, \ldots, n-4, n-3, n-2, n-2)$, for an n player tournament. See Appendix A, and Harary and Moser [7] for more details. To get the lowest possible n for a given d, we consider the reverse equations, which give the largest possible d for a given n. These equations were stated in Kendall and Babington Smith [9], and they are that

$$d \leq \frac{n^3 - n}{24},$$

if n is odd, and

$$d \leq \frac{n^3 - 4n}{24},$$

if n is even. Then the equations that must be solved for $I_0(d)$ and $I_e(d)$ are merely the inversions of these two inequalities for the largest possible d for a given n. □

Note that the value of $L(d) = \min(I_0(d), I_e(d))$ follows a pattern which is simpler than one would expect from the definition of $L(d)$. The sequence of the $L(d)$ is $3, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, \ldots$.

Further insight into $L(d)$ can be found by studying the roots of the two cubic equations $x^3 - x - 24d = 0$ and $x^3 - 4x - 24d = 0$. The first of the roots is, using Cardano's result, given by

$$r_1(d) = \sqrt[3]{12d + \sqrt{144d^2 - 1/27}} + \sqrt[3]{12d - \sqrt{144d^2 - 1/27}}.$$

The second root is

$$r_2(d) = \sqrt[3]{12d + \sqrt{144d^2 - 64/27}} + \sqrt[3]{12d - \sqrt{144d^2 - 64/27}}.$$

Now define $I_1(d) = \lceil r_1(d) \rceil$, and $I_2(d) = \lceil r_2(d) \rceil$. If I_1 is odd then $I_0(d) = I_1(d)$, otherwise $I_0(d) = I_1(d) + 1$, and if I_2 is even then $I_e(d) = I_2(d)$, otherwise $I_e(d) = I_2(d) + 1$. Clearly, asymptotically in d, both $r_1(d), r_2(d) \to \sqrt[3]{24d}$, and thus we can state that

$$L(d) \to \sqrt[3]{24d}.$$

Definition 9. The *atomic multipliers* are the coefficients of x^n in the exponential generating functions $A_d(x)$.

Using the definition, we can see that the atomic multipliers for $d = 5$ are $1/5, 14/3, 8$.

Kadane [8] described the process of obtaining the atomic multipliers. This process requires counting the $N(k, d)$ by listing all tournaments, and then dividing the counts by the total number of tournaments. The chief utility in atomic multipliers is in constructing an explicit formula for $g_d(n)$, as explained in Section 5.

The atomic generators (and implicitly the atomic multipliers) for $d = 1, \ldots, 14$ are listed in Appendix B.

4 Building tournaments

All tournaments can be built by putting together simple chains of trivial atoms combined with non-trivial atoms. Below we explain how to find the general formula for $G_d(x)$, the EGF of $(g_d(n))_{n \geq 1}$.

We index a partition of d by the *multiplicities* of the partition, $\mathbf{r} = (r_1, \ldots r_d)$, where $r_i \in (1, \ldots, d)$ is the number of times i appears in the partition. Thus $d = \sum_{i=1}^{d} i r_i$. For example, for the partition $5 = 2 + 2 + 1$, we have $\mathbf{r} = (1, 2, 0, 0, 0)$. Note that although partitions are traditionally listed with largest components first, when counting multiplicities, it is easier to count from smaller components to larger.

Now let $P_d = \{\mathbf{r}\}$ denote the *set of partitions of* d, and for each partition define the *size* of the partition to be $r = \sum_{i=1}^{d} r_i$.

The following theorem now describes the formula for $G_d(x)$.

Theorem 10.

$$G_d(x) = \sum_{\mathbf{r}\in P_d} \frac{r!}{r_1!\cdots r_d!}(1-x)^{-(1+r)} \prod_{i=1}^{d} A_i^{r_i}(x). \tag{2}$$

Proof. For each partition, $\mathbf{r} = (r_1, \ldots r_d)$, with size r, there are $r-1$ chains of trivial atoms arranged between the non trivial atoms, with a chain above and a chain below, and thus $r+1$ chains altogether. By Theorem 1 these chains contribute $(1-x)^{-(1+r)}$ to the EGF $G_d(x)$. The partitions give atomic components of circular triads, each of which has generating function $A_i(x)$, which allows for isotope choices, and the atoms are arranged in $\frac{r!}{r_1!\cdots r_d!}$ ways. By Proposition 3 we multiply the chain generators by the atom generators, and thus the contribution of the partition to $G_d(x)$ is

$$\frac{r!}{r_1!\cdots r_d!}(1-x)^{-(1+r)} \prod_{i=1}^{d} A_i^{r_i}(x).$$

Then, using Proposition 2 we add these contributions of the partitions over P_d to get the result. □

Below we give some examples to illustrate how to use Equation (2).

Example 11. Here $d = 0$. This corresponds to a chain, as shown in Figure 2, with $g_0(n) = n!$. For simplicity, we define $g_0(0) = 0! = 1$, thus the EGF for $(g_0(n))_{n\geq 0}$ is

$$G_0(x) = (1-x)^{-1}.$$

Example 12. Here $d = 1$. This corresponds to a circular triad, 1_3, with score sequence $(1, 1, 1)$, preceded by a chain, and followed by a chain (with one or both of the chains being empty). The diagram in Figure 5 illustrates this. In Figure 5, the EGF of the green (superior) chain of A, B, C is $(1-x)^{-1}$, the EGF of the pink (inferior) chain of G, H, I is also $(1-x)^{-1}$, and the EGF of the yellow circular triad of D, E, F is $A_1(x)$. Thus by Proposition 3, we have

$$G_1(x) = (1-x)^{-1}A_1(x)(1-x)^{-1} = \frac{x^3}{3}(1-x)^{-2}.$$

Example 13. Here $d = 2$. There are two different ways to build a tournament with exactly two circular triads, as $P_2 = \{(2, 0), (0, 1)\}$. The first ($\mathbf{r} = (2, 0)$) is to put together two circular triad atoms, as shown in Figure 6, which in atomic mass notation is $1_3 + 1_3 = 2 \cdot 1_3$. In the two separate circular triads, in Figure 6, we need a term of $(1-x)^{-1}$ for each of the chains (green, gray, blue), and for each of the circular triad atoms (pink, yellow), we need $A_1(x)$. Thus by Proposition 3, we get a contribution of $A_1^2(x)(1-x)^{-3}$ to the EGF.

The second type of tournament ($\mathbf{r} = (0, 1)$) with $d = 2$ is a single atom on 4 players, which in atomic mass notation is 2_4. The atom is the one shown in Figure 3. This atom is

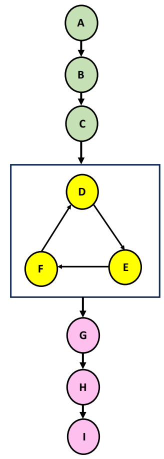

Figure 5: $d = 1$.

then combined with a chain both "above" and "below" the atom, as shown in Figure 7. The EGF for the $\mathbf{r} = (0, 1)$ tournament is then $A_2(x)(1 - x)^{-2}$.

Thus by Proposition 2, we add the generating functions for the two types of $d = 2$ tournaments to get the following expression for (2),

$$G_2(x) = A_2(x)(1 - x)^{-2} + A_1^2(x)(1 - x)^{-3}.$$

Example 14. Here $d = 3$. We have $R_3 = \{(3, 0, 0), (1, 1, 0), (0, 0, 1)\}$. In our atomic mass notation these are $3 \cdot 1_3, 2_4 + 1_3, 3_5$ respectively. The contributions to the EGF of these are shown in Table 2. Note that the contribution to $2_4 + 1_3$ is multiplied by 2, as we could have $2_4 > 1_3$ or $2_4 < 1_3$. Using Proposition 2, we add these to get

$$G_3(x) = (1 - x)^{-2} A_3(x) + 2(1 - x)^{-3} A_2(x) A_1(x) + (1 - x)^{-4} A_1^3(x).$$

Partition	EGF
$3_5 = (0, 0, 1)$	$(1 - x)^{-2} A_3(x)$
$2_4 + 1_3 = (1, 1, 0)$	$2(1 - x)^{-3} A_2(x) A_1(x)$
$3 \cdot 1_3 = (3, 0, 0)$	$(1 - x)^{-4} A_1^3(x)$

Table 2: Contributions to EGF for $d = 3$.

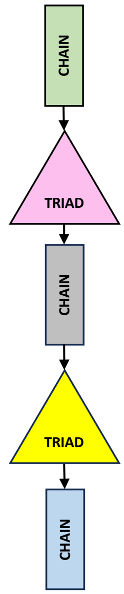

Figure 6: $d = 2,\ 2 \cdot 1_3$.

Example 15. Here $d = 8$. We choose this example to illustrate the effect of isotopes. We examine the contribution to Equation (2) of the partition corresponding to $(0, 0, 0, 2, 0, 0, 0, 0)$, that is $8 = 4 + 4$. Since there are two isotopes of 4, labeled $4_5, 4_6$, we could have $4_5 + 4_6$, $2 \cdot 4_5$, or $2 \cdot 4_6$. The atomic generator for 4 has two terms, $A_4(x) = \frac{7}{3}x^5 + 4x^6$, the first term corresponding to 4_5, and the second to 4_6. We can account for the choice as to which isotope we have for each of the partitions of size 4 by multiplying the A_4 by itself. Thus $\frac{2!}{(0!)^7 2!}(1-x)^{-3}A_4^2(x)$ is the contribution to $G_8(x)$ from the partition $8 = 4 + 4$.

