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Sums over Primes II

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Abstract

In this paper, we give explicit asymptotic formulas for some sums over primes involving three types of generalized alternating hyperharmonic numbers. We also consider analogous results for numbers with k prime factors.

1 Introduction and preliminaries

The prime numbers (see the sequence A000040 in the OEIS [16]) play an essential role in number theory. Let $\pi(x)$ denote the number of primes up to x. Gauss and Legendre proposed independently that the ratio $\pi(x)/\frac{x}{\log x}$ approaches 1 as x approaches ∞. With the help of analytic tools, Hadamard [5] and de la Vallée Poussin [17] independently and almost simultaneously proved the prime number theorem, i.e.,

$$\pi(x) \sim \frac{x}{\log x}.$$

Let p_n be the n-th prime number, and let α be a non-negative integer. It is natural to consider asymptotic formulas for more general sums of type $\sum_{p_n \leq x} p_n^{\alpha}$. We restate the prime number theorem as

$$\pi(x) = \sum_{p_n \leq x} p_n^0 \sim \frac{x}{\log x}.$$

An exercise in Granville's book [4] states that $\sum_{p \leq x} p \sim \frac{x^2}{2 \log x}$. In fact, Šalát and Znám [15] proved more general sums $\sum_{p_n \leq x} p_n^{\alpha} \sim \frac{x^{1+\alpha}}{(1+\alpha) \log x}$. Later, Jakimczuk [7, 8] extended this kind of summation to numbers with k prime factors and functions of slow increase. Gerard and Washington [3] also gave accurate estimates for $\sum_{p_n \le x} p_n^{\alpha} - \frac{x^{1+\alpha}}{(1+\alpha)\log x}$ by using the prime number theorem with error terms.

We now recall some definitions and notation. Let $k \ge 1$, and let n be the product of just k prime factors (p_i and p_j are allowed to be the same), i.e.,

$$n = p_1 p_2 \cdots p_k. \tag{1}$$

We write $\tau_k(x)$ for the number of such $n \le x$. If we impose the additional restriction that all the prime divisors p in equation(1) are different, n is squarefree. We write $\pi_k(x)$ for the number of these (squarefree) $n \le x$. Landau [6, 9] proved that

$$\pi_k(x) \sim \tau_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \quad (k \ge 2).$$

For $k = 1$, this result reduces to the prime number theorem, if, as usual, we take $0! = 1$.

Conway and Guy [1] introduced the conception of hyperharmonic numbers as

$$h_n^{(r)} := \sum_{j=1}^{n} h_j^{(r-1)} \quad (n, r \in \mathbb{N} := \{1, 2, 3, \ldots\}) \quad \text{with} \quad h_n^{(1)} = H_n := \sum_{j=1}^{n} 1/j.$$

Dil, Mező, and Cenkci [2] introduced the notion of generalized hyperharmonic numbers as

$$H_n^{(p,r)} := \sum_{j=1}^{n} H_j^{(p,r-1)} \quad (n, p, r \in \mathbb{N}),$$

and studied the Euler sums of hyperharmonic numbers. Ömür and Koparal [14] introduced the generalized hyperharmonic numbers $H_n^{(p,r)}$ independently and almost simultaneously from a combinatorial point of view, and defined two $n \times n$ matrices A_n and B_n with $a_{i,j} = H_i^{(j,r)}$ and $b_{i,j} = H_i^{(p,j)}$, respectively. Ömür and Koparal also gave some interesting factorizations and determinant properties of the matrices A_n and B_n. The author [12] proved that the generalized hyperharmonic numbers $H_n^{(p,r)}$ are linear combinations of n's power times generalized harmonic numbers.

The author [10] introduced the conception of generalized alternating hyperharmonic numbers $H_n^{(p,r)}$. Define the notion of the generalized alternating hyperharmonic numbers of types I, II, and III, respectively, as

$$H_n^{(p,r,1)} := \sum_{k=1}^{n} (-1)^{k-1} H_k^{(p,r-1,1)} \quad (H_n^{(p,1,1)} = H_n^{(p)}),$$

$$H_n^{(p,r,2)} := \sum_{k=1}^{n} H_k^{(p,r-1,2)} \quad (H_n^{(p,1,2)} = \overline{H}_n^{(p)} := \sum_{j=1}^{n} (-1)^{j-1}/j^p),$$

$$H_n^{(p,r,3)} := \sum_{k=1}^{n} (-1)^{k-1} H_k^{(p,r-1,3)} \quad (H_n^{(p,1,3)} = \overline{H}_n^{(p)}).$$

Let $\mathbb{N}_0$ denote the set of nonnegative integers. If $p \in \mathbb{N}_0$, then $H_n^{(-p)}$ and $\overline{H}_n^{(-p)}$ are the sum $\sum_{j=1}^{n} j^p$ and $\sum_{j=1}^{n}(-1)^{j-1}j^p$, respectively. The author [10] proved that Euler sums of the generalized alternating hyperharmonic numbers of types I, II, and III are linear combinations of classical (alternating) Euler sums.

Let $f(n)$ denote an arithmetical function. It is interesting to consider asymptotic formulas for sums over primes of type $\sum_{p_n \leq x} p_n^{\alpha} f(n)^m$. The author [11] gave explicit asymptotic formulas for sums over primes involving generalized hyperharmonic numbers of type $\sum_{p_n \leq x} p_n^{\alpha}(H_n^{(p,r)})^m$. The author [11] also considered analogous results for numbers with k prime factors.

