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On Integrality and Asymptotic Behavior of the (k, l)-Göbel Sequences

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Abstract

Recently, Matsuhira, Matsusaka, and Tsuchida revisited old studies of the integrality of k-Göbel sequences and showed that the first 19 terms are always integers for every integer $k \geq 2$. In this article, we further explore two topics: Ibstedt's (k, l)-Göbel sequences and Zagier's asymptotic formula for the 2-Göbel sequence, and extend their results.

1 Introduction

Sloane's collection of integer sequences [8] inspired Göbel to introduce a sequence defined by the recursion

$$g_n = \frac{1 + g_0^2 + g_1^2 + \cdots + g_{n-1}^2}{n},$$

starting with $g_0 = 1$. Despite the initial terms $(g_n)_n = (1, 2, 3, 5, 10, 28, 154, 3520, \ldots)$ appearing to follow an integer sequence pattern, the sequence's integrality was not immediately

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clear, raising doubts about its suitability for inclusion in Sloane's collection. In 1975, Lenstra resolved this problem by showing that

$$g_n \in \mathbb{Z} \Longleftrightarrow 0 \leq n \leq 42.$$

Unfortunately, Göbel's sequence is not an integer sequence. Nevertheless, it has since been registered as an intriguing exception in Sloane's collection [7, A003504]. (About its history, also see Matsuhira, Matsusaka, and Tsuchida [6]).

After that, several aspects of Göbel's sequence were investigated. In 1990, Ibstedt [4] focused on an alternative recursion:

$$(n+1)g_{n+1} = g_n(n + g_n)$$

with the initial value $g_1 = 2$, and introduced a generalization.

Definition 1. For integers $k, l \geq 2$, the (k,l)-*Göbel sequence* $g_{k,l}(n)$ is defined by the recursion

$$(n+1)g_{k,l}(n+1) = g_{k,l}(n)(n + g_{k,l}(n)^{k-1})$$

with the initial value $g_{k,l}(1) = l$.

We can pose the same question as Lenstra: when does its integrality break? To address the question, we introduce the notation

$$N_{k,l} \coloneqq \inf\{n \in \mathbb{Z}_{>0} : g_{k,l}(n) \notin \mathbb{Z}\} \tag{1}$$

for $k, l \geq 2$. Lenstra's result is stated as $N_{2,2} = 43$. Then Ibstedt provided a method to compute the values of $N_{k,l}$ and presented the list of $N_{k,l}$ for $2 \leq k, l \leq 11$ as follows. Here we extend the list to include cases where $2 \leq k, l \leq 17$. We provide Mathematica codes to compute $N_{k,l}$ in Appendix A.

In 1996, Zagier [10] considered the asymptotic behavior of Göbel's sequence and described it as

$$g_{2,2}(n) \sim C^{2^n} n \left(1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \cdots\right) \quad (n \to \infty) \tag{2}$$

without proof, where $C = 1.0478314475764112295599\ldots$ is a constant (A115632). Finch [2, Section 6.10] notes that this asymptotic formula shares the same coefficients (A116603) as that for the sequence (s_n) introduced by Somos. Here, the sequence (s_n) is defined by the recursion

$$s_n = n s_{n-1}^2 \tag{3}$$

starting with $s_0 = 1$ (A052129). It satisfies the relation

$$s_n \sim \sigma^{2^n} n^{-1} \left(1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \cdots\right)^{-1} \quad (n \to \infty),$$

$l\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	43	89	97	214	19	239	37	79	83	239	31	431	19	79	23	827
3	7	89	17	43	83	191	7	127	31	389	109	431	7	79	83	683
4	17	89	23	139	13	359	23	158	41	169	103	643	31	79	167	118
5	34	89	97	107	19	419	37	79	83	137	31	431	19	41	23	59
6	17	31	149	269	13	127	23	103	71	239	41	431	31	79	23	499
7	17	151	13	107	37	127	37	103	83	239	101	167	19	79	13	59
8	51	79	13	214	13	239	17	163	71	239	41	431	31	79	13	118
9	17	89	83	139	37	191	23	103	23	239	41	431	47	79	29	177
10	7	79	23	251	347	239	7	163	41	239	53	251	7	251	23	59
11	34	601	13	107	19	461	37	79	31	389	101	479	19	79	13	59
12	17	197	97	263	37	191	17	79	41	263	82	167	29	79	53	59
13	17	151	23	263	37	127	37	158	31	137	61	431	19	41	83	271
14	43	158	67	139	37	191	23	158	41	239	29	383	29	79	23	683
15	67	197	173	139	37	239	37	127	31	1097	82	431	31	419	23	347
16	59	151	157	107	59	359	37	103	46	137	29	431	29	79	23	607
17	7	89	67	43	13	127	7	179	41	263	31	431	7	79	59	59

Table 1: The list of $N_{k,l}$. As noted in OEIS (A097398), the articles of Ibstedt [4] and Guy [3, E15] contain some mistakes in the values.

where

$$\sigma := \prod_{n=1}^{\infty} n^{1/2^n} = 1.6616879496\ldots \tag{4}$$

is called the *Somos constant* (A112302).