Example 16. Here $d = 6$. Note that $|R_6| = 11$. Table 3 lists the partitions and their associated contribution to $G_6(x)$. Adding the contributions together then gives

$$\begin{aligned} G_6(x) &= (1-x)^{-2}A_6(x) + 2(1-x)^{-3}A_5(x)A_1(x) + 2(1-x)^{-3}A_4(x)A_2(x) \\ &+ (1-x)^{-3}A_3^2(x) + 3(1-x)^{-4}A_4(x)A_1^2(x) + 6(1-x)^{-4}A_3(x)A_2(x)A_1(x) \\ &+ (1-x)^{-4}A_2^3(x) + 4(1-x)^{-5}A_3(x)A_1^3(x) + \frac{4!}{(2!)^2}(1-x)^{-5}A_2^2(x)A_1^2(x) \\ &+ 5(1-x)^{-6}A_2(x)A_1^4(x) + (1-x)^{-7}A_1^6(x). \end{aligned}$$

Note we substitute the expressions obtained for the atomic generators (Appendix B), and

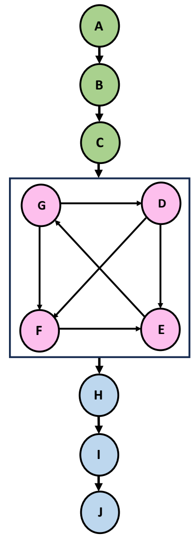

Figure 7: $d = 2$ with 2_4.

collect terms, to get

$$\begin{aligned} G_6(x) = {} & 12x^6(1-x)^{-2} + \frac{35}{3}x^7(1-x)^{-2} + 16x^8(1-x)^{-2} + \frac{2}{15}x^8(1-x)^{-3} \\ & + \frac{70}{9}x^9(1-x)^{-3} + \frac{52}{3}x^{10}(1-x)^{-3} + \frac{7}{9}x^{11}(1-x)^{-4} + \frac{19}{3}x^{12}(1-x)^{-4} \\ & + \frac{26}{27}x^{14}(1-x)^{-5} + \frac{5}{81}x^{16}(1-x)^{-6} + \frac{1}{729}x^{18}(1-x)^{-7}. \end{aligned}$$

Appendix C lists the $G_d(x)$ for $d = 0, \ldots 14$.

5 Obtaining the sequences from the generating functions

First define a modified binomial coefficient $\left\langle {n \atop k} \right\rangle$, defined on integers n, k, such that $\left\langle {n \atop k} \right\rangle = \binom{n}{k}$ if $n \geq k \geq 0$ and $\left\langle {n \atop k} \right\rangle = 0$ if $n < k$, with $k \geq 0$. Note that the modification extends the property of binomial coefficients that $\binom{n}{k} = 0$ if $0 \leq n < k$ to negative integer n such that $n < k$. The purpose of the definition of $\left\langle {n \atop k} \right\rangle$ is to avoid the use of the Iverson bracket, and thus produce simpler expressions for $g_d(n)$.

Partition	Partition multiplicities	Contribution to $G_6(x)$
6	$(0,0,0,0,0,1)$	$(1-x)^{-2}A_6(x)$
$5+1$	$(1,0,0,0,1,0$	$2(1-x)^{-3}A_5(x)A_1(x)$
$4+2$	$(0,1,0,1,0,0)$	$2(1-x)^{-3}A_4(x)A_2(x)$
$4+1+1$	$(2,0,0,1,0,0)$	$3(1-x)^{-4}A_4(x)A_1^2(x)$
$3+3$	$(0,0,2,0,0,0)$	$(1-x)^{-3}A_3^2(x)$
$3+2+1$	$(1,1,1,0,0,0)$	$6(1-x)^{-4}A_3(x)A_2(x)A_1(x)$
$2+2+2$	$(0,3,0,0,0,0)$	$(1-x)^{-4}A_2^3(x)$
$3+1+1+1$	$(3,0,1,0,0,0)$	$4(1-x)^{-5}A_3(x)A_1^3(x)$
$2+2+1+1$	$(2,2,0,0,0,0)$	$\frac{4!}{(2!)^2}(1-x)^{-5}A_2^2(x)A_1^2(x)$
$2+1+1+1+1$	$(4,1,0,0,0,0)$	$5(1-x)^{-6}A_2(x)A_1^4(x)$
$1+1+1+1+1+1$	$(6,0,0,0,0,0)$	$(1-x)^{-7}A_1^6(x)$.

Table 3: Partition contributions to the EGF for $d=6$.

To obtain a formula for $g_d(n)$ from $G_d(x)$ extract the coefficient of $\frac{x^n}{n!}$ in $x^a(1-x)^{-(1+b)}$, which by Theorem 1 is $n!\binom{n-a+b}{b}$ if $n \geq a$, and 0 otherwise. Thus the coefficient is $n!\left\langle {n-a+b \atop b} \right\rangle$. For example, for $d=6$,

$$\begin{aligned} g_6(n) = n!(12\left\langle {n-6+1 \atop 1} \right\rangle + \frac{35}{3}\left\langle {n-7+1 \atop 1} \right\rangle + 16\left\langle {n-8+1 \atop 1} \right\rangle + \frac{2}{15}\left\langle {n-8+2 \atop 2} \right\rangle \\ + \frac{70}{9}\left\langle {n-9+2 \atop 2} \right\rangle + \frac{52}{3}\left\langle {n-10+2 \atop 2} \right\rangle + \frac{7}{9}\left\langle {n-11+3 \atop 3} \right\rangle + \frac{19}{3}\left\langle {n-12+3 \atop 3} \right\rangle \\ + \frac{26}{27}\left\langle {n-14+4 \atop 4} \right\rangle + \frac{5}{81}\left\langle {n-16+5 \atop 5} \right\rangle + \frac{1}{729}\left\langle {n-18+6 \atop 6} \right\rangle). \end{aligned}$$

One can also obtain a general expression for $g_d(n)$, although some of the simplicity of the EGF approach is lost. Specifically, write

$$g_d(n) = n! \sum_{\mathbf{r} \in S_d} r! \left(\prod_{i=1}^{d} \prod_{m_{ij} \in N_i} \frac{(m_{ij})^{r_{ij}}}{r_{ij}!} \right) \left\langle {n - M_{\mathbf{r}} + r \atop r} \right\rangle. \tag{3}$$

Here S_d is the set of partitions of d, accounting for isotopes, N_i is the set of atomic masses for a given circular triad count i, and $M_{\mathbf{r}} = \sum_{i=1}^{d} \sum_{j \in N_i} r_{ij} j$ is the sum of the masses of the compound in question. For example, for $d=5$, $|R_5| = 7$ but $|S_5| = 10$, with the extra partitions coming from accounting for the two isotopes of 4 in partition $4+1$, and the three isotopes of 5. Staying with $d=5$, for circular triad count $i=4$ the atomic masses are $N_4 = \{5,6\}$, and for $i=5$ they are $N_5 = \{5,6,7\}$.

Equation (3) is the same as the one obtained by Kadane [8], accounting for differences in notation.

6 Discussion

We use an exponential generating function approach to derive a formula for the count of the number of tournaments with a specified number, d, of circular triads. Specific expressions are given for the generating functions for $d = 0, \ldots, 14$. We illustrate how to find the formulae for the counts from the generating functions, and list the formulae for $d = 0, \ldots 14$. The formulae for $d = 6, \ldots 14$ have not previously been published.

Extending to larger values of d is currently not possible as the atomic multipliers have to be found by brute force counting of tournaments. Finding a formula for such counts, or at least an approximation to the counts, is a remaining major challenge.

A Counting atomic tournaments

For simple atomic structures, direct counts via combinatorial arguments are possible. One of the simplest possible atomic tournaments has score sequence $(1, 1, 2, 3, \ldots, n-4, n-3, n-2, n-2)$, which for n players generates $d = n-2$ circular triads. The score sequence $(1, 1, 2, 3, \ldots, n-4, n-3, n-2, n-2)$ is obtained by taking a simple chain of n elements with score sequence $(0, 1, \ldots, n-1)$, and then reversing the connection between the lowest ranked player and highest ranked player. The easiest way to do this is directly, reversing the direction between the first and last player. The connection can also be reversed indirectly, by "visiting" players in the middle. The diagram in Figure 8 shows the four ways to create the $(1, 1, 2, 3, 4, 4)$ tournament in the case of $n = 6$. As one can see, we can "journey" from F to A either directly, or by visiting one of the two middle players, or both, that is $2^{6-4} = 4$ ways. Note that visiting players B or E does not result in a reversal as one can simply relabel B to A or E to F and obtain the original tournament without a reversal. Each of the four different structures has $n!$ choices of permutations of players, and thus the total count $N(n, n-2)$ for the $(1, 1, 2, 3, \ldots, n-4, n-3, n-2, n-2)$ tournament is $2^{n-4}n!$ for $n \geq 4$.

For more complex tournaments, finding the counts is made more complicated by two factors. The first is that a particular n, d combination can be composed of more than one score sequence. We saw this previously in the case of $d = 6, n = 7$, which is associated with 3 different score sequences, $(1, 2, 2, 2, 4, 5, 5)$, $(1, 1, 2, 4, 4, 4, 5)$, and $(1, 1, 3, 3, 3, 5, 5)$.

A second issue is that score sequences are not in 1-1 correspondence with unique classes of isomorphic graphs. We can see this in the example above of $(1, 1, 2, 3, 4, 4)$, where there are 4 different isomorphisms, each of which has 6! permutations. For other score sequences, we typically do not get equal counts for the isomorphisms. For example, the atomic tournament with score sequence $(2, 2, 2, 3, 3, 3)$ for $n = 6$ is associated with 5 different isomorphisms, with three of these having a count of 6!, and the remaining two having a count of $\frac{1}{3}6!$.