The motivation of this paper arises from an exercise in Granville's book [4] and the author's recent work [10] on generalized alternating hyperharmonic numbers. This paper is a continuation of the previous paper of the author with the same title [11]. In this paper, we derive explicit asymptotic formulas for some sums over primes involving three types of generalized alternating hyperharmonic numbers. We also consider analogous results for numbers with k prime factors.

2 Some notation and lemmas

We now recall some notation and lemmas.

Lemma 1 ([13])**.** *For all $n \in \mathbb{N}$ and a fixed order $r \geq 1$, we have*

$$h_n^{(r)} \sim \frac{1}{(r-1)!} n^{r-1} \log(n).$$

Lemma 2 ([11])**.** *For $r, n, p \in \mathbb{N}$ with $p \geq 2$, we have*

$$H_n^{(p,r)} \sim \frac{1}{(r-1)!} n^{r-1} \zeta(p),$$

where $\zeta(p) := \sum_{n=1}^{\infty} n^{-p}$ is the Riemann zeta function.

Lemma 3 ([12])**.** *For $r, n, p \in \mathbb{N}$, we have*

$$H_n^{(p,r,2)} = \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r,m,j) n^j \overline{H}_n^{(p-m)}.$$

The coefficients $a(r,m,j)$ satisfy the following recurrence formulas:

$$a(r+1,r,0) = -\sum_{m=0}^{r-1} a(r,m,r-m-1) \frac{1}{r-m},$$

$$a(r+1,m,\ell) = \sum_{j=\ell-1}^{r-1-m} \frac{a(r,m,j)}{j+1} \binom{j+1}{j-\ell+1} B_{j-\ell+1}$$

$$(0 \leq m \leq r-1, 1 \leq \ell \leq r-m),$$

$$a(r+1,m,0) = -\sum_{y=0}^{m} \sum_{j=\max\{0,m-y-1\}}^{r-1-y} a(r,y,j)D(r,m,j,y) \quad (0 \le m \le r-1),$$

where

$$D(r,m,j,y) = \sum_{\ell=\max\{0,m-y-1\}}^{j} \frac{1}{j+1}\binom{j+1}{j-\ell} B_{j-\ell}\binom{\ell+1}{m-y}(-1)^{1+\ell-m+y}.$$

The Bernoulli numbers B_n satisfy the following recurrence formula

$$\sum_{j=0}^{k} \binom{k+1}{j} B_j = k+1 \quad (k \ge 0).$$

The initial value is $a(1,0,0) = 1$.

Definition 4. For $m, j \in \mathbb{N}_0$, define the quantities $c(m,j)$, $d(m,j)$, $c_1(m,j)$, and $d_1(m,j)$ as

$$c(m,j) = \frac{1}{m+1}\binom{m+1}{m+1-j} B_{m+1-j},$$

$$d(m,j) = \frac{1}{m+1}\sum_{k=j-1}^{m}\binom{m+1}{m-k} B_{m-k}\binom{1+k}{j}(-1)^{1+k-j},$$

$$c_1(m,j) = \frac{1}{2(m+1)}\sum_{k=0}^{m-j}\binom{m+1}{k} B_k 2^k \binom{m+1-k}{j}(-1)^{m-k-j},$$

$$d_1(m,j) = \sum_{k=j}^{m}\binom{k}{j}(-1)^{k-j} c_1(m,k).$$

Definition 5. Let $g(r) := (2r - (-1)^r - 3)/4$. For $r \in \mathbb{N}$, define the boundary values of the quantities $b_1(r,m,j,k)$, $k = 0,1,2,3$ as

- $b_1(1,0,0,2) = 1, \quad b_1(1,0,0,3) = 0;$
- $b_1(r,m,j,0) = b_1(r,m,j,1) = 0 \quad$ (r odd);
- $b_1(r,m,j,2) = b_1(r,m,j,3) = 0 \quad$ (r even);
- $b_1(r,m,j,3) = 0 \quad$ (r odd, $\quad m + j = g(r)$).

For $k = 0,1,2,3$, the quantities $b_1(r,m,j,k)$ satisfy the following recurrence formulas:

When r is odd,

- $b_1(r+1,m,j,0) = \sum_{\ell=m}^{g(r)} b_1(r,\ell,j,2)c_1(\ell,m) \quad (1 \le m \le g(r), \quad 0 \le j \le g(r) - m);$

- $b_1(r+1,0,j,0) = \sum_{\ell=0}^{g(r)-j} b_1(r,\ell,j,2)c_1(\ell,0) \quad (0 \le j \le g(r));$

- $b_1(r+1,m,j,1) = \sum_{\ell=m-1}^{g(r)-1} b_1(r,\ell,j,3)c(\ell,m) \quad (1 \le m \le g(r), \quad 0 \le j \le g(r)-m);$

- $$b_1(r+1,0,j,1) = \sum_{m=0}^{g(r)} \sum_{\substack{j_1+\ell=j \\ 0\le j_1\le g(r)-m \\ 1\le\ell\le m}} b_1(r,m,j_1,2)d_1(m,\ell) + \sum_{m=0}^{g(r)-j} b_1(r,m,j,2)d_1(m,0) +$$
  $$b_1(r,0,j,3) - \sum_{m=0}^{g(r)-1} \sum_{\substack{j_1+\ell=j \\ 0\le j_1\le g(r)-m-1 \\ 1\le\ell\le m+1}} b_1(r,m,j_1,3)d(m,\ell) \quad (0 \le j \le g(r)).$$