After a period of silence, in 2023, inspired by the Japanese manga "Seisu-tan" [5], Matsuhira, Tsuchida, and the second author [6] addressed the problem of determining the minimum value of $N_{k,2}$ and showed that

$$\min_{k\geq 2} N_{k,2} = 19. \tag{5}$$

Once again, Göbel's sequence returned as a subject of research.

In this article, we combine the above results and extend them as suggested by the previous work [6] in the last remarks. First, we consider the minimum value of all $N_{k,l}$ and show the following:

Theorem 2. *We have* $\min_{k,l\geq 2} N_{k,l} = 7$*, which implies that* $g_{k,l}(n) \in \mathbb{Z}$ *for all integers* $k, l \geq 2$ *and* $1 \leq n \leq 6$*. Moreover, we have* $N_{k,l} = 7$ *if and only if* $k \equiv 2 \pmod 6$ *and* $l \equiv 3 \pmod 7$*.*

Secondly, we give a complete proof of Zagier's asymptotic formula (2) and generalize it for (k, l)-Göbel sequences. Before stating the theorem, we recall the definition of asymptotic expansions.

Definition 3. Assume that the sequence $(\lambda_r(n))_r$ satisfies $\lambda_{r+1}(n) = o(\lambda_r(n))$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \frac{\lambda_{r+1}(n)}{\lambda_r(n)} = 0$$

for every r. For a sequence $(c_n)_n$, we call $(a_r)_r$ its *asymptotic coefficients* and write

$$c_n \sim \sum_{r=0}^{\infty} a_r \lambda_r(n)$$

if

$$c_n - \sum_{r=0}^{R} a_r \lambda_r(n) = O(\lambda_{R+1}(n)) \quad (n \to \infty)$$

holds for every $R \geq 0$.

By adapting the sequence $\lambda_r(n) = C_{k,l}^{k^n} n^{\frac{1}{k-1}} n^{-r}$, we obtain the following asymptotic expansion.

Theorem 4. *For integers $k, l \geq 2$, there exist a constant $C_{k,l} > 1$ and a sequence $(a_{k,r})_r$ such that*

$$g_{k,l}(n) \sim C_{k,l}^{k^n} n^{\frac{1}{k-1}} \left(1 + \sum_{r=1}^{\infty} \frac{a_{k,r}}{n^r}\right) \quad (n \to \infty).$$

The constant $C_{k,l}$ and the sequence $(a_{k,r})_r$ is explicitly defined in Proposition 10 and Theorem 15, respectively.

In Section 2 and Section 3, we give proofs of the above theorems, respectively. In Section 4, we provide further observations on a variability of $g_{k,l}(n)$ modulo a higher power of p.

2 How long can (k, l)-Göbel sequences remain integers?

In this section, we provide a proof of Theorem 2, drawing on Ibstedt's method for computing $N_{k,l}$ and the argument presented by Matsuhira, Matsusaka, and Tsuchida [6]. First, we prepare some notation.

2.1 Notation and key properties

Let $\mathcal{P}$ be the set of all prime numbers. For each $p \in \mathcal{P}$, we let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at the prime ideal (p), that is, $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$. By the fact that

$$\bigcap_{p \in \mathcal{P}} \mathbb{Z}_{(p)} = \mathbb{Z}, \tag{6}$$

for $x \in \mathbb{Q}$, we have $x \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_{(p)}$ for all $p \in \mathcal{P}$. Let $\nu_p(x)$ be the p-adic valuation of $x \in \mathbb{Q}$. More precisely, for an integer n, $\nu_p(n)$ is the exponent of the largest power of p that divides n, and it is extended to rational numbers by $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$. The following lemma is the key for our proof of Theorem 2. For convenience, we also include the cases when $k = 1$ or $l = 1$, in which $g_{k,l}(n)$ is a constant sequence.

Lemma 5. *Let N be a positive integer. For each $p \in \mathcal{P}$, we put $r = \nu_p(N!)$. Let $k, k_1, k_2, l, l_1, l_2 \geq 1$ and $1 \leq n \leq N$ be integers. Then we have the following.*

(1) *If $p > N$, then $g_{k,l}(N) \in \mathbb{Z}_{(p)}$.*

(2) *If $g_{k,l}(n) \notin \mathbb{Z}_{(p)}$, then $g_{k,l}(n+1) \notin \mathbb{Z}_{(p)}$.*

(3) *Assume that $p \leq N$, that is, $r \geq 1$. If $k_1, k_2 \geq r$ and $k_1 \equiv k_2 \pmod{\varphi(p^r)}$, then $g_{k_1,l}(n) \in \mathbb{Z}_{(p)}$ if and only if $g_{k_2,l}(n) \in \mathbb{Z}_{(p)}$, where $\varphi(n)$ is the Euler totient function. Moreover, in this case, we have $g_{k_1,l}(n) - g_{k_2,l}(n) \in p^{r-\nu_p(n!)}\mathbb{Z}_{(p)}$.*

(4) *Assume that $p \leq N$. If $l_1 \equiv l_2 \pmod{p^r}$, then $g_{k,l_1}(n) \in \mathbb{Z}_{(p)}$ if and only if $g_{k,l_2}(n) \in \mathbb{Z}_{(p)}$. Moreover, in this case, we have $g_{k,l_1}(n) - g_{k,l_2}(n) \in p^{r-\nu_p(n!)}\mathbb{Z}_{(p)}$.*

Proof.