An efficient method of counting must be able to generate all possible isomorphism classes, and also assign a rule for counting how many permutations there are for each isomorphism class. Such a method will almost certainly have to work directly with the adjacency matrices. Currently we have no such method, and so apart from the very simplest cases we have found

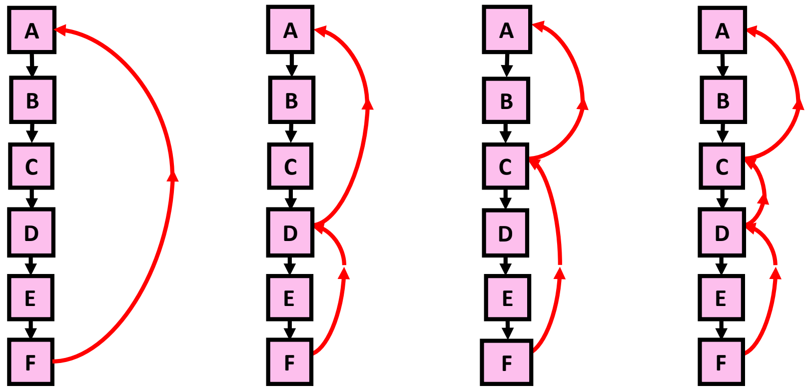

Figure 8: The four ways to create $(1, 1, 2, 3, 4, 4)$.

the $N(n, d)$ by "brute-force" counting. For more details see McShane[14]. It is possible that the techniques of Stockmeyer [17] might be adapted to this counting problem.

B Atomic generators

As explained in Section 3, the atomic generators have a simple structure. The atomic multipliers are obtained through counting the number of atomic tournaments with a specific score sequence and then dividing the counts by $n!$. As an example, consider the generator for $d = 4$. There are two different score sequences for atomic tournaments with 4 circular triads, which we label 4_5 and 4_6. The first has score sequence $(1, 2, 2, 2, 3)$, and is atomic with $n = 5$. An examination of all $2^{\binom{5}{2}} = 1024$ tournaments of 5 players shows that exactly 280 of the tournaments have score sequence $(1, 2, 2, 2, 3)$, and thus the coefficient is $280/5! = 7/3$. The second type of $d = 4$ atomic tournament has score sequence $(1, 1, 2, 3, 4, 4)$, and has $n = 6$. Looking at all $2^{\binom{6}{2}} = 32768$ tournaments of size 6, we find there are exactly 2880 with score sequence $(1, 1, 2, 3, 4, 4)$, and so the coefficient on the generator is $2880/6! = 4$. Combining these two coefficients gives the atomic generator $A_4(x) = \frac{7}{3}x^5 + 4x^6$. As there is currently no simple way to enumerate the atomic tournaments, there is no simple way to obtain the atomic generators.

This appendix lists the generators for $d = 1, \ldots, 14$.

$$A_1(x) = \frac{x^3}{3}.$$
$$A_2(x) = x^4.$$
$$A_3(x) = 2x^5.$$
$$A_4(x) = \frac{7}{3}x^5 + 4x^6.$$
$$A_5(x) = \frac{1}{5}x^5 + \frac{14}{3}x^6 + 8x^7.$$
$$A_6(x) = 12x^6 + \frac{35}{3}x^7 + 16x^8.$$
$$A_7(x) = \frac{20}{3}x^6 + 24x^7 + 28x^8 + 32x^9.$$
$$A_8(x) = \frac{11}{3}x^6 + \frac{143}{3}x^7 + \frac{589}{9}x^8 + \frac{196}{3}x^9 + 64x^{10}.$$
$$A_9(x) = \frac{167}{3}x^7 + \frac{286}{3}x^8 + \frac{1492}{9}x^9 + \frac{448}{3}x^{10} + 128x^{11}.$$
$$A_{10}(x) = \frac{381}{5}x^7 + 240x^8 + \frac{863}{3}x^9 + \frac{3661}{9}x^{10} + 336x^{11} + 256x^{12}.$$
$$A_{11}(x) = \frac{436}{9}x^7 + \frac{4036}{15}x^8 + \frac{4859}{9}x^9 + \frac{2308}{3}x^{10} + \frac{8774}{9}x^{11} + \frac{2240}{3}x^{12} + 512x^{13}.$$
$$A_{12}(x) = \frac{139}{3}x^7 + \frac{1544}{3}x^8 + 981x^9 + \frac{4978}{3}x^{10} + \frac{53875}{27}x^{11} + \frac{20648}{9}x^{12} + \frac{4928}{3}x^{13} + 1024x^{14}.$$

$$A_{13}(x) = \frac{43}{3}x^7 + \frac{1738}{3}x^8 + \frac{5024}{3}x^9 + \frac{36496}{15}x^{10} + \frac{40300}{9}x^{11} + \frac{45374}{9}x^{12} + \frac{47888}{9}x^{13} + 3584x^{14} + 2048x^{15}.$$

$$A_{14}(x) = \frac{11}{21}x^7 + \frac{7750}{9}x^8 + 2490x^9 + \frac{243292}{45}x^{10} + \frac{351772}{45}x^{11} + \frac{106246}{9}x^{12} + \frac{336751}{27}x^{13} + \frac{109744}{9}x^{14} + \frac{50176}{3}x^{15} + 4096x^{16}.$$

C The generating functions

This appendix lists the generating functions for $(g_d(n))_{n\geq 1}$. In the equations below, we have replaced the term $(1-x)^{-1}$ by the symbol y, for simplicity of display.

$$G_0(x) = y.$$

$$G_1(x) = \frac{1}{3}x^3y^2.$$

$$G_2(x) = x^4y^2 + \frac{1}{9}x^6y^3.$$

$$G_3(x) = 2x^5y^2 + \frac{2}{3}x^7y^3 + \frac{1}{27}x^9y^4.$$

$$G_4(x) = \frac{7}{3}x^5y^2 + 4x^6y^2 + \frac{7}{3}x^8y^3 + \frac{1}{3}x^{10}y^4 + \frac{1}{81}x^{12}y^5.$$

$$\begin{aligned} G_5(x) = {} & \frac{1}{5}x^5y^2 + \frac{14}{3}x^6y^2 + 8x^7y^2 + \frac{14}{9}x^8y^3 + \frac{20}{3}x^9y^3 \\ & + \frac{5}{3}x^{11}y^4 + \frac{4}{27}x^{13}y^5 + \frac{1}{243}x^{15}y^6. \end{aligned}$$

$$\begin{aligned} G_6(x) = {} & 12x^6y^2 + \frac{35}{3}x^7y^2 + 16x^8y^2 + \frac{2}{15}x^8y^3 \\ & + \frac{70}{9}x^9y^3 + \frac{52}{3}x^{10}y^3 + \frac{7}{9}x^{11}y^4 + \frac{19}{3}x^{12}y^4 \\ & + \frac{26}{27}x^{14}y^5 + \frac{5}{81}x^{16}y^6 + \frac{1}{729}x^{18}y^7. \end{aligned}$$

$$\begin{aligned} G_7(x) = {} & \frac{20}{3}x^6y^2 + 24x^7y^2 + 28x^8y^2 + 32x^9y^2 + \frac{42}{5}x^9y^3 \\ & + \frac{238}{9}x^{10}y^3 + \frac{128}{3}x^{11}y^3 + \frac{1}{15}x^{11}y^4 + \frac{56}{9}x^{12}y^4 \\ & + \frac{62}{3}x^{13}y^4 + \frac{28}{81}x^{14}y^5 + \frac{124}{27}x^{15}y^5 + \frac{40}{81}x^{17}y^6 \\ & + \frac{2}{81}x^{19}y^7 + \frac{1}{2187}x^{21}y^8. \end{aligned}$$

$$\begin{aligned} G_8(x) = {} & \frac{11}{3}x^6y^2 + \frac{143}{3}x^7y^2 + \frac{589}{9}x^8y^2 + \frac{196}{3}x^9y^2 + \frac{40}{9}x^9y^3 + 64x^{10}y^2 + \frac{2081}{45}x^{10}y^3 \\ & + \frac{238}{3}x^{11}y^3 + \frac{304}{3}x^{12}y^3 + \frac{22}{5}x^{12}y^4 + \frac{266}{9}x^{13}y^4 + \frac{184}{3}x^{14}y^4 \\ & + \frac{4}{135}x^{14}y^5 + \frac{308}{81}x^{15}y^5 + \frac{491}{27}x^{16}y^5 + \frac{35}{243}x^{17}y^6 + \frac{230}{81}x^{18}y^6 \\ & + \frac{19}{81}x^{20}y^7 + \frac{7}{729}x^{22}y^8 + \frac{1}{6561}x^{24}y^9. \end{aligned}$$

$$
\begin{aligned}
G_9(x) = {} & \frac{167}{3}x^7y^2 + \frac{286}{3}x^8y^2 + \frac{1492}{9}x^9y^2 + \frac{22}{9}x^9y^3 + \frac{448}{3}x^{10}y^2 + \frac{2072}{45}x^{10}y^3 \\
& + 128x^{11}y^2 + \frac{22\,006}{135}x^{11}y^3 + \frac{1988}{9}x^{12}y^3 + \frac{20}{9}x^{12}y^4 + \frac{704}{3}x^{13}y^3 \\
& + \frac{1748}{45}x^{13}y^4 + 112x^{14}y^4 + \frac{512}{3}x^{15}y^4 + \frac{92}{45}x^{15}y^5 + \frac{1904}{81}x^{16}y^5 \\
& + \frac{1720}{27}x^{17}y^5 + \frac{1}{81}x^{17}y^6 + \frac{490}{243}x^{18}y^6 + \frac{1075}{81}x^{19}y^6 + \frac{14}{243}x^{20}y^7 \\
& + \frac{128}{81}x^{21}y^7 + \frac{77}{729}x^{23}y^8 + \frac{8}{2187}x^{25}y^9 + \frac{1}{19\,683}x^{27}y^{10}.
\end{aligned}
$$