When r is even,

- $b_1(r+1,m,j,2) = \sum_{\ell=m-1}^{g(r)} b_1(r,\ell,j,0)c(\ell,m) \quad (1 \le m \le g(r)+1, \quad 0 \le j \le g(r)+1-m);$

- $$b_1(r+1,0,j,2) = -\sum_{m=0}^{g(r)} \sum_{\substack{j_1+\ell=j \\ 0\le j_1\le g(r)-m \\ 1\le\ell\le m+1}} b_1(r,m,j_1,0)d(m,\ell) + \sum_{m=0}^{g(r)-j} b_1(r,m,j,1)d_1(m,0) +$$
  $$b_1(r,0,j,0) + \sum_{m=0}^{g(r)} \sum_{\substack{j_1+\ell=j \\ 0\le j_1\le g(r)-m \\ 1\le\ell\le m}} b_1(r,m,j_1,1)d_1(m,\ell) \quad (0 \le j \le g(r)+1);$$

- $b_1(r+1,m,j,3) = \sum_{\ell=m}^{g(r)} b_1(r,\ell,j,1)c_1(\ell,m) \quad (1 \le m \le g(r), \quad 0 \le j \le g(r)-m);$

- $b_1(r+1,0,j,3) = \sum_{\ell=0}^{g(r)-j} b_1(r,\ell,j,1)c_1(\ell,0) \quad (0 \le j \le g(r)).$

Lemma 6 ([10])**.** *For $r,n,p \in \mathbb{N}$, we have*

$$H_n^{(p,r,1)} = \sum_{m=0}^{\frac{2r-(-1)^r-3}{4}} \sum_{j=0}^{\frac{2r-(-1)^r-3}{4}-m} \Big(b_1(r,j,m,0)(-1)^{n-1}H_n^{(p-m)} + b_1(r,j,m,1)\overline{H}_n^{(p-m)}$$
$$+ b_1(r,j,m,2)H_n^{(p-m)} + b_1(r,j,m,3)(-1)^{n-1}\overline{H}_n^{(p-m)}\Big)n^j,$$

$$H_n^{(p,r,3)} = \sum_{m=0}^{\frac{2r-(-1)^r-3}{4}} \sum_{j=0}^{\frac{2r-(-1)^r-3}{4}-m} \Big(b_1(r,j,m,0)(-1)^{n-1}\overline{H}_n^{(p-m)} + b_1(r,j,m,1)H_n^{(p-m)}$$
$$+ b_1(r,j,m,2)\overline{H}_n^{(p-m)} + b_1(r,j,m,3)(-1)^{n-1}H_n^{(p-m)}\Big) n^j.$$

Lemma 7 ([7, 8])**.** *Let* $\sum_{i=1}^{\infty} a_i$ *and* $\sum_{i=1}^{\infty} b_i$ *be two series of positive terms such that* $a_i \sim b_i$*. Then if* $\sum_{i=1}^{\infty} b_i$ *is divergent, the following result holds:*

$$\sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i.$$

Lemma 8 ([6, 11])**.** *Let* $p_{n,k}$ *denote the nth squarefree number with just k prime factors and* $q_{n,k}$ *denote the nth number with just k prime factors. Then the following asymptotic formulas hold:*

$$p_{n,k} \sim q_{n,k} \sim (k-1)! \frac{n\log(n)}{(\log\log(n))^{k-1}},$$
$$p_{n,k}(\log\log(p_{n,k}))^{k-1} \sim q_{n,k}(\log\log(q_{n,k}))^{k-1} \sim (k-1)! n\log(n).$$

For $k=1$*, we have* $p_n \sim n\log(n)$*.*

Lemma 9 ([11])**.** *For* $m,n,k,x \in \mathbb{N}$*, we have*

$$\sum_{\ell=1}^{x} \ell^m (\log(\ell))^n \sim \frac{x^{m+1}(\log(x))^n}{m+1},$$
$$\sum_{\ell=1}^{x} \frac{\ell^m (\log(\ell))^n}{(\log\log(\ell))^k} \sim \frac{x^{m+1}(\log(x))^n}{(m+1)(\log\log(x))^k}.$$

3 Sums over primes involving generalized alternating hyperharmonic numbers of type $H_n^{(p,r,1)}$

Now we provide the asymptotic formulas for the generalized alternating hyperharmonic numbers of type $H_n^{(p,r,1)}$.