(1) Obvious from Definition 1.

(2) If $g_{k,l}(n) \notin \mathbb{Z}_{(p)}$, that is, $\nu_p(g_{k,l}(n)) < 0$, then we have

$$\begin{aligned}\nu_p(g_{k,l}(n+1)) &= \nu_p(g_{k,l}(n)) + \nu_p(n + g_{k,l}(n)^{k-1}) - \nu_p(n+1) \\ &= k\nu_p(g_{k,l}(n)) - \nu_p(n+1) < 0.\end{aligned}$$

(3) follows from Euler's theorem, (a generalization of Fermat's little theorem), and induction on n. For the initial case, we have $g_{k_1,l}(1) = l = g_{k_2,l}(1)$. Assume that the claim holds for some $1 \leq n < N$. If $g_{k_1,l}(n) \notin \mathbb{Z}_{(p)}$, then both of $g_{k_1,l}(n+1)$ and $g_{k_2,l}(n+1)$ are not in $\mathbb{Z}_{(p)}$ by the induction hypothesis and (2). On the other hand, if $g_{k_1,l}(n) \in \mathbb{Z}_{(p)}$, then

$$(n+1)\Big(g_{k_1,l}(n+1) - g_{k_2,l}(n+1)\Big) = n(g_{k_1,l}(n) - g_{k_2,l}(n)) + (g_{k_1,l}(n)^{k_1} - g_{k_2,l}(n)^{k_2}).$$

By the induction hypothesis, the first term is in $p^{r-\nu_p(n!)}\mathbb{Z}_{(p)}$. As for the second term, if $g_{k_1,l}(n) \notin p\mathbb{Z}_{(p)}$, then by applying Euler's theorem, it belongs to $p^{r-\nu_p(n!)}\mathbb{Z}_{(p)}$. If $g_{k_1,l}(n) \in p\mathbb{Z}_{(p)}$, then since $k_1, k_2 \geq r$, it is also in $p^{r-\nu_p(n!)}\mathbb{Z}_{(p)}$. Therefore, by dividing the both sides by $(n+1)$, we get $g_{k_1,l}(n+1) - g_{k_2,l}(n+1) \in p^{r-\nu_p((n+1)!)}\mathbb{Z}_{(p)}$. In particular, if $g_{k_1,l}(n+1) \notin \mathbb{Z}_{(p)}$, then $g_{k_2,l}(n+1) \notin \mathbb{Z}_{(p)}$, and vice versa.

(4) Since $g_{k,l_1}(n)$ and $g_{k,l_2}(n)$ satisfy the same recursion with the same initial value modulo p^r, the claim immediately follows in a similar manner to (3).

□

2.2 Proof of Theorem 2

We have to check the following two claims.

1. For integers $k, l \geq 2$ and $1 \leq n \leq 6$, $g_{k,l}(n) \in \mathbb{Z}$.
2. $g_{k,l}(7) \notin \mathbb{Z}$ if and only if $k \equiv 2 \pmod 6$ and $l \equiv 3 \pmod 7$.

By applying Lemma 5 with $N = 7$ and combining it with (6), these claims can be translated as follows.

Lemma 6. *Theorem 2 is equivalent to the following claims for $p = 2, 3, 5, 7$.*

1. *For $2 \leq k \leq 11$ and $1 \leq l \leq 16$, we have $g_{k,l}(7) \in \mathbb{Z}_{(2)}$.*
2. *For $2 \leq k \leq 7$ and $1 \leq l \leq 9$, we have $g_{k,l}(7) \in \mathbb{Z}_{(3)}$.*
3. *For $1 \leq k \leq 4$ and $1 \leq l \leq 5$, we have $g_{k,l}(7) \in \mathbb{Z}_{(5)}$.*
4. *For $1 \leq k \leq 6$ and $1 \leq l \leq 7$, $g_{k,l}(7) \notin \mathbb{Z}_{(7)}$ if and only if $k = 2$ and $l = 3$.*