$$
\begin{aligned}
G_{10}(x) = {} & \frac{381}{5}x^7y^2 + 240x^8y^2 + \frac{863}{3}x^9y^2 + \frac{3661}{9}x^{10}y^2 + 336x^{11}y^2 + 256x^{12}y^2 \\
& + \frac{10\,009}{225}x^{10}y^3 + \frac{10\,954}{45}x^{11}y^3 + \frac{69\,232}{135}x^{12}y^3 + \frac{5264}{9}x^{13}y^3 + \frac{1600}{3}x^{14}y^3 \\
& + \frac{11}{9}x^{12}y^4 + \frac{1357}{45}x^{13}y^4 + \frac{5290}{27}x^{14}y^4 + \frac{3367}{9}x^{15}y^4 + \frac{1360}{3}x^{16}y^4 \\
& + \frac{80}{81}x^{15}y^5 + \frac{662}{27}x^{16}y^5 + \frac{2968}{27}x^{17}y^5 + \frac{5528}{27}x^{18}y^5 + \frac{8}{9}x^{18}y^6 + \frac{3745}{243}x^{19}y^6 \\
& + \frac{4361}{81}x^{20}y^6 + \frac{2}{405}x^{20}y^7 + \frac{238}{243}x^{21}y^7 + \frac{691}{81}x^{22}y^7 + \frac{49}{2187}x^{23}y^8 \\
& + \frac{595}{729}x^{24}y^8 + \frac{100}{2187}x^{26}y^9 + \frac{1}{729}x^{28}y^{10} + \frac{1}{59\,049}x^{30}y^{11}.
\end{aligned}
$$

$$
\begin{aligned}
G_{11}(x) = {} & \frac{436}{9}x^7y^2 + \frac{4036}{15}x^8y^2 + \frac{4859}{9}x^9y^2 + \frac{2308}{3}x^{10}y^2 + \frac{8774}{9}x^{11}y^2 + \frac{2240}{3}x^{12}y^2 \\
& + 512x^{13}y^2 + \frac{254}{5}x^{10}y^3 + \frac{14\,486}{45}x^{11}y^3 + \frac{7696}{9}x^{12}y^3 + \frac{201\,754}{135}x^{13}y^3 + \frac{4480}{3}x^{14}y^3 \\
& + \frac{3584}{3}x^{15}y^3 + \frac{5834}{225}x^{13}y^4 + \frac{2542}{9}x^{14}y^4 + \frac{91\,994}{135}x^{15}y^4 + \frac{9676}{9}x^{16}y^4 + \frac{2912}{3}x^{17}y^4 \\
& + \frac{44}{81}x^{15}y^5 + \frac{6712}{405}x^{16}y^5 + \frac{198\,116}{1215}x^{17}y^5 + \frac{35\,308}{81}x^{18}y^5 + \frac{16\,672}{27}x^{19}y^5 \\
& + \frac{100}{243}x^{18}y^6 + \frac{3244}{243}x^{19}y^6 + \frac{7000}{81}x^{20}y^6 + \frac{15\,970}{81}x^{21}y^6 + \frac{10}{27}x^{21}y^7 + \frac{2170}{243}x^{22}y^7 \\
& + \frac{3194}{81}x^{23}y^7 + \frac{7}{3645}x^{23}y^8 + \frac{980}{2187}x^{24}y^8 + \frac{3647}{729}x^{25}y^8 + \frac{56}{6561}x^{26}y^9 + \frac{872}{2187}x^{27}y^9 \\
& + \frac{14}{729}x^{29}y^{10} + \frac{10}{19\,683}x^{31}y^{11} + \frac{1}{177147}x^{33}y^{12}.
\end{aligned}
$$

$$
\begin{aligned}
G_{12}(x) &= \frac{139}{3}x^7y^2 + \frac{1544}{3}x^8y^2 + 981x^9y^2 + \frac{4978}{3}x^{10}y^2 + \frac{53\,875}{27}x^{11}y^2 + \frac{20\,648}{9}x^{12}y^2 \\
&+ \frac{4928}{3}x^{13}y^2 + 1024x^{14}y^2 + \frac{872}{27}x^{10}y^3 + \frac{3164}{9}x^{11}y^3 + \frac{206\,576}{135}x^{12}y^3 + \frac{375\,052}{135}x^{13}y^3 \\
&+ \frac{557\,993}{135}x^{14}y^3 + \frac{33\,376}{9}x^{15}y^3 + \frac{7936}{3}x^{16}y^3 + \frac{127}{5}x^{13}y^4 + \frac{56\,932}{225}x^{14}y^4 + \frac{158\,507}{135}x^{15}y^4 \\
&+ \frac{367\,076}{135}x^{16}y^4 + \frac{10\,024}{3}x^{17}y^4 + \frac{8704}{3}x^{18}y^4 + \frac{26\,654}{2025}x^{16}y^5 + \frac{66\,596}{405}x^{17}y^5 \\
&+ \frac{979\,898}{1215}x^{18}y^5 + \frac{125\,524}{81}x^{19}y^5 + \frac{47\,888}{27}x^{20}y^5 + \frac{55}{243}x^{18}y^6 + \frac{1999}{243}x^{19}y^6 + \frac{80\,483}{729}x^{20}y^6 \\
&+ \frac{97\,265}{243}x^{21}y^6 + \frac{54\,140}{81}x^{22}y^6 + \frac{40}{243}x^{21}y^7 + \frac{1613}{243}x^{22}y^7 + \frac{1582}{27}x^{23}y^7 + \frac{13\,165}{81}x^{24}y^7 \\
&+ \frac{182}{1215}x^{24}y^8 + \frac{10\,388}{2187}x^{25}y^8 + \frac{18\,949}{729}x^{26}y^8 + \frac{8}{10\,935}x^{26}y^9 + \frac{1288}{6561}x^{27}y^9 + \frac{5986}{2187}x^{28}y^9 \\
&+ \frac{7}{2187}x^{29}y^{10} + \frac{136}{729}x^{30}y^{10} + \frac{155}{19\,683}x^{32}y^{11} + \frac{11}{59\,049}x^{34}y^{12} + \frac{1}{531\,441}x^{36}y^{13}.
\end{aligned}
$$

$$
\begin{aligned}
G_{13}(x) &= \frac{43}{3}x^7y^2 + \frac{1738}{3}x^8y^2 + \frac{5024}{3}x^9y^2 + \frac{36\,496}{15}x^{10}y^2 + \frac{40\,300}{9}x^{11}y^2 + \frac{45\,374}{9}x^{12}y^2 \\
&+ \frac{47\,888}{9}x^{13}y^2 + 3584x^{14}y^2 + 2048x^{15}y^2 + \frac{278}{9}x^{10}y^3 + \frac{6622}{15}x^{11}y^3 + 1970x^{12}y^3 \\
&+ \frac{238\,388}{45}x^{13}y^3 + \frac{3402\,334}{405}x^{14}y^3 + \frac{1485\,176}{135}x^{15}y^3 + \frac{81\,088}{9}x^{16}y^3 + \frac{17\,408}{3}x^{17}y^3 \\
&+ \frac{436}{27}x^{13}y^4 + \frac{3928}{15}x^{14}y^4 + \frac{1128\,917}{675}x^{15}y^4 + \frac{639\,946}{135}x^{16}y^4 + \frac{1177\,603}{135}x^{17}y^4 \\
&+ \frac{83\,552}{9}x^{18}y^4 + \frac{21\,248}{3}x^{19}y^4 + \frac{508}{45}x^{16}y^5 + \frac{106\,868}{675}x^{17}y^5 + \frac{48\,164}{45}x^{18}y^5 \\
&+ \frac{4075\,016}{1215}x^{19}y^5 + 5096x^{20}y^5 + \frac{132\,352}{27}x^{21}y^5 + \frac{7493}{1215}x^{19}y^6 + \frac{23\,774}{243}x^{20}y^6 \\
&+ \frac{480\,827}{729}x^{21}y^6 + \frac{395\,570}{243}x^{22}y^6 + \frac{172\,840}{81}x^{23}y^6 + \frac{22}{243}x^{21}y^7 \\
&+ \frac{928}{243}x^{22}y^7 + \frac{47\,822}{729}x^{23}y^7 + \frac{75\,782}{243}x^{24}y^7 + \frac{49\,700}{81}x^{25}y^7 + \frac{140}{2187}x^{24}y^8 + \frac{11\,312}{3645}x^{25}y^8 \\
&+ \frac{26\,068}{729}x^{26}y^8 + \frac{86\,975}{729}x^{27}y^8 + \frac{8}{135}x^{27}y^9 + \frac{15\,568}{6561}x^{28}y^9 + \frac{34\,568}{2187}x^{29}y^9 + \frac{1}{3645}x^{29}y^{10} \\
&+ \frac{182}{2187}x^{30}y^{10} + \frac{1034}{729}x^{31}y^{10} + \frac{7}{59\,049}x^{32}y^{11} + \frac{1624}{19\,683}x^{33}y^{11} + \frac{187}{59\,049}x^{35}y^{12} \\
&+ \frac{4}{59\,049}x^{37}y^{13} + \frac{1}{1594\,323}x^{39}y^{14}.
\end{aligned}
$$