Lemma 10. *Let* $y,p \in \mathbb{N}$ *with* $p \geq 2$*, the following asymptotic formulas hold:*

$$H_n^{(1,2y+1,1)} \sim \frac{1}{2^y \cdot y!} n^y \log(n), \quad H_n^{(p,2y+1,1)} \sim \frac{1}{2^y \cdot y!} n^y \zeta(p),$$
$$H_n^{(1,2y,1)} \sim \frac{1}{2^y \cdot (y-1)!} n^{y-1} (-1)^{n-1} \log(n),$$
$$H_{2n}^{(p,2y,1)} \sim -\frac{1}{2 \cdot (y-1)!} n^{y-1} (\zeta(p) - \overline{\zeta}(p)),$$
$$H_{2n-1}^{(p,2y,1)} \sim \frac{1}{2 \cdot (y-1)!} n^{y-1} (\zeta(p) + \overline{\zeta}(p)),$$

where $\overline{\zeta}(s)$ *is the well-known alternating zeta function*

$$\overline{\zeta}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s) \quad \textit{with} \quad \overline{\zeta}(1) = \log 2.$$

Proof. By applying Definition 5 and Lemma 6, we have the following identities:
when r is odd,

$$H_n^{(p,r,1)} = \sum_{m=0}^{g(r)} \sum_{j=0}^{g(r)-m} \left(b_1(r,j,m,2) H_n^{(p-m)} + b_1(r,j,m,3)(-1)^{n-1} \overline{H}_n^{(p-m)} \right) n^j;$$

when r is even,

$$H_n^{(p,r,1)} = \sum_{m=0}^{g(r)} \sum_{j=0}^{g(r)-m} \left(b_1(r,j,m,0)(-1)^{n-1} H_n^{(p-m)} + b_1(r,j,m,1) \overline{H}_n^{(p-m)} \right) n^j.$$

When r is odd, by $b_1(r,m,j,3) = 0 \quad (m+j = g(r))$, we know that the main term of $H_n^{(p,r,1)}$ is $b_1(r,g(r),0,2)H_n^{(p)} n^{g(r)}$.

When r is even and $p=1$, we know that the main term of $H_n^{(1,r,1)}$ is

$$b_1(r,g(r),0,0)(-1)^{n-1} H_n n^{g(r)}.$$

When r is even and $p \geq 2$, we know that the main term of $H_n^{(p,r,1)}$ is

$$\left(b_1(r,g(r),0,0)(-1)^{n-1} H_n^{(p-m)} + b_1(r,g(r),0,1) \overline{H}_n^{(p)} \right) n^{g(r)}.$$

By applying Definition 5, we can obtain the following recursive formulas:
When r is odd with $r \geq 3$,

$$b_1(r+1, g(r+1), 0, 0) = b_1(r, g(r), 0, 2) \frac{1}{2},$$
$$b_1(r+1, g(r+1), 0, 1) = b_1(r, g(r)-1, 0, 3) \frac{1}{g(r)}.$$

When r is even,

$$b_1(r+1, g(r+1), 0, 2) = b_1(r, g(r), 0, 0) \frac{1}{g(r)+1},$$
$$b_1(r+1, g(r+1)-1, 0, 3) = b_1(r, g(r)-1, 0, 1) \frac{1}{2}.$$

Let $y \in \mathbb{N}$. By using the initial values $b_1(1,0,0,2) = 1$ and $b_1(1,0,0,3) = 0$, and the above recursive formulas, we can obtain the following explicit formulas:

$$b_1(2y+1, y, 0, 2) = \frac{1}{2^y \cdot y!},$$
$$b_1(2y+1, y-1, 0, 3) = \frac{1}{2^{y+1} \cdot (y-1)!},$$
$$b_1(2y, y-1, 0, 0) = b_1(2y, y-1, 0, 1) = \frac{1}{2^y \cdot (y-1)!}.$$

Thus we get the desired results. □

Now we state our main theorems of this section.

Theorem 11. *For $\alpha, m, q, y \in \mathbb{N}$ with $q \geq 2$, we have*

- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha (H_\ell^{(1,2y+1,1)})^m \sim \frac{x^{\alpha+my+1}(\log(x))^{\alpha+m}}{(2^y \cdot y!)^m(\alpha+my+1)};$
- $\displaystyle\sum_{p_\ell \leq x} p_\ell^\alpha (H_\ell^{(1,2y+1,1)})^m \sim \frac{x^{\alpha+my+1}}{(2^y \cdot y!)^m(\alpha+my+1)(\log(x))^{m(y-1)+1}};$
- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha (H_\ell^{(q,2y+1,1)})^m \sim \frac{\zeta(q)^m x^{\alpha+my+1}(\log(x))^{\alpha}}{(2^y \cdot y!)^m(\alpha+my+1)};$
- $\displaystyle\sum_{p_\ell \leq x} p_\ell^\alpha (H_\ell^{(q,2y+1,1)})^m \sim \frac{\zeta(q)^m x^{\alpha+my+1}}{(2^y \cdot y!)^m(\alpha+my+1)(\log(x))^{my+1}};$
- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha ((-1)^{\ell-1} H_\ell^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2^y \cdot (y-1)!)^m(\alpha+m(y-1)+1)};$
- $\displaystyle\sum_{p_\ell \leq x} p_\ell^\alpha ((-1)^{\ell-1} H_\ell^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}}{(2^y \cdot (y-1)!)^m(\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}};$
- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha (H_{2\ell-1}^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2 \cdot (y-1)!)^m(\alpha+m(y-1)+1)};$
- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha (-H_{2\ell}^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2 \cdot (y-1)!)^m(\alpha+m(y-1)+1)};$
- $\displaystyle\sum_{p_\ell \leq x} p_\ell^\alpha (H_{2\ell-1}^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}}{(2 \cdot (y-1)!)^m(\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}};$

- $\displaystyle\sum_{p_\ell\le x} p_\ell^\alpha(-H_{2\ell}^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}}$;

- $\displaystyle\sum_{\ell\le x} p_\ell^\alpha(H_{2\ell-1}^{(q,2y,1)})^m \sim \frac{(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^\alpha}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)}$;

- $\displaystyle\sum_{p_\ell\le x} p_\ell^\alpha(H_{2\ell-1}^{(q,2y,1)})^m \sim \frac{(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}}$;

- $\displaystyle\sum_{\ell\le x} p_\ell^\alpha(-H_{2\ell}^{(q,2y,1)})^m \sim \frac{(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^\alpha}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)}$;

- $\displaystyle\sum_{p_\ell\le x} p_\ell^\alpha(-H_{2\ell}^{(q,2y,1)})^m \sim \frac{(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}}$.