Proof. Since it is obvious that Theorem 2 implies the claims, we now show the converse implication. By Lemma 5 (1), $g_{k,l}(7) \in \mathbb{Z}_{(p)}$ for every $p > 7$. Since the discussion remains similar for the remaining cases of $p = 2, 3, 5$, and 7, let us focus here on explaining the case when $p = 2$. By Lemma 5 (2), $g_{k,l}(7) \in \mathbb{Z}_{(2)}$ if and only if $g_{k,l}(n) \in \mathbb{Z}_{(2)}$ for all $1 \leq n \leq 7$. By the periodicity shown in Lemma 5 (3) and (4), it is enough to show that $g_{k,l}(7) \in \mathbb{Z}_{(2)}$ for $k = 2, 3$, and $4 \leq k < 4 + \varphi(2^4)$, and $1 \leq l \leq 2^4$, where we note that $\nu_2(7!) = 4$. This is the claim for $p = 2$. □

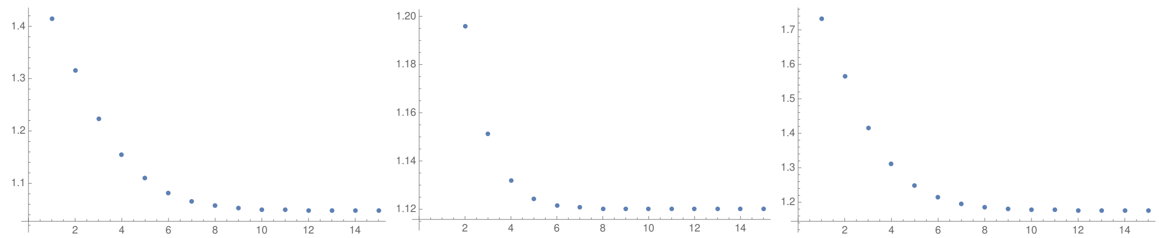
Proof of Theorem 2. Since there are only a finite (and relatively small) number of cases to consider, it can be checked by using Mathematica. The codes to compute them are available in Appendix A. Alternatively, it is enough (and possible) to check that $g_{k,l}(7) \in \mathbb{Z}$ if and only if $(k, l) \neq (2, 3), (2, 10), (8, 3), (8, 10)$ for $1 \leq k \leq 11$ and $1 \leq l \leq 16$ because $g_{k,l}(7)$ is not too large in these cases. □

3 Zagier's asymptotic formula and its generalization

In this section, we prove Theorem 4 by defining the constant $C_{k,l}$ and the asymptotic coefficients $a_{k,r}$ explicitly.

3.1 The constant $C_{k,l}$

We first show the monotonic behavior of $C_{k,l}(n) \coloneqq g_{k,l}(n)^{1/k^n}$.



$(k,l) = (2,2)$ $(k,l) = (3,2)$ $(k,l) = (2,3)$

Figure 1: The plots of $C_{k,l}(n)$ for $1 \le n \le 15$.

Lemma 7. *For integers $k, l \ge 2$ and $n \ge 1$, we have $C_{k,l}(n) > C_{k,l}(n+1) > 1$.*

Proof. First, we check that $C_{k,l}(n) > 1$, that is, $g_{k,l}(n) > 1$ by induction on n. The initial condition is satisfied by $g_{k,l}(1) = l \ge 2$. Assume that $g_{k,l}(n) > 1$. Then we have

$$g_{k,l}(n+1) = \frac{1}{n+1}\left(n g_{k,l}(n) + g_{k,l}(n)^k\right) > \frac{n+1}{n+1} = 1.$$

Next, we check for monotonicity. Since $g_{k,l}(n) > 1$, we have $g_{k,l}(n)^{k-1} > 1$, and then

$$\frac{1}{n+1}\left(\frac{n}{g_{k,l}(n)^{k-1}} + 1\right) < 1.$$

Hence, we have

$$\left(\frac{C_{k,l}(n+1)}{C_{k,l}(n)}\right)^{k^{n+1}} = \frac{g_{k,l}(n+1)}{g_{k,l}(n)^k} = \frac{1}{n+1}\frac{n g_{k,l}(n) + g_{k,l}(n)^k}{g_{k,l}(n)^k} < 1, \tag{7}$$

which concludes the proof. □

The above lemma shows that $C_{k,l}(n)$ converges. We denote the limit as

$$C_{k,l} \coloneqq \lim_{n\to\infty} C_{k,l}(n) \ge 1. \tag{8}$$

Next, we introduce some notation to show the strict inequality $C_{k,l} > 1$. For $k \geq 2$, we define the k-*Somos constant* σ_k by

$$\sigma_k \coloneqq \exp\left(\sum_{m=1}^{\infty} \frac{\log m}{k^m}\right) > 1, \tag{9}$$

(see also Sondow-Hadjicostas's article [9] and A123852). For a real number $t_0 > 1$, we define a Somos-like sequence $t_k(n)$ by the recursion:

$$t_k(n+1) = \frac{1}{n+1} t_k(n)^k \tag{10}$$

with the initial value $t_k(0) = t_0$.