$$\begin{aligned}G_{14}(x) = {} & \frac{11}{21}x^7y^2 + \frac{7750}{9}x^8y^2 + 2490x^9y^2 + \frac{243\,292}{45}x^{10}y^2 + \frac{351\,772}{45}x^{11}y^2 + \frac{106\,246}{9}x^{12}y^2 \\ & + \frac{336\,751}{27}x^{13}y^2 + \frac{109\,744}{9}x^{14}y^2 + \frac{50\,176}{3}x^{15}y^2 + 4096x^{16}y^2 + \frac{86}{9}x^{10}y^3 + \frac{4310}{9}x^{11}y^3 \\ & + \frac{42\,748}{15}x^{12}y^3 + \frac{382\,372}{45}x^{13}y^3 + \frac{2338\,492}{135}x^{14}y^3 + \frac{3260\,534}{135}x^{15}y^3 + \frac{3837\,992}{135}x^{16}y^3 \\ & + 21\,504x^{17}y^3 + \frac{37\,888}{3}x^{18}y^3 + \frac{139}{9}x^{13}y^4 + \frac{12\,146}{45}x^{14}y^4 + \frac{144\,841}{75}x^{15}y^4 \\ & + \frac{600\,316}{75}x^{16}y^4 + \frac{6881\,891}{405}x^{17}y^4 + \frac{3556\,498}{135}x^{18}y^4 + \frac{224\,336}{9}x^{19}y^4 + \frac{50\,944}{3}x^{20}y^4 \\ & + \frac{1744}{243}x^{16}y^5 + \frac{62\,632}{405}x^{17}y^5 + \frac{7926\,182}{6075}x^{18}y^5 + \frac{2168\,128}{405}x^{19}y^5 + \frac{15\,121\,078}{1215}x^{20}y^5 \\ & + \frac{1280\,048}{81}x^{21}y^5 + \frac{354\,560}{27}x^{22}y^5 + \frac{127}{27}x^{19}y^6 + \frac{105\,872}{1215}x^{20}y^6 + \frac{189\,107}{243}x^{21}y^6 \\ & + \frac{2755\,265}{729}x^{22}y^6 + \frac{565\,765}{81}x^{23}y^6 + \frac{525\,920}{81}x^{24}y^6 + \frac{3329}{1215}x^{22}y^7 + \frac{12\,770}{243}x^{23}y^7 \\ & + \frac{336\,044}{729}x^{24}y^7 + \frac{303\,338}{243}x^{25}y^7 + \frac{149\,140}{81}x^{26}y^7 + \frac{77}{2187}x^{24}y^8 + \frac{18\,487}{10\,935}x^{25}y^8 \\ & + \frac{1168\,916}{32\,805}x^{26}y^8 + \frac{471\,478}{2187}x^{27}y^8 + \frac{362\,545}{729}x^{28}y^8 + \frac{160}{6561}x^{27}y^9 + \frac{45\,524}{32\,805}x^{28}y^9 \\ & + \frac{44\,128}{2187}x^{29}y^9 + \frac{175\,060}{2187}x^{30}y^9 + \frac{28}{1215}x^{30}y^{10} + \frac{2471}{2187}x^{31}y^{10} + \frac{6574}{729}x^{32}y^{10} \\ & + \frac{2}{19\,683}x^{32}y^{11} + \frac{2030}{59\,049}x^{33}y^{11} + \frac{13\,850}{19\,683}x^{34}y^{11} + \frac{77}{177\,147}x^{35}y^{12} + \frac{2189}{59\,049}x^{36}y^{12} \\ & + \frac{74}{59\,049}x^{38}y^{13} + \frac{13}{531\,441}x^{40}y^{14} + \frac{1}{4782\,969}x^{42}y^{15}.\end{aligned}$$

D The sequences

As explained in 5, to obtain an explicit formula for $(g_d(n))_{n\geq 1}$ from $G_d(x)$ we extract the coefficient of $\frac{x^n}{n!}$ in $x^a(1-x)^{-(1+b)}$ as $n!\left\langle{n-a+b \atop b}\right\rangle$. The formulae are found by adding together the appropriate terms. Note that the formulae for $d = 0, \ldots, 5$ are identical to the ones in [8] allowing for differences in notation.

Zero circular triads, $d = 0$. The formula for the sequence is

$$g_0(n) = n!,$$

with the first non-zero term at $n = 1$. The first terms are $1, 2, 6, 24, \ldots$, related to sequence number A000142.

One circular triad, $d = 1$. The formula for the sequence is

$$g_1(n) = n!\frac{1}{3}\left\langle{n-2 \atop 1}\right\rangle,$$

with the first non-zero term at $n = 3$. The first non-zero terms are $2, 16, 120, 960, 8400, \dots,$ related to sequence number A090672.

Two circular triads, $d = 2$. The formula for the sequence is

$$g_2(n) = n!\left(\left\langle {n-3 \atop 1} \right\rangle + \frac{1}{9}\left\langle {n-4 \atop 2} \right\rangle\right),$$

with the first non-zero term at $n = 4$. The first non-zero terms are $24, 240, 2240, 21840, \dots,$ with sequence number A357242.

Three circular triads, $d = 3$. The formula for the sequence is

$$g_3(n) = n!\left(2\left\langle {n-4 \atop 1} \right\rangle + \frac{2}{3}\left\langle {n-5 \atop 2} \right\rangle + \frac{1}{3^3}\left\langle {n-6 \atop 3} \right\rangle\right),$$

with the first non-zero term at $n = 5$. The first non-zero terms are $240, 2880, 33600, 403200,$ $\dots,$ with sequence number A357257.

Four circular triads, $d = 4$. The formula for the sequence is

$$g_4(n) = n!\left(\frac{7}{3}\left\langle {n-4 \atop 1} \right\rangle + 4\left\langle {n-5 \atop 1} \right\rangle + \frac{7}{3}\left\langle {n-6 \atop 2} \right\rangle + \frac{1}{3}\left\langle {n-7 \atop 3} \right\rangle + \frac{1}{81}\left\langle {n-8 \atop 4} \right\rangle\right),$$

with the first non-zero term at $n = 5$. The first non-zero terms are $280, 6240, 75600, 954240,$ $\dots,$ with sequence number A357248.

Five circular triads, $d = 5$. The formula for the sequence is

$$g_5(n) = n!\left(\frac{1}{5}\left\langle {n-4 \atop 1} \right\rangle + \frac{14}{3}\left\langle {n-5 \atop 1} \right\rangle + 8\left\langle {x-6 \atop 1} \right\rangle + \frac{14}{9}\left\langle {n-6 \atop 2} \right\rangle + \frac{20}{3}\left\langle {n-7 \atop 2} \right\rangle\right.$$
$$\left. + \frac{5}{3}\left\langle {n-8 \atop 3} \right\rangle + \frac{4}{27}\left\langle {n-9 \atop 4} \right\rangle + \frac{1}{243}\left\langle {n-10 \atop 5} \right\rangle\right),$$

with the first non-zero term at $n = 5$. The first non-zero terms are $24, 3648, 90384, 1304576,$ $\dots,$ with sequence number A357266.

Six circular triads, $d = 6$. The formula for the sequence is

$$g_6(n) = n!(12\left\langle {n-5 \atop 1} \right\rangle + \frac{35}{3}\left\langle {n-6 \atop 1} \right\rangle + 16\left\langle {n-7 \atop 1} \right\rangle + \frac{2}{15}\left\langle {n-6 \atop 2} \right\rangle + \frac{70}{9}\left\langle {n-7 \atop 2} \right\rangle$$
$$+ \frac{52}{3}\left\langle {n-8 \atop 2} \right\rangle + \frac{7}{9}\left\langle {n-8 \atop 3} \right\rangle + \frac{19}{3}\left\langle {n-9 \atop 3} \right\rangle + \frac{26}{27}\left\langle {n-10 \atop 4} \right\rangle + \frac{5}{81}\left\langle {n-11 \atop 5} \right\rangle$$
$$+ \frac{1}{729}\left\langle {n-12 \atop 6} \right\rangle),$$

with the first non-zero term at $n = 6$. The first non-zero terms are $8640, 179760, 3042816, 44698752, \ldots$.

Seven circular triads, $d = 7$. The formula for the sequence is

$$\begin{aligned} g_7(n) = n!\Big(&\frac{20}{3}\left\langle {n-5 \atop 1} \right\rangle + 24\left\langle {n-6 \atop 1} \right\rangle + 28\left\langle {n-7 \atop 1} \right\rangle + 32\left\langle {n-8 \atop 1} \right\rangle + \frac{42}{5}\left\langle {n-7 \atop 2} \right\rangle \\ &+ \frac{238}{9}\left\langle {n-8 \atop 2} \right\rangle + \frac{128}{3}\left\langle {n-9 \atop 2} \right\rangle + \frac{1}{15}\left\langle {n-8 \atop 3} \right\rangle + \frac{56}{9}\left\langle {n-9 \atop 3} \right\rangle + \frac{62}{3}\left\langle {n-10 \atop 3} \right\rangle \\ &+ \frac{28}{81}\left\langle {n-10 \atop 4} \right\rangle + \frac{124}{27}\left\langle {n-11 \atop 4} \right\rangle + \frac{40}{81}\left\langle {n-12 \atop 5} \right\rangle + \frac{2}{81}\left\langle {n-13 \atop 6} \right\rangle + \frac{1}{2187}\left\langle {n-14 \atop 7} \right\rangle\Big), \end{aligned}$$

with the first non-zero term at $n = 6$. The first non-zero terms are $4800, 188160, 3870720, 70785792, \ldots$.