Proof. By using Lemmas 7, 8, 9, and 10, we have

$$\sum_{\ell\le x} p_\ell^\alpha(H_\ell^{(1,2y+1,1)})^m \sim \sum_{\ell\le x}\frac{\ell^{\alpha+my}(\log(\ell))^{\alpha+m}}{(2^y\cdot y!)^m} \sim \frac{x^{\alpha+my+1}(\log(x))^{\alpha+m}}{(2^y\cdot y!)^m(\alpha+my+1)}.$$

We can prove thirteen additional asymptotic formulas in a similar manner. □

Theorem 12. *For $\alpha,m,k,q,y\in\mathbb{N}$ with $q\ge 2$, we have*

- $\displaystyle\sum_{\ell\le x} p_{\ell,k}^\alpha(H_\ell^{(1,2y+1,1)})^m \sim \frac{((k-1)!)^\alpha x^{\alpha+my+1}(\log(x))^{\alpha+m}}{(2^y\cdot y!)^m(\alpha+my+1)(\log\log(x))^{\alpha(k-1)}}$;

- $\displaystyle\sum_{p_{\ell,k}\le x} p_{\ell,k}^\alpha(H_\ell^{(1,2y+1,1)})^m \sim \frac{x^{\alpha+my+1}(\log\log(x))^{(my+1)(k-1)}}{(2^y\cdot y!)^m((k-1)!)^{my+1}(\alpha+my+1)(\log(x))^{m(y-1)+1}}$;

- $\displaystyle\sum_{\ell\le x} p_{\ell,k}^\alpha(H_\ell^{q,2y+1,1)})^m \sim \frac{((k-1)!)^\alpha\zeta(q)^m x^{\alpha+my+1}(\log(x))^\alpha}{(2^y\cdot y!)^m(\alpha+my+1)(\log\log(x))^{\alpha(k-1)}}$;

- $\displaystyle\sum_{p_{\ell,k}\le x} p_{\ell,k}^\alpha(H_\ell^{(q,2y+1,1)})^m \sim \frac{\zeta(q)^m x^{\alpha+my+1}(\log\log(x))^{(my+1)(k-1)}}{(2^y\cdot y!)^m((k-1)!)^{my+1}(\alpha+my+1)(\log(x))^{my+1}}$;

- $\displaystyle\sum_{\ell\le x} p_{\ell,k}^\alpha((-1)^{\ell-1}H_\ell^{1,2y,1)})^m \sim \frac{((k-1)!)^\alpha x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2^y\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}}$;

- $\displaystyle\sum_{p_{\ell,k}\le x} p_{\ell,k}^\alpha((-1)^{\ell-1}H_\ell^{(1,2y,1)})^m \sim \frac{x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2^y\cdot(y-1)!)^m((k-1)!)^{m(y-1)+1}}$
  $\displaystyle\times\frac{1}{(\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}}$;

- $$\sum_{\ell\le x} p_{\ell,k}^{\alpha}(H_{2\ell-1}^{1,2y,1})^m \sim \sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(-H_{2\ell}^{(1,2y,1)})^m$$
$$\sim \frac{((k-1)!)^{\alpha}x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}};$$
- $$\sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(H_{2\ell-1}^{(1,2y,1)})^m \sim \sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(-H_{2\ell}^{(1,2y,1)})^m$$
$$\sim \frac{x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2\cdot(y-1)!)^m((k-1)!)^{m(y-1)+1}(\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}};$$
- $$\sum_{\ell\le x} p_{\ell,k}^{\alpha}(H_{2\ell-1}^{(q,2y,1)})^m \sim \frac{((k-1)!)^{\alpha}(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^{\alpha}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}};$$
- $$\sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(H_{2\ell-1}^{(q,2y,1)})^m \sim \frac{(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2\cdot(y-1)!)^m((k-1)!)^{m(y-1)+1}}$$
$$\times \frac{1}{(\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}};$$
- $$\sum_{\ell\le x} p_{\ell,k}^{\alpha}(-H_{2\ell}^{(q,2y,1)})^m \sim \frac{((k-1)!)^{\alpha}(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^{\alpha}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}};$$
- $$\sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(-H_{2\ell}^{(q,2y,1)})^m \sim \frac{(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2\cdot(y-1)!)^m((k-1)!)^{m(y-1)+1}}$$
$$\times \frac{1}{(\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}}.$$

Proof. By using Lemmas 7, 8, 9, and 10, we have

$$\sum_{\ell\le x} p_{\ell,k}^{\alpha}(H_{\ell}^{(1,2y+1,1)})^m \sim \sum_{\ell\le x}\frac{((k-1)!)^{\alpha}\ell^{\alpha+my}(\log(\ell))^{\alpha+m}}{(2^y\cdot y!)^m(\log\log(\ell))^{\alpha(k-1)}}$$
$$\sim \frac{((k-1)!)^{\alpha}x^{\alpha+my+1}(\log(x))^{\alpha+m}}{(2^y\cdot y!)^m(\alpha+my+1)(\log\log(x))^{\alpha(k-1)}}.$$

We can prove eleven additional asymptotic formulas in a similar manner. □

4 Sums over primes involving generalized alternating hyperharmonic numbers of type $H_n^{(p,r,2)}$

Now we provide the asymptotic formulas for the generalized alternating hyperharmonic numbers of type $H_n^{(p,r,2)}$.