Lemma 8. *If $t_0 > \sigma_k$, then there exists a constant $c > 1$ such that $t_k(n)^{1/k^n}$ decreases monotonically and tends to c as $n \to \infty$.*

Proof. It is equivalent to show that $a_k(n) \coloneqq k^{-n} \log t_k(n)$ decreases and tends to a positive constant. Since $a_k(n)$ satisfies the recursion

$$a_k(n) = a_k(n-1) - \frac{\log n}{k^n}$$

for $n \geq 1$, the sequence decreases monotonically. By our assumption, we have $a_k(0) > \log \sigma_k$, which implies that

$$\lim_{n\to\infty} a_k(n) = a_k(0) - \sum_{n=1}^{\infty} \frac{\log n}{k^n} > 0.$$

□

We can estimate (k,l)-Göbel sequences $g_{k,l}(n)$ by using the sequence $t_k(n)$ as follows.

Lemma 9. *If $l \geq t_0^k$, then $g_{k,l}(n) \geq t_k(n)$ for all $n \geq 1$.*

Proof. It immediately follows from induction on n. Indeed, we have $g_{k,l}(1) = l \geq t_0^k = t_k(1)$. Assume that $g_{k,l}(n) \geq t_k(n)$. Then we obtain

$$g_{k,l}(n+1) > \frac{1}{n+1} g_{k,l}(n)^k \geq \frac{1}{n+1} t_k(n)^k = t_k(n+1).$$

□

Proposition 10. *For integers $k, l \geq 2$, the constant $C_{k,l}$ defined in (8) satisfies $C_{k,l} > 1$.*

Proof. We prove the claim by considering three cases.

(1) For $k \geq 3$ and $l \geq 2$, since $\sigma_3^3 = 1.5462\ldots$ and σ_k^k decreases monotonically, there exists a real number $t_0 > 1$ such that $l \geq t_0^k > \sigma_k^k$. For such a t_0, by applying Lemma 8 and Lemma 9, we obtain

$$C_{k,l}(n) = g_{k,l}(n)^{1/k^n} \geq t_k(n)^{1/k^n} \geq c > 1,$$

which implies that $C_{k,l} > 1$.

(2) For $k = 2$ and $l \geq 3$, since $\sigma_2^2 = 2.7612\ldots$, there exists a real number $t_0 > 1$ such that $l \geq t_0^2 > \sigma_2^2$. By the same argument, we also obtain $C_{k,l} > 1$.

(3) For $k = l = 2$, we need to modify the argument. We define another sequence $t_2'(n)$ by the same recursion as in (10) with the initial value $t_2'(3) = 5$. Then the inequality $g_{2,2}(n) \geq t_2'(n)$ for $n \geq 3$ is shown in a manner similar to Lemma 9. Thus, we obtain

$$C_{2,2}(n) = g_{2,2}(n)^{1/2^n} \geq t_2'(n)^{1/2^n}$$

and

$$\lim_{n\to\infty} \frac{1}{2^n} \log t_2'(n) = \frac{1}{2^3} \log 5 - \sum_{n=4}^{\infty} \frac{\log n}{2^n} = 0.00395\ldots > 0.$$

Hence, we conclude that $C_{2,2} > 1$. □

3.2 Asymptotic behavior

In the previous subsection, we defined the constant $C_{k,l} > 1$. By definition, we obtain

$$\frac{g_{k,l}(n)}{C_{k,l}^{k^n}} = \left(\frac{C_{k,l}(n)}{C_{k,l}}\right)^{k^n}. \tag{11}$$

Thus, it is sufficient to evaluate the right-hand side to prove Theorem 4. The aim of this subsection is to establish a connection to a simpler sequence.

Theorem 11. *For every real number $R > 0$, we have*

$$\exp\left(\sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m}\right) - \left(\frac{C_{k,l}(n)}{C_{k,l}}\right)^{k^n} = O\left(\frac{1}{n^R}\right)$$

as $n \to \infty$.

First, we prepare a lemma with the aim of proving this theorem. Let

$$\epsilon_{k,l}(n) \coloneqq \sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m} - k^n(\log C_{k,l}(n) - \log C_{k,l}).$$

Lemma 12. *We have*

$$\epsilon_{k,l}(n) = \sum_{m=1}^{\infty} \frac{1}{k^m} \log\left(1 + \frac{m+n-1}{g_{k,l}(m+n-1)^{k-1}}\right) \le \frac{2n}{C_{k,l}^{(k-1)k^n}}.$$

Proof. By (7), we have

$$\log C_{k,l}(m+n) - \log C_{k,l}(m+n-1) = \frac{1}{k^{m+n}} \log\left(\frac{m+n-1}{g_{k,l}(m+n-1)^{k-1}} + 1\right) - \frac{\log(m+n)}{k^{m+n}}.$$

By summing each side over m, we obtain

$$k^n(\log C_{k,l} - \log C_{k,l}(n)) = \sum_{m=1}^{\infty} \frac{1}{k^m} \log\left(\frac{m+n-1}{g_{k,l}(m+n-1)^{k-1}} + 1\right) - \sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m},$$

which implies the first equality.