Eight circular triads, $d = 8$. The formula for the sequence is

$$\begin{aligned} g_8(n) = n!\Big(&\frac{11}{3}\left\langle {n-5 \atop 1} \right\rangle + \frac{143}{3}\left\langle {n-6 \atop 1} \right\rangle + \frac{589}{9}\left\langle {n-7 \atop 1} \right\rangle + \frac{196}{3}\left\langle {n-8 \atop 1} \right\rangle + \frac{40}{9}\left\langle {n-7 \atop 2} \right\rangle \\ &+ 64\left\langle {n-9 \atop 1} \right\rangle + \frac{2081}{45}\left\langle {n-8 \atop 2} \right\rangle + \frac{238}{3}\left\langle {n-9 \atop 2} \right\rangle + \frac{304}{3}\left\langle {n-10 \atop 2} \right\rangle + \frac{22}{5}\left\langle {n-9 \atop 3} \right\rangle \\ &+ \frac{266}{9}\left\langle {n-10 \atop 3} \right\rangle + \frac{184}{3}\left\langle {n-11 \atop 3} \right\rangle + \frac{4}{135}\left\langle {n-10 \atop 4} \right\rangle + \frac{308}{81}\left\langle {n-11 \atop 4} \right\rangle + \frac{491}{27}\left\langle {n-12 \atop 4} \right\rangle \\ &+ \frac{35}{243}\left\langle {n-12 \atop 5} \right\rangle + \frac{230}{81}\left\langle {n-13 \atop 5} \right\rangle + \frac{19}{81}\left\langle {n-14 \atop 6} \right\rangle + \frac{7}{729}\left\langle {n-15 \atop 7} \right\rangle + \frac{1}{6561}\left\langle {n-16 \atop 8} \right\rangle\Big), \end{aligned}$$

with the first non-zero term at $n = 6$. The first non-zero terms are $2640, 277200, 6926080, 130032000, \ldots$.

Nine circular triads, $d = 9$. The formula for the sequence is

$$\begin{aligned} g_9(n) = n!\Big(&\frac{167}{3}\left\langle {n-6 \atop 1} \right\rangle + \frac{286}{3}\left\langle {n-7 \atop 1} \right\rangle + \frac{1492}{9}\left\langle {n-8 \atop 1} \right\rangle + \frac{22}{9}\left\langle {n-7 \atop 2} \right\rangle + \frac{448}{3}\left\langle {n-9 \atop 1} \right\rangle \\ &+ \frac{2072}{45}\left\langle {n-8 \atop 2} \right\rangle + 128\left\langle {n-10 \atop 1} \right\rangle + \frac{22\,006}{135}\left\langle {n-9 \atop 2} \right\rangle + \frac{1988}{9}\left\langle {n-10 \atop 2} \right\rangle + \frac{20}{9}\left\langle {n-9 \atop 3} \right\rangle \\ &+ \frac{704}{3}\left\langle {n-11 \atop 2} \right\rangle + \frac{1748}{45}\left\langle {n-10 \atop 3} \right\rangle + 112\left\langle {n-11 \atop 3} \right\rangle + \frac{512}{3}\left\langle {n-12 \atop 3} \right\rangle + \frac{92}{45}\left\langle {n-11 \atop 4} \right\rangle \\ &+ \frac{1904}{81}\left\langle {n-12 \atop 4} \right\rangle + \frac{1720}{27}\left\langle {n-13 \atop 4} \right\rangle + \frac{1}{81}\left\langle {n-12 \atop 5} \right\rangle + \frac{490}{243}\left\langle {n-13 \atop 5} \right\rangle + \frac{1075}{81}\left\langle {n-14 \atop 5} \right\rangle \\ &+ \frac{14}{243}\left\langle {n-14 \atop 6} \right\rangle + \frac{128}{81}\left\langle {n-15 \atop 6} \right\rangle + \frac{77}{729}\left\langle {n-16 \atop 7} \right\rangle + \frac{8}{2187}\left\langle {n-17 \atop 8} \right\rangle + \frac{1}{19\,683}\left\langle {n-18 \atop 9} \right\rangle\Big), \end{aligned}$$

with the first non-zero term at $n = 7$. The first non-zero terms are $280560, 8332800, 190834560, 3784596480, \ldots$.

Ten circular triads, $d = 10$. The formula for the sequence is

$$
\begin{aligned}
g_{10}(n) = n!(&\frac{381}{5}\left\langle {n-6 \atop 1} \right\rangle + 240\left\langle {n-7 \atop 1} \right\rangle + \frac{863}{3}\left\langle {n-8 \atop 1} \right\rangle + \frac{3661}{9}\left\langle {n-9 \atop 1} \right\rangle + 336\left\langle {n-10 \atop 1} \right\rangle \\
&+ 256\left\langle {n-11 \atop 1} \right\rangle + \frac{10\,009}{225}\left\langle {n-8 \atop 2} \right\rangle + \frac{10\,954}{45}\left\langle {n-9 \atop 2} \right\rangle + \frac{69\,232}{135}\left\langle {n-10 \atop 2} \right\rangle \\
&+ \frac{5264}{9}\left\langle {n-11 \atop 2} \right\rangle + \frac{1600}{3}\left\langle {n-12 \atop 2} \right\rangle + \frac{11}{9}\left\langle {n-9 \atop 3} \right\rangle + \frac{1357}{45}\left\langle {n-10 \atop 3} \right\rangle + \frac{5290}{27}\left\langle {n-11 \atop 3} \right\rangle \\
&+ \frac{3367}{9}\left\langle {n-12 \atop 3} \right\rangle + \frac{1360}{3}\left\langle {n-13 \atop 3} \right\rangle + \frac{80}{81}\left\langle {n-11 \atop 4} \right\rangle + \frac{662}{27}\left\langle {n-12 \atop 4} \right\rangle + \frac{2968}{27}\left\langle {n-13 \atop 4} \right\rangle \\
&+ \frac{5528}{27}\left\langle {n-14 \atop 4} \right\rangle + \frac{8}{9}\left\langle {n-13 \atop 5} \right\rangle + \frac{3745}{243}\left\langle {n-14 \atop 5} \right\rangle + \frac{4361}{81}\left\langle {n-15 \atop 5} \right\rangle + \frac{2}{405}\left\langle {n-14 \atop 6} \right\rangle \\
&+ \frac{238}{243}\left\langle {n-15 \atop 6} \right\rangle + \frac{691}{81}\left\langle {n-16 \atop 6} \right\rangle + \frac{49}{2187}\left\langle {n-16 \atop 7} \right\rangle + \frac{595}{729}\left\langle {n-17 \atop 7} \right\rangle + \frac{100}{2187}\left\langle {n-18 \atop 8} \right\rangle \\
&+ \frac{1}{729}\left\langle {n-19 \atop 9} \right\rangle + \frac{1}{59\,049}\left\langle {n-20 \atop 10} \right\rangle),
\end{aligned}
$$

with the first non-zero term at $n = 7$. The first non-zero terms are $384048, 15821568, 361525248, 7444104192, \ldots$.

Eleven circular triads, $d = 11$. The formula for the sequence is

$$
\begin{aligned}
g_{11}(n) = n!(&\frac{436}{9}\left\langle {n-6 \atop 1} \right\rangle + \frac{4036}{15}\left\langle {n-7 \atop 1} \right\rangle + \frac{4859}{9}\left\langle {n-8 \atop 1} \right\rangle + \frac{2308}{3}\left\langle {n-9 \atop 1} \right\rangle + \frac{8774}{9}\left\langle {n-10 \atop 1} \right\rangle \\
&+ \frac{2240}{3}\left\langle {n-11 \atop 1} \right\rangle + 512\left\langle {n-12 \atop 1} \right\rangle + \frac{254}{5}\left\langle {n-8 \atop 2} \right\rangle + \frac{14\,486}{45}\left\langle {n-9 \atop 2} \right\rangle + \frac{7696}{9}\left\langle {n-10 \atop 2} \right\rangle \\
&+ \frac{201\,754}{135}\left\langle {n-11 \atop 2} \right\rangle + \frac{4480}{3}\left\langle {n-12 \atop 2} \right\rangle + \frac{3584}{3}\left\langle {n-13 \atop 2} \right\rangle + \frac{5834}{225}\left\langle {n-10 \atop 3} \right\rangle \\
&+ \frac{2542}{9}\left\langle {n-11 \atop 3} \right\rangle + \frac{91\,994}{135}\left\langle {n-12 \atop 3} \right\rangle + \frac{9676}{9}\left\langle {n-13 \atop 3} \right\rangle + \frac{2912}{3}\left\langle {n-14 \atop 3} \right\rangle \\
&+ \frac{44}{81}\left\langle {n-11 \atop 4} \right\rangle + \frac{6712}{405}\left\langle {n-12 \atop 4} \right\rangle + \frac{198\,116}{1215}\left\langle {n-13 \atop 4} \right\rangle + \frac{35\,308}{81}\left\langle {n-14 \atop 4} \right\rangle \\
&+ \frac{16\,672}{27}\left\langle {n-15 \atop 4} \right\rangle + \frac{100}{243}\left\langle {n-13 \atop 5} \right\rangle + \frac{3244}{243}\left\langle {n-14 \atop 5} \right\rangle + \frac{7000}{81}\left\langle {n-15 \atop 5} \right\rangle \\
&+ \frac{15\,970}{81}\left\langle {n-16 \atop 5} \right\rangle + \frac{10}{27}\left\langle {n-15 \atop 6} \right\rangle + \frac{2170}{243}\left\langle {n-16 \atop 6} \right\rangle + \frac{3194}{81}\left\langle {n-17 \atop 6} \right\rangle \\
&+ \frac{7}{3645}\left\langle {n-16 \atop 7} \right\rangle + \frac{980}{2187}\left\langle {n-17 \atop 7} \right\rangle + \frac{3647}{729}\left\langle {n-18 \atop 7} \right\rangle + \frac{56}{6561}\left\langle {n-18 \atop 8} \right\rangle \\
&+ \frac{872}{2187}\left\langle {n-19 \atop 8} \right\rangle + \frac{14}{729}\left\langle {n-20 \atop 9} \right\rangle + \frac{10}{19\,683}\left\langle {n-21 \atop 10} \right\rangle + \frac{1}{177147}\left\langle {n-22 \atop 11} \right\rangle),
\end{aligned}
$$

with the first non-zero term at $n = 7$. The first non-zero terms are $244160, 14755328, 443931264, 10526745600, \ldots$.