Lemma 13. *For $r, n, p \in \mathbb{N}$, we have*

$$H_n^{(p,r,2)} \sim \frac{1}{(r-1)!} n^{r-1} \overline{\zeta}(p).$$

Proof. By using Lemma 3, we know that the main term of $H_n^{(p,r,2)}$ is $a(r,0,r-1)n^{r-1}\overline{H}_n^{(p)}$. The author [11] proves that $a(r,0,r-1) = \frac{1}{(r-1)!}$ and $\overline{H}_n^{(p)} \sim \overline{\zeta}(p)$. Thus we get the desired result. □

Now we show our main theorems of this section.

Theorem 14. *For $\alpha, m, q, k, r \in \mathbb{N}$, we have*

- $\displaystyle\sum_{\ell \le x} p_\ell^\alpha (H_\ell^{(q,r,2)})^m \sim \frac{\overline{\zeta}(q)^m x^{\alpha+m(r-1)+1}(\log(x))^\alpha}{((r-1)!)^m(\alpha+m(r-1)+1)};$

- $\displaystyle\sum_{p_\ell \le x} p_\ell^\alpha (H_\ell^{(q,r,2)})^m \sim \frac{\overline{\zeta}(q)^m x^{\alpha+m(r-1)+1}}{((r-1)!)^m(\alpha+m(r-1)+1)(\log(x))^{m(r-1)+1}};$

- $\displaystyle\sum_{\ell \le x} p_{\ell,k}^\alpha (H_\ell^{(q,r,2)})^m \sim \frac{((k-1)!)^\alpha \overline{\zeta}(q)^m x^{\alpha+m(r-1)+1}(\log(x))^\alpha}{((r-1)!)^m(\alpha+m(r-1)+1)(\log\log(x))^{\alpha(k-1)}};$

- $\displaystyle\sum_{p_{\ell,k} \le x} p_{\ell,k}^\alpha (H_\ell^{(q,r,2)})^m \sim \frac{\overline{\zeta}(q)^m x^{\alpha+m(r-1)+1}(\log\log(x))^{(m(r-1)+1)(k-1)}}{((k-1)!)^{m(r-1)+1}((r-1)!)^m}$
  $\displaystyle\times \frac{1}{(\alpha+m(r-1)+1)(\log(x))^{m(r-1)+1}}.$

Proof. By using Lemmas 7, 8, 9, and 13, we have

$$\begin{aligned}\sum_{\ell \le x} p_{\ell,k}^\alpha (H_\ell^{(q,r,2)})^m &\sim \sum_{\ell \le x} \frac{((k-1)!)^\alpha \overline{\zeta}(q)^m \ell^{\alpha+m(r-1)}(\log(\ell))^\alpha}{((r-1)!)^m(\log\log(\ell))^{\alpha(k-1)}} \\ &\sim \frac{((k-1)!)^\alpha \overline{\zeta}(q)^m x^{\alpha+m(r-1)+1}(\log(x))^\alpha}{((r-1)!)^m(\alpha+m(r-1)+1)(\log\log(x))^{\alpha(k-1)}}.\end{aligned}$$

We can prove three other asymptotic formulas in a similar manner. □

Theorem 15. *For $q_1, q_2, \alpha, \beta, m, k, s, n, r_1, r_2 \in \mathbb{N}$ with $q_1 \ge 2$, we have*

- $\displaystyle\sum_{\ell \le x} p_{\ell,k}^\alpha (H_\ell^{(q_1,r_1)})^m (h_\ell^{(s)})^n (H_\ell^{(q_2,r_2,2)})^\beta \sim \frac{((k-1)!)^\alpha \zeta(q_1)^m \overline{\zeta}(q_2)^\beta (\log(x))^{\alpha+n}}{((r_1-1)!)^m((s-1)!)^n((r_2-1)!)^\beta}$
  $\displaystyle\times \frac{x^{\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)+1}}{(\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)+1)(\log\log(x))^{\alpha(k-1)}};$

- $$\sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(H_\ell^{(q_1,r_1)})^m(h_\ell^{(s)})^n(H_\ell^{(q_2,r_2,2)})^\beta \sim \frac{\zeta(q_1)^m\overline{\zeta}(q_2)^\beta}{(\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)+1)}$$
$$\times \frac{x^{\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)+1}(\log\log(x))^{(m(r_1-1)+n(s-1)+\beta(r_2-1)+1)(k-1)}}{((r_1-1)!)^m((s-1)!)^n((r_2-1)!)^\beta((k-1)!)^{m(r_1-1)+n(s-1)+\beta(r_2-1)+1}}$$
$$\times \frac{1}{(\log(x))^{m(r_1-1)+n(s-2)+\beta(r_2-1)+1}}.$$