Next, the inequality $\log(1+x) < x$ for $x > 0$ implies that

$$\sum_{m=1}^{\infty} \frac{1}{k^m} \log\left(1 + \frac{m+n-1}{g_{k,l}(m+n-1)^{k-1}}\right) < \sum_{m=1}^{\infty} \frac{1}{k^m} \frac{m+n-1}{g_{k,l}(m+n-1)^{k-1}}.$$

Since $C_{k,l}(n) \ge C_{k,l} > 1$ and $g_{k,l}(n) = C_{k,l}(n)^{k^n}$, we have

$$g_{k,l}(m+n-1)^{k-1} = C_{k,l}(m+n-1)^{(k-1)k^{m+n-1}} \ge C_{k,l}^{(k-1)k^n}.$$

Therefore, by using $m+n-1 \le mn$ for $m, n \ge 1$, we obtain

$$\epsilon_{k,l}(n) < \frac{n}{C_{k,l}^{(k-1)k^n}} \sum_{m=1}^{\infty} \frac{m}{k^m} = \frac{n}{C_{k,l}^{(k-1)k^n}} \frac{k}{(k-1)^2} \le \frac{2n}{C_{k,l}^{(k-1)k^n}}$$

for $k \ge 2$. □

Proof of Theorem 11. By using the expression

$$\left(\frac{C_{k,l}(n)}{C_{k,l}}\right)^{k^n} = \exp\left(\sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m} - \epsilon_{k,l}(n)\right)$$

and the inequality $e^x - e^{x-\epsilon} \le \epsilon e^x$, we have

$$\exp\left(\sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m}\right) - \left(\frac{C_{k,l}(n)}{C_{k,l}}\right)^{k^n} \le \epsilon_{k,l}(n) \exp\left(\sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m}\right).$$

Moreover, by applying $m+n \le m(n+1)$ for $m, n \ge 1$, it is bounded by

$$\le \epsilon_{k,l}(n)\sigma_k \cdot \exp\left(\sum_{m=1}^{\infty} \frac{\log(n+1)}{k^m}\right) = \epsilon_{k,l}(n)\sigma_k \cdot (n+1)^{\frac{1}{k-1}},$$

where σ_k is the k-Somos constant defined in (9). Finally, by Lemma 12, we obtain the theorem. □

3.3 The asymptotic coefficients $a_{k,r}$

Finally, to complete the statement of Theorem 4 and its proof, we provide the asymptotic expansion of the first term of Theorem 11. First, we review the relevant parts of the studies by Sondow and Hadjicostas [9] regarding a generalization of Somos's sequence introduced in (3). Then we establish a connection between their results and our (k,l)-Göbel sequences. To state their claims explicitly, we recall the Eulerian polynomials.

Definition 13. For any integer $r \geq 0$ and $t > 1$, there exists a polynomial $A_r(t)$ such that

$$\sum_{m=1}^{\infty} \frac{m^r}{t^m} = \frac{A_r(t)}{(t-1)^{r+1}}.$$

We call the polynomial $A_r(t)$ the *Eulerian polynomial*.

Example 14. The first few examples are given by $A_0(t) = 1$ and

$$\begin{aligned}
A_1(t) &= t, \\
A_2(t) &= t^2 + t, \\
A_3(t) &= t^3 + 4t^2 + t, \\
A_4(t) &= t^4 + 11t^3 + 11t^2 + t, \\
A_5(t) &= t^5 + 26t^4 + 66t^3 + 26t^2 + t.
\end{aligned}$$

For more examples, see A008292.

Then the following is known.

Theorem 15 ([9, Theorem 9 and Lemma 1])**.** *For an integer $k \geq 2$, we define the sequence $(a_{k,r})_r$ to be*

$$a_{k,r} \coloneqq \sum_{\substack{m_1,\dots,m_r \geq 0 \\ m_1+2m_2+\cdots+rm_r=r}} \prod_{j=1}^{r} \frac{1}{m_j!} \left(\frac{(-1)^{j-1}}{j} \frac{A_j(k)}{(k-1)^{j+1}} \right)^{m_j}. \tag{12}$$

Then we have

$$\exp\left(\sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m} \right) \sim n^{\frac{1}{k-1}} \left(1 + \sum_{r=1}^{\infty} \frac{a_{k,r}}{n^r} \right) \quad (n \to \infty).$$

Example 16. The first several terms are calculated as follows.

$$
\begin{aligned}
a_{k,1} &= \frac{k}{(k-1)^2},\\
a_{k,2} &= -\frac{k(k^2-k-1)}{2(k-1)^4},\\
a_{k,3} &= \frac{k(2k^4+k^3-11k^2+7k+2)}{6(k-1)^6},\\
a_{k,4} &= -\frac{k(6k^6+37k^5-124k^4+53k^3+92k^2-59k-6)}{24(k-1)^8},\\
a_{k,5} &= \frac{k(24k^8+478k^7-1013k^6-1324k^5+4411k^4-2724k^3-453k^2+578k+24)}{120(k-1)^{10}}.
\end{aligned}
$$

In particular, when $k = 2$, we observe that $(a_{2,r})_{r=1}^{5} = (2, -1, 4, -21, 138)$, which matches the asymptotic coefficients in (2).