Twelve circular triads, $d = 12$. The formula for the sequence is

$$\begin{aligned}
g_{12}(n) = n!(&\frac{139}{3}\left\langle {n-6 \atop 1} \right\rangle + \frac{1544}{3}\left\langle {n-7 \atop 1} \right\rangle + 981\left\langle {n-8 \atop 1} \right\rangle + \frac{4978}{3}\left\langle {n-9 \atop 1} \right\rangle + \frac{53\,875}{27}\left\langle {n-10 \atop 1} \right\rangle \\
&+ \frac{20\,648}{9}\left\langle {n-11 \atop 1} \right\rangle + \frac{4928}{3}\left\langle {n-12 \atop 1} \right\rangle + 1024\left\langle {n-13 \atop 1} \right\rangle + \frac{872}{27}\left\langle {n-8 \atop 2} \right\rangle + \frac{3164}{9}\left\langle {n-9 \atop 2} \right\rangle \\
&+ \frac{206\,576}{135}\left\langle {n-10 \atop 2} \right\rangle + \frac{375\,052}{135}\left\langle {n-11 \atop 2} \right\rangle + \frac{557\,993}{135}\left\langle {n-12 \atop 2} \right\rangle + \frac{33\,376}{9}\left\langle {n-13 \atop 2} \right\rangle \\
&+ \frac{7936}{3}\left\langle {n-14 \atop 2} \right\rangle + \frac{127}{5}\left\langle {n-10 \atop 3} \right\rangle + \frac{56\,932}{225}\left\langle {n-11 \atop 3} \right\rangle + \frac{158\,507}{135}\left\langle {n-12 \atop 3} \right\rangle \\
&+ \frac{367\,076}{135}\left\langle {n-13 \atop 3} \right\rangle + \frac{10\,024}{3}\left\langle {n-14 \atop 3} \right\rangle + \frac{8704}{3}\left\langle {n-15 \atop 3} \right\rangle + \frac{26\,654}{2025}\left\langle {n-12 \atop 4} \right\rangle \\
&+ \frac{66\,596}{405}\left\langle {n-13 \atop 4} \right\rangle + \frac{979\,898}{1215}\left\langle {n-14 \atop 4} \right\rangle + \frac{125\,524}{81}\left\langle {n-15 \atop 4} \right\rangle + \frac{47\,888}{27}\left\langle {n-16 \atop 4} \right\rangle \\
&+ \frac{55}{243}\left\langle {n-13 \atop 5} \right\rangle + \frac{1999}{243}\left\langle {n-14 \atop 5} \right\rangle + \frac{80\,483}{729}\left\langle {n-15 \atop 5} \right\rangle + \frac{97\,265}{243}\left\langle {n-16 \atop 5} \right\rangle \\
&+ \frac{54\,140}{81}\left\langle {n-17 \atop 5} \right\rangle + \frac{40}{243}\left\langle {n-15 \atop 6} \right\rangle + \frac{1613}{243}\left\langle {n-16 \atop 6} \right\rangle + \frac{1582}{27}\left\langle {n-17 \atop 6} \right\rangle + \frac{13\,165}{81}\left\langle {n-18 \atop 6} \right\rangle \\
&+ \frac{182}{1215}\left\langle {n-17 \atop 7} \right\rangle + \frac{10\,388}{2187}\left\langle {n-18 \atop 7} \right\rangle + \frac{18\,949}{729}\left\langle {n-19 \atop 7} \right\rangle + \frac{8}{10\,935}\left\langle {n-18 \atop 8} \right\rangle \\
&+ \frac{1288}{6561}\left\langle {n-19 \atop 8} \right\rangle + \frac{5986}{2187}\left\langle {n-20 \atop 8} \right\rangle + \frac{7}{2187}\left\langle {n-20 \atop 9} \right\rangle + \frac{136}{729}\left\langle {n-21 \atop 9} \right\rangle + \frac{155}{19\,683}\left\langle {n-22 \atop 10} \right\rangle \\
&+ \frac{11}{59\,049}\left\langle {n-23 \atop 11} \right\rangle + \frac{1}{531\,441}\left\langle {n-24 \atop 12} \right\rangle),
\end{aligned}$$

with the first non-zero term at $n = 7$. The first non-zero terms are $233520, 24487680, 779950080, 19533696000, \ldots$.

Thirteen circular triads, $d = 13$. The formula for the sequence is

$$\begin{aligned}
g_{13}(n) = n!(&\frac{43}{3}\left\langle {n-6 \atop 1} \right\rangle + \frac{1738}{3}\left\langle {n-7 \atop 1} \right\rangle + \frac{5024}{3}\left\langle {n-8 \atop 1} \right\rangle + \frac{36\,496}{15}\left\langle {n-9 \atop 1} \right\rangle + \frac{40\,300}{9}\left\langle {n-10 \atop 1} \right\rangle \\
&+ \frac{45\,374}{9}\left\langle {n-11 \atop 1} \right\rangle + \frac{47\,888}{9}\left\langle {n-12 \atop 1} \right\rangle + 3584\left\langle {n-13 \atop 1} \right\rangle + 2048\left\langle {n-14 \atop 1} \right\rangle + \frac{278}{9}\left\langle {n-8 \atop 2} \right\rangle \\
&+ \frac{6622}{15}\left\langle {n-9 \atop 2} \right\rangle + 1970\left\langle {n-10 \atop 2} \right\rangle + \frac{238\,388}{45}\left\langle {n-11 \atop 2} \right\rangle + \frac{3402\,334}{405}\left\langle {n-12 \atop 2} \right\rangle \\
&+ \frac{1485\,176}{135}\left\langle {n-13 \atop 2} \right\rangle + \frac{81\,088}{9}\left\langle {n-14 \atop 2} \right\rangle + \frac{17\,408}{3}\left\langle {n-15 \atop 2} \right\rangle + \frac{436}{27}\left\langle {n-10 \atop 3} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3928}{15}\genfrac{\langle}{\rangle}{0pt}{}{n-11}{3}+\frac{1\,128\,917}{675}\genfrac{\langle}{\rangle}{0pt}{}{n-12}{3}+\frac{639\,946}{135}\genfrac{\langle}{\rangle}{0pt}{}{n-13}{3}+\frac{1\,177\,603}{135}\genfrac{\langle}{\rangle}{0pt}{}{n-14}{3}\\
&+\frac{83\,552}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-15}{3}+\frac{21\,248}{3}\genfrac{\langle}{\rangle}{0pt}{}{n-16}{3}+\frac{508}{45}\genfrac{\langle}{\rangle}{0pt}{}{n-12}{4}+\frac{106\,868}{675}\genfrac{\langle}{\rangle}{0pt}{}{n-13}{4}\\
&+\frac{48\,164}{45}\genfrac{\langle}{\rangle}{0pt}{}{n-14}{4}+\frac{4075\,016}{1215}\genfrac{\langle}{\rangle}{0pt}{}{n-15}{4}+5096\genfrac{\langle}{\rangle}{0pt}{}{n-16}{4}+\frac{132\,352}{27}\genfrac{\langle}{\rangle}{0pt}{}{n-17}{4}\\
&+\frac{7493}{1215}\genfrac{\langle}{\rangle}{0pt}{}{n-14}{5}+\frac{23\,774}{243}\genfrac{\langle}{\rangle}{0pt}{}{n-15}{5}+\frac{480\,827}{729}\genfrac{\langle}{\rangle}{0pt}{}{n-16}{5}+\frac{395\,570}{243}\genfrac{\langle}{\rangle}{0pt}{}{n-17}{5}\\
&+\frac{172\,840}{81}\genfrac{\langle}{\rangle}{0pt}{}{n-18}{5}+\frac{22}{243}\genfrac{\langle}{\rangle}{0pt}{}{n-15}{6}+\frac{928}{243}\genfrac{\langle}{\rangle}{0pt}{}{n-16}{6}+\frac{47\,822}{729}\genfrac{\langle}{\rangle}{0pt}{}{n-17}{6}\\
&+\frac{75\,782}{243}\genfrac{\langle}{\rangle}{0pt}{}{n-18}{6}+\frac{49\,700}{81}\genfrac{\langle}{\rangle}{0pt}{}{n-19}{6}+\frac{140}{2187}\genfrac{\langle}{\rangle}{0pt}{}{n-17}{7}+\frac{11\,312}{3645}\genfrac{\langle}{\rangle}{0pt}{}{n-18}{7}\\
&+\frac{26\,068}{729}\genfrac{\langle}{\rangle}{0pt}{}{n-19}{7}+\frac{86\,975}{729}\genfrac{\langle}{\rangle}{0pt}{}{n-20}{7}+\frac{8}{135}\genfrac{\langle}{\rangle}{0pt}{}{n-19}{8}+\frac{15\,568}{6561}\genfrac{\langle}{\rangle}{0pt}{}{n-20}{8}\\
&+\frac{34\,568}{2187}\genfrac{\langle}{\rangle}{0pt}{}{n-21}{8}+\frac{1}{3645}\genfrac{\langle}{\rangle}{0pt}{}{n-20}{9}+\frac{182}{2187}\genfrac{\langle}{\rangle}{0pt}{}{n-21}{9}+\frac{1034}{729}\genfrac{\langle}{\rangle}{0pt}{}{n-22}{9}\\
&+\frac{7}{59\,049}\genfrac{\langle}{\rangle}{0pt}{}{n-22}{10}+\frac{1624}{19\,683}\genfrac{\langle}{\rangle}{0pt}{}{n-23}{10}+\frac{187}{59\,049}\genfrac{\langle}{\rangle}{0pt}{}{n-24}{11}+\frac{4}{59\,049}\genfrac{\langle}{\rangle}{0pt}{}{n-25}{12}\\
&+\frac{1}{1594\,323}\genfrac{\langle}{\rangle}{0pt}{}{n-26}{13}),
\end{aligned}
$$

with the first non-zero term at $n = 7$. The first non-zero terms are $72240, 24514560, 1043763840, 27610168320, \ldots$.