Proof. By using Lemmas 1, 2, 7, 8, 9, and 13, we have

$$\begin{aligned}
&\sum_{\ell\le x} p_{\ell,k}^{\alpha}(H_\ell^{(q_1,r_1)})^m(h_\ell^{(s)})^n(H_\ell^{(q_2,r_2,2)})^\beta \\
&\sim \sum_{\ell\le x} \frac{((k-1)!)^\alpha\zeta(q_1)^m\overline{\zeta}(q_2)^\beta\ell^{\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)}(\log(\ell))^{\alpha+n}}{((r_1-1)!)^m((s-1)!)^n((r_2-1)!)^\beta(\log\log(\ell))^{\alpha(k-1)}} \\
&\sim \frac{((k-1)!)^\alpha\zeta(q_1)^m\overline{\zeta}(q_2)^\beta(\log(x))^{\alpha+n}}{((r_1-1)!)^m((s-1)!)^n((r_2-1)!)^\beta(\log\log(x))^{\alpha(k-1)}} \\
&\times \frac{x^{\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)+1}}{(\alpha+m(r_1-1)+n(s-1)+\beta(r_2-1)+1)}.
\end{aligned}$$

We can prove the other asymptotic formula in a similar manner. □

5 Sums over primes involving generalized alternating hyperharmonic numbers of type $H_n^{(p,r,3)}$

Now we provide the asymptotic formulas for the generalized alternating hyperharmonic numbers of type $H_n^{(p,r,3)}$.

Lemma 16. *Let $y, p \in \mathbb{N}$, the following asymptotic formulas hold:*

- $H_n^{(p,2y+1,3)} \sim \frac{1}{2^y\cdot y!}n^y\overline{\zeta}(p)$;
- $H_n^{(1,2y,3)} \sim \frac{1}{2^y\cdot (y-1)!}n^{y-1}\log(n)$;
- $H_{2n}^{(p,2y,3)} \sim \frac{1}{2\cdot (y-1)!}n^{y-1}(\zeta(p)-\overline{\zeta}(p)) \quad (p\ge 2)$;
- $H_{2n-1}^{(p,2y,3)} \sim \frac{1}{2\cdot (y-1)!}n^{y-1}(\zeta(p)+\overline{\zeta}(p)) \quad (p\ge 2)$.

Proof. By applying Definition 5 and Lemma 6, we have the following identities:
when r is odd,

$$H_n^{(p,r,3)} = \sum_{m=0}^{g(r)} \sum_{j=0}^{g(r)-m} \left(b_1(r,j,m,2)\overline{H}_n^{(p-m)} + b_1(r,j,m,3)(-1)^{n-1} H_n^{(p-m)} \right) n^j;$$

when r is even,

$$H_n^{(p,r,1)} = \sum_{m=0}^{g(r)} \sum_{j=0}^{g(r)-m} \left(b_1(r,j,m,0)(-1)^{n-1}\overline{H}_n^{(p-m)} + b_1(r,j,m,1) H_n^{(p-m)} \right) n^j.$$

When r is odd, by $b_1(r,m,j,3) = 0 \quad (m+j = g(r))$, we know that the main term of $H_n^{(p,r,3)}$ is $b_1(r,g(r),0,2)\overline{H}_n^{(p)} n^{g(r)}$.

When r is even and $p = 1$, we know that the main term of $H_n^{(1,r,3)}$ is $b_1(r,g(r),0,1)H_n n^{g(r)}$.

When r is even and $p \geq 2$, we know that the main term of $H_n^{(p,r,3)}$ is

$$\left(b_1(r,g(r),0,0)(-1)^{n-1}\overline{H}_n^{(p-m)} + b_1(r,g(r),0,1) H_n^{(p)} \right) n^{g(r)}.$$

Let $y \in \mathbb{N}$. By applying Lemma 10, we have the following explicit formulas:

$$b_1(2y+1,y,0,2) = \frac{1}{2^y \cdot y!},$$
$$b_1(2y+1,y-1,0,3) = \frac{1}{2^{y+1} \cdot (y-1)!},$$
$$b_1(2y,y-1,0,0) = b_1(2y,y-1,0,1) = \frac{1}{2^y \cdot (y-1)!}.$$

Thus we get the desired results. □

Now we show our main theorems of this section.

Theorem 17. *For $\alpha, m, p, q, y \in \mathbb{N}$ with $q \geq 2$, we have*

- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha (H_\ell^{(p,2y+1,3)})^m \sim \frac{\overline{\zeta}(p)^m x^{\alpha+my+1}(\log(x))^\alpha}{(2^y \cdot y!)^m(\alpha+my+1)};$
- $\displaystyle\sum_{p_\ell \leq x} p_\ell^\alpha (H_\ell^{(p,2y+1,3)})^m \sim \frac{\overline{\zeta}(p)^m x^{\alpha+my+1}}{(2^y \cdot y!)^m(\alpha+my+1)(\log(x))^{my+1}};$
- $\displaystyle\sum_{\ell \leq x} p_\ell^\alpha (H_\ell^{(1,2y,3)})^m \sim \frac{x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2^y \cdot (y-1)!)^m(\alpha+m(y-1)+1)};$

- $\displaystyle\sum_{p_\ell \le x} p_\ell^\alpha (H_\ell^{(1,2y,3)})^m \sim \frac{x^{\alpha+m(y-1)+1}}{(2^y \cdot (y-1)!)^m (\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}};$