Proof of Theorem 4. We show that

$$\frac{g_{k,l}(n)}{C_{k,l}^{k^n}} - n^{\frac{1}{k-1}}\left(1 + \sum_{r=1}^{R} \frac{a_{k,r}}{n^r}\right) = O\left(\frac{n^{\frac{1}{k-1}}}{n^{R+1}}\right) \quad (n \to \infty)$$

for every $R \geq 0$. It immediately follows by applying (11), Theorem 11, and Theorem 15. □

Remark 17. Let $G_{k,l}(n) \coloneqq \log g_{k,l}(n)$ denote the logarithm of the (k,l)-Göbel sequence. The recurrence relation in Definition 1 can be rewritten as

$$G_{n+1} = kG_n + F_n,$$

where we define

$$F_n \coloneqq \log\left(\frac{1}{n+1}\left(1 + \frac{n}{g_{k,l}(n)^{k-1}}\right)\right).$$

As the reviewer noted, this recursion closely resembles the one considered by Aho and Sloane [1]. Likewise, the sequence

$$H_n \coloneqq \sum_{m=1}^{\infty} \frac{\log(m+n)}{k^m}$$

discussed in Theorem 11 also satisfies a similar recurrence relation:

$$H_{n+1} = kH_n - \log(n+1).$$

Aho and Sloane analyzed the asymptotic behavior of sequences defined by recurrence relations of this form in the case of $k = 2$. Our analysis in Section 3 can be seen as following their approach, extending it to provide a rigorous treatment for the (k,l)-Göbel sequences.

4 Further observations

Zagier [10] observed not only the asymptotic formula but also a heuristic explaining why $N_{2,2} = 43$ is unexpectedly large, assuming a certain "randomness" of the values $g_{2,2}(n)$ modulo p for $1 \le n < p$. Inspired by his heuristic argument, we can ask the question: for any pair of integers $k, l \ge 2$, does there exist (infinitely many) $p \in \mathcal{P}$ such that $g_{k,l}(p) \notin \mathbb{Z}_{(p)}$. For instance, $g_{2,2}(p) \notin \mathbb{Z}_{(p)}$ holds when $p = 43, 61, 67, 83, \ldots$. We do not have an answer to this question, but we obtained a result concerning "randomness", which we present as a final remark.

Theorem 18. *For any prime number $p \in \mathcal{P}$ and integers $k, l, r \ge 2$, the set*

$$G^r_{k,l,p} := \{g_{k,l}(n) \bmod p^r : 1 \le n < p, g_{k,l}(n) \equiv 0 \pmod{p^{r-1}}\}$$

is a singleton $\{0 \bmod p^r\}$ or has the same cardinality as $\{1 \le n < p : g_{k,l}(n) \equiv 0 \pmod{p^{r-1}}\}$. Here the congruence is considered in $\mathbb{Z}_{(p)}$.

Example 19. Let $r = 2$. For $(k, l) = (4, 4)$ and $p = 13$, we see that $g_{4,4}(1) \not\equiv 0 \pmod{13}$ and $g_{4,4}(n) \equiv 0 \pmod{13}$ for $2 \le n \le 12$. Moreover, we can check that all entries of

$$(g_{4,4}(n) \bmod 13^2)_{2 \le n \le 12} = (130, 143, 65, 52, 156, 13, 117, 104, 26, 39, 78)$$

are distinct from each other.

On the other hand, for $(k, l) = (3, 2)$ and $p = 13$, we see that $g_{3,2}(n) \not\equiv 0 \pmod{13}$ for $1 \le n \le 3$ and $g_{3,2}(n) \equiv 0 \pmod{13^2}$ for $4 \le n \le 12$. Thus, $G^2_{3,2,13} = \{0 \bmod 13^2\}$ is a singleton.

To prove the theorem, we first show a lemma.

Lemma 20. *Let $k, l, r \ge 2$ be integers and $p \in \mathcal{P}$. Assume that there exists $1 \le a < p$ and $0 \le b < p$ such that $g_{k,l}(a) \equiv bp^{r-1} \pmod{p^r}$. Then, for every $a \le n < p$, we have*

$$ng_{k,l}(n) \equiv abp^{r-1} \pmod{p^r}.$$

Proof. It follows from induction on n. The first condition $ag_{k,l}(a) \equiv abp^{r-1} \pmod{p^r}$ is clearly satisfied by our assumption. Assume that $ng_{k,l}(n) \equiv abp^{r-1} \pmod{p^r}$ for some $a \le n < p - 1$. By definition,

$$(n+1)g_{k,l}(n+1) = g_{k,l}(n)^k + ng_{k,l}(n) \equiv (n^{-1}ab)^k p^{k(r-1)} + abp^{r-1} \pmod{p^r},$$

which implies that $(n+1)g_{k,l}(n+1) \equiv abp^{r-1} \pmod{p^r}$ because $k(r-1) \ge r$. □