Fourteen circular triads, $d = 14$. The formula for the sequence is

$$
\begin{aligned}
g_{14}(n) = n!(&\frac{11}{21}\genfrac{\langle}{\rangle}{0pt}{}{n-6}{1}+\frac{7750}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-7}{1}+2490\genfrac{\langle}{\rangle}{0pt}{}{n-8}{1}+\frac{243\,292}{45}\genfrac{\langle}{\rangle}{0pt}{}{n-9}{1}+\frac{351\,772}{45}\genfrac{\langle}{\rangle}{0pt}{}{n-10}{1}\\
&+\frac{106\,246}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-11}{1}+\frac{336\,751}{27}\genfrac{\langle}{\rangle}{0pt}{}{n-12}{1}+\frac{109\,744}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-13}{1}+\frac{50\,176}{3}\genfrac{\langle}{\rangle}{0pt}{}{n-14}{1}\\
&+4096\genfrac{\langle}{\rangle}{0pt}{}{n-15}{1}+\frac{86}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-8}{2}+\frac{4310}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-9}{2}+\frac{42\,748}{15}\genfrac{\langle}{\rangle}{0pt}{}{n-10}{2}+\frac{382\,372}{45}\genfrac{\langle}{\rangle}{0pt}{}{n-11}{2}\\
&+\frac{2338\,492}{135}\genfrac{\langle}{\rangle}{0pt}{}{n-12}{2}+\frac{3260\,534}{135}\genfrac{\langle}{\rangle}{0pt}{}{n-13}{2}+\frac{3837\,992}{135}\genfrac{\langle}{\rangle}{0pt}{}{n-14}{2}+21\,504\genfrac{\langle}{\rangle}{0pt}{}{n-15}{2}\\
&+\frac{37\,888}{3}\genfrac{\langle}{\rangle}{0pt}{}{n-16}{2}+\frac{139}{9}\genfrac{\langle}{\rangle}{0pt}{}{n-10}{3}+\frac{12\,146}{45}\genfrac{\langle}{\rangle}{0pt}{}{n-11}{3}+\frac{144\,841}{75}\genfrac{\langle}{\rangle}{0pt}{}{n-12}{3}
\end{aligned}
$$

$$+ \frac{600\,316}{75}\left\langle {n-13 \atop 3} \right\rangle + \frac{6881\,891}{405}\left\langle {n-14 \atop 3} \right\rangle + \frac{3556\,498}{135}\left\langle {n-15 \atop 3} \right\rangle + \frac{224\,336}{9}\left\langle {n-16 \atop 3} \right\rangle$$

$$+ \frac{50\,944}{3}\left\langle {n-17 \atop 3} \right\rangle + \frac{1744}{243}\left\langle {n-12 \atop 4} \right\rangle + \frac{62\,632}{405}\left\langle {n-13 \atop 4} \right\rangle + \frac{7926\,182}{6075}\left\langle {n-14 \atop 4} \right\rangle$$

$$+ \frac{2168\,128}{405}\left\langle {n-15 \atop 4} \right\rangle + \frac{15\,121\,078}{1215}\left\langle {n-16 \atop 4} \right\rangle + \frac{1280\,048}{81}\left\langle {n-17 \atop 4} \right\rangle + \frac{354\,560}{27}\left\langle {n-18 \atop 4} \right\rangle$$

$$+ \frac{127}{27}\left\langle {n-14 \atop 5} \right\rangle + \frac{105\,872}{1215}\left\langle {n-15 \atop 5} \right\rangle + \frac{189\,107}{243}\left\langle {n-16 \atop 5} \right\rangle + \frac{2755\,265}{729}\left\langle {n-17 \atop 5} \right\rangle$$

$$+ \frac{565\,765}{81}\left\langle {n-18 \atop 5} \right\rangle + \frac{525\,920}{81}\left\langle {n-19 \atop 5} \right\rangle + \frac{3329}{1215}\left\langle {n-16 \atop 6} \right\rangle + \frac{12\,770}{243}\left\langle {n-17 \atop 6} \right\rangle$$

$$+ \frac{336\,044}{729}\left\langle {n-18 \atop 6} \right\rangle + \frac{303\,338}{243}\left\langle {n-19 \atop 6} \right\rangle + \frac{149\,140}{81}\left\langle {n-20 \atop 6} \right\rangle + \frac{77}{2187}\left\langle {n-17 \atop 7} \right\rangle$$

$$+ \frac{18\,487}{10\,935}\left\langle {n-18 \atop 7} \right\rangle + \frac{1168\,916}{32\,805}\left\langle {n-19 \atop 7} \right\rangle + \frac{471\,478}{2187}\left\langle {n-20 \atop 7} \right\rangle + \frac{362\,545}{729}\left\langle {n-21 \atop 7} \right\rangle$$

$$+ \frac{160}{6561}\left\langle {n-19 \atop 8} \right\rangle + \frac{45\,524}{32\,805}\left\langle {n-20 \atop 8} \right\rangle + \frac{44\,128}{2187}\left\langle {n-21 \atop 8} \right\rangle + \frac{175\,060}{2187}\left\langle {n-22 \atop 8} \right\rangle$$

$$+ \frac{28}{1215}\left\langle {n-21 \atop 9} \right\rangle + \frac{2471}{2187}\left\langle {n-22 \atop 9} \right\rangle + \frac{6574}{729}\left\langle {n-23 \atop 9} \right\rangle + \frac{2}{19\,683}\left\langle {n-22 \atop 10} \right\rangle$$

$$+ \frac{2030}{59\,049}\left\langle {n-23 \atop 10} \right\rangle + \frac{13\,850}{19\,683}\left\langle {n-24 \atop 10} \right\rangle + \frac{77}{177\,147}\left\langle {n-24 \atop 11} \right\rangle + \frac{2189}{59\,049}\left\langle {n-25 \atop 11} \right\rangle$$

$$+ \frac{74}{59\,049}\left\langle {n-26 \atop 12} \right\rangle + \frac{13}{531\,441}\left\langle {n-27 \atop 13} \right\rangle + \frac{1}{4782\,969}\left\langle {n-28 \atop 14} \right\rangle \Big),$$

with the first non-zero term at $n = 7$. The first non-zero terms are $2640, 34\,762\,240, 1\,529\,101\,440, 47\,107\,169\,280, \ldots$.

We can rearrange the sequence values generated by these formulae in two tables, which present the values indexed by d for rows, and n for columns. Table 4 gives the values for $n = 1, \ldots, 7$, and Table 5 for $n = 8, 9, 10$. Examination of the values in Table 4 show they match the values in Table II of Kendall and Babington Smith [9].

For some of the sequences, the values listed in Table 5 include extra values for higher n that were not listed in the individual sequence descriptions. Examination of Table 5 shows that the values match the values for the rows up to $d = 14$ in Table 2 in Alway [1].

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$d=0$	1	2	6	24	120	720	5040
$d=1$			2	16	120	960	8400
$d=2$				24	240	2240	21840
$d=3$					240	2880	33600
$d=4$					280	6240	75600
$d=5$					24	3648	90384
$d=6$						8640	179760
$d=7$						4800	188160
$d=8$						2640	277200
$d=9$							280560
$d=10$							384048
$d=11$							244160
$d=12$							233520
$d=13$							72240
$d=14$							2640

Table 4: Circular triad tournament counts for $n \leq 7$.

	$n=8$	$n=9$	$n=10$
$d=0$	40320	362880	3628800
$d=1$	80640	846720	9676800
$d=2$	228480	2580480	31449600
$d=3$	403200	5093760	68275200
$d=4$	954240	12579840	175392000
$d=5$	1304576	19958400	311592960
$d=6$	3042816	44698752	711728640
$d=7$	3870720	70785792	1193794560
$d=8$	6926080	130032000	2393475840
$d=9$	8332800	190834560	3784596480
$d=10$	15821568	361525248	7444104192
$d=11$	14755328	443931264	10526745600
$d=12$	24487680	779950080	19533696000
$d=13$	24514560	1043763840	27610168320
$d=14$	34762240	1529101440	47107169280

Table 5: Circular triad tournament counts for $n = 8, 9, 10$.

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2020 *Mathematics Subject Classification*: Primary 05C20; Secondary 05A15.
Keywords: exponential generating function, three-cycle, cyclic triple.

(Concerned with sequences A000142, A090672, A357242, A357248, A357257, and A357266.)

Received December 14 2023; revised versions received December 15 2023; July 26 2024; December 18 2024. Published in *Journal of Integer Sequences*, December 18 2024.

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