- $\displaystyle\sum_{\ell \le x} p_\ell^\alpha (H_{2\ell-1}^{(q,2y,3)})^m \sim \frac{(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^\alpha}{(2 \cdot (y-1)!)^m (\alpha+m(y-1)+1)};$

- $\displaystyle\sum_{p_\ell \le x} p_\ell^\alpha (H_{2\ell-1}^{(q,2y,3)})^m \sim \frac{(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}}{(2 \cdot (y-1)!)^m (\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}};$

- $\displaystyle\sum_{\ell \le x} p_\ell^\alpha (H_{2\ell}^{(q,2y,3)})^m \sim \frac{(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^\alpha}{(2 \cdot (y-1)!)^m (\alpha+m(y-1)+1)};$

- $\displaystyle\sum_{p_\ell \le x} p_\ell^\alpha (H_{2\ell}^{(q,2y,3)})^m \sim \frac{(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}}{(2 \cdot (y-1)!)^m (\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}}.$

Proof. By using Lemmas 7, 8, 9, and 16, we have

$$\sum_{\ell \le x} p_\ell^\alpha (H_\ell^{(p,2y+1,3)})^m \sim \sum_{\ell \le x} \frac{\overline{\zeta}(p)^m \ell^{\alpha+my}(\log(\ell))^\alpha}{(2^y \cdot y!)^m} \sim \frac{\overline{\zeta}(p)^m x^{\alpha+my+1}(\log(x))^\alpha}{(2^y \cdot y!)^m (\alpha+my+1)}.$$

We can prove seven other asymptotic formulas in a similar manner. □

Theorem 18. *For $\alpha, m, k, p, q, y \in \mathbb{N}$ with $q \ge 2$, we have*

- $\displaystyle\sum_{\ell \le x} p_{\ell,k}^\alpha (H_\ell^{p,2y+1,1)})^m \sim \frac{((k-1)!)^\alpha \overline{\zeta}(p)^m x^{\alpha+my+1}(\log(x))^\alpha}{(2^y \cdot y!)^m (\alpha+my+1)(\log\log(x))^{\alpha(k-1)}};$

- $\displaystyle\sum_{p_{\ell,k} \le x} p_{\ell,k}^\alpha (H_\ell^{(p,2y+1,1)})^m \sim \frac{\overline{\zeta}(p)^m x^{\alpha+my+1}(\log\log(x))^{(my+1)(k-1)}}{(2^y \cdot y!)^m ((k-1)!)^{my+1}(\alpha+my+1)(\log(x))^{my+1}};$

- $\displaystyle\sum_{\ell \le x} p_{\ell,k}^\alpha (H_\ell^{1,2y,3)})^m \sim \frac{((k-1)!)^\alpha x^{\alpha+m(y-1)+1}(\log(x))^{\alpha+m}}{(2^y \cdot (y-1)!)^m (\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}};$

- $\displaystyle\sum_{p_{\ell,k} \le x} p_{\ell,k}^\alpha (H_\ell^{(1,2y,3)})^m \sim \frac{x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2^y \cdot (y-1)!)^m ((k-1)!)^{m(y-1)+1}}$
  $\displaystyle\times \frac{1}{(\alpha+m(y-1)+1)(\log(x))^{m(y-2)+1}};$

- $\displaystyle\sum_{\ell \le x} p_{\ell,k}^\alpha (H_{2\ell-1}^{(q,2y,3)})^m \sim \frac{((k-1)!)^\alpha (\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^\alpha}{(2 \cdot (y-1)!)^m (\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}};$

- $$\sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(H_{2\ell-1}^{(q,2y,3)})^m \sim \frac{(\zeta(q)+\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2\cdot(y-1)!)^m((k-1)!)^{m(y-1)+1}} \times \frac{1}{(\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}};$$
- $$\sum_{\ell\le x} p_{\ell,k}^{\alpha}(H_{2\ell}^{(q,2y,3)})^m \sim \frac{((k-1)!)^{\alpha}(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log(x))^{\alpha}}{(2\cdot(y-1)!)^m(\alpha+m(y-1)+1)(\log\log(x))^{\alpha(k-1)}};$$
- $$\sum_{p_{\ell,k}\le x} p_{\ell,k}^{\alpha}(H_{2\ell}^{(q,2y,3)})^m \sim \frac{(\zeta(q)-\overline{\zeta}(q))^m x^{\alpha+m(y-1)+1}(\log\log(x))^{(m(y-1)+1)(k-1)}}{(2\cdot(y-1)!)^m((k-1)!)^{m(y-1)+1}} \times \frac{1}{(\alpha+m(y-1)+1)(\log(x))^{m(y-1)+1}}.$$

Proof. By using Lemmas 7, 8, 9, and 16, we have

$$\sum_{\ell\le x} p_{\ell,k}^{\alpha}(H_{\ell}^{(p,2y+1,3)})^m \sim \sum_{\ell\le x}\frac{((k-1)!)^{\alpha}\overline{\zeta}(p)^m \ell^{\alpha+my}(\log(\ell))^{\alpha}}{(2^y\cdot y!)^m(\log\log(\ell))^{\alpha(k-1)}}$$
$$\sim \frac{((k-1)!)^{\alpha}\overline{\zeta}(p)^m x^{\alpha+my+1}(\log(x))^{\alpha}}{(2^y\cdot y!)^m(\alpha+my+1)(\log\log(x))^{\alpha(k-1)}}.$$

We can prove seven other asymptotic formulas in a similar manner. □

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