Proof of Theorem 18. It is sufficient to consider the case where $I := \{1 \le n < p : g_{k,l}(n) \equiv 0 \pmod{p^{r-1}}\} \ne \emptyset$. Then we take $a := \min I$ and $0 \le b < p$ satisfying $g_{k,l}(a) \equiv bp^{r-1} \pmod{p^r}$. If $b = 0$, then Lemma 20 tells us that $g_{k,l}(n) \equiv 0 \pmod{p^r}$ for all $n \in I$, that is, $G^r_{k,l,p} = \{0 \bmod p^r\}$. We now assume that $b \neq 0$. If there exist $n_1, n_2 \in I$ satisfying $g_{k,l}(n_1) \equiv g_{k,l}(n_2) \pmod{p^r}$, Lemma 20 implies that

$$n_1 g_{k,l}(n_1) \equiv abp^{r-1} \equiv n_2 g_{k,l}(n_2) \pmod{p^r},$$

and then $n_1 = n_2$. □

As a further observation, Table 1 suggests a tendency for $N_{k,l}$ to increase when k is prime. However, elucidating this phenomenon is a subject for future investigation.

A Methods for Computing $N_{k,l}$

In this appendix, we provide a method to compute $N_{k,l}$ and the Mathematica codes we used to generate Table 1. First, we recall the following sequence $g_{k,l,p,r}(n)$ introduced in [6].

Definition 21. Let $k, l \geq 2, r \geq 1$ be integers, and p a prime. We define $b(n) = b_{p,r}(n) = r - \nu_p(n!)$ and use the symbol F to represent "False". For any positive integer n with $\nu_p(n!) \leq r$, we define $g_{k,l,p,r}(n) \in \mathbb{Z}/p^{b(n)}\mathbb{Z} \cup \{\mathsf{F}\}$ as follows.

- Initial condition: $g_{k,l,p,r}(1) = l \bmod p^r \in \mathbb{Z}/p^r\mathbb{Z}$.
- For $n \geq 2$: When $g_{k,l,p,r}(n-1) = \mathsf{F}$, $g_{k,l,p,r}(n) = \mathsf{F}$.
- For $n \geq 2$: When $g_{k,l,p,r}(n-1) = a \bmod p^{b(n-1)}$,
  - if $a(n-1+a^{k-1}) \not\equiv 0 \pmod{p^{\nu_p(n)}}$, then $g_{k,l,p,r}(n) = \mathsf{F}$.
  - if $a(n-1+a^{k-1}) \equiv 0 \pmod{p^{\nu_p(n)}}$, then letting $c \in \mathbb{Z}$ such that $c \cdot (n/p^{\nu_p(n)}) \equiv 1 \pmod{p^{b(n)}}$, we define

$$g_{k,l,p,r}(n) = \frac{a(n-1+a^{k-1})}{p^{\nu_p(n)}} \cdot c \bmod p^{b(n)}.$$

The recursion above, defining $g_{k,l,p,r}(n)$, is a translation of Definition 1 modulo a power of p. Since $g_{k,l}(n) \notin \mathbb{Z}_{(p)}$ if and only if $g_{k,l,p,\nu_p(n!)}(n) = \mathsf{F}$, the number $N_{k,l}$ can be expressed as

$$N_{k,l} = \inf\{n \in \mathbb{Z}_{>0} : \text{there exists } p \in \mathcal{P}_{\leq n} \text{ such that } g_{k,l,p,\nu_p(n!)}(n) = \mathsf{F}\}.$$

The following code implements this argument in Mathematica.

```
nu[p_, n_] := FirstCase[FactorInteger[n], {p, r_} -> r, 0];
inv[n_, M_] := If[M == 1, 1, ModularInverse[n, M]];

g[k_, l_, p_, r_, 1] := {Mod[l, p^r], r};
g[k_, l_, p_, r_, n_] :=
```

```
  g[k, l, p, r, n] =
   Module[{a, b},
    If[g[k, l, p, r, n - 1] === "F",
     "F", {a, b} = g[k, l, p, r, n - 1];
     If[Mod[a (n - 1 + a^(k - 1)), p^nu[p, n]] != 0,
      "F", {Mod[
        a (n - 1 + a^(k - 1))/p^nu[p, n] inv[n/p^nu[p, n],
          p^(b - nu[p, n])], p^(b - nu[p, n])], b - nu[p, n]}]]];

NN[k_, l_] := Module[{n}, n = 2;
   While[
    FreeQ[Table[
      g[k, l, Prime[m], nu[Prime[m], n!], n], {m, 1, PrimePi[n]}],
     "F"], n++]; n];
```

Here, the output `g[k,l,p,r,n] = {a, b}` means that $g_{k,l,p,r}(n) = a \bmod p^b$ with $b = b(n)$, that is, $g_{k,l}(n) \equiv a \pmod{p^b}$, and `NN[k,l]` gives the value of $N_{k,l}$.

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