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Incomplete Finite Binomial Sums of Harmonic Numbers

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Abstract

We explore a straightforward recursive relation for an incomplete binomial series. Through this approach, we establish novel identities for the incomplete finite binomial sum of harmonic numbers. Additionally, we introduce a new proof for an identity related to the incomplete finite alternating binomial sum of harmonic numbers. These identities act as analogues to their respective well-established formulas for complete series and enable the characterization of the asymptotic behavior of the incomplete binomial series of harmonic numbers.

1 Introduction

Let N be a positive integer, and let m be an integer satisfying $0 \le m \le N$. We define *the incomplete finite binomial sum of harmonic numbers*, and *the incomplete finite alternating binomial sum of harmonic numbers*, respectively, as follows:

$$S_{N,m} = \sum_{k=1}^{m} \binom{N}{k} H_k, \qquad \text{and} \qquad I_{N,m} = \sum_{k=1}^{m} \binom{N}{k} (-1)^{k+1} H_k.$$

Here, H_k represents the harmonic number, defined by

$$H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \quad \text{for } k \ge 1, \qquad \text{and} \qquad H_0 = 0. \tag{1}$$

Furthermore, for $m = \lfloor N/2 \rfloor$, we introduce the following notation:

$$S_N = S_{N,\lfloor N/2 \rfloor}, \qquad \text{and} \qquad I_N = I_{N,\lfloor N/2 \rfloor}.$$

For $n \geq 1$, we prove the following new identities:

$$S_{2n-1} = 2^{2n-2} \left(H_{2n-1} - 2C_{2n-1} + D_{n-1} \right), \tag{2}$$

$$S_{2n} = 2^{2n-1} \left(H_{2n} - 2C_{2n} + D_n \right) + \frac{1}{2}\binom{2n}{n} H_n. \tag{3}$$

In addition, for all positive integers N, and for all integers m satisfying $0 \leq m \leq N$, we present a new proof of the identity originally proved by Batir [2]:

$$I_{N,m} = \frac{1}{N} + (-1)^{m+1}\binom{N-1}{m}\left(H_m + \frac{1}{N}\right). \tag{4}$$

Identities (2)–(3) are incomplete binomial sum analogues of the well-known identity for the complete binomial sum of harmonic numbers:

$$\sum_{k=0}^{N} \binom{N}{k} H_k = 2^N \left(H_N - C_N \right), \tag{5}$$

proved by various techniques by Spivey [16], Paule and Schneider [14], Boyadzhiev [4], Gonzalez [9], Batir [1, 2], and Batir and Sofo [3]. Identity (4) is the incomplete analogue of the complete alternating binomial sum of harmonic numbers:

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^{k+1} H_k = \frac{1}{N},$$

proved by various techniques by Euler [8], Jin and Du [11], Wang [17], Paule and Schneider [14], Gonzalez [9], and Batir [1, 2]. Here we introduce the following notation:

$$C_n = \sum_{k=1}^{n} \frac{1}{k2^k}, \quad \text{and} \quad D_n = \sum_{k=1}^{n} \frac{1}{(2k)2^{2k}}\binom{2k}{k}, \tag{6}$$

for $n \geq 1$, with $C_0 = D_0 = 0$.

Combinatorial identities involving harmonic numbers, including their binomial sums, have been extensively studied. In addition to the references mentioned previously, significant contributions have been made by Spiess [15], Liu and Wang [13], Chu [6], Choi [5], and Wei and Wang [18]. Batir [2] provides a comprehensive survey of the literature.

The term *incomplete finite sum* refers to a finite sum where the summation is taken over only a part of the full index range, not over all integer indices k such that $0 \leq k \leq N$.

Incomplete binomial sums related to those presented in this work have been studied by Batir [2, Corollaries 3–4], who proved the following identities for all $m \in \mathbb{N}$ and $s \in \mathbb{C} \setminus \mathbb{Z}^-$:

$$\sum_{k=1}^{m} \binom{m+s}{k} H_k = 2^m \sum_{k=1}^{m} \binom{s+k-1}{k} \frac{1}{2^k} H_k + 2^m \sum_{k=1}^{m} \frac{1}{2^k} \sum_{j=0}^{k-1} \binom{s+k-1}{j} \frac{1}{m+1},$$

$$\sum_{k=1}^{m} \binom{m+s}{k} (-1)^k H_k = (-1)^m \binom{s+m-1}{m} H_m + \frac{(-1)^m}{s+m} \binom{s+m-1}{m} - \frac{1}{s+m}.$$

By setting $s = N - m$ and $N \geq m$, the second identity reduces to (4), while the first one transforms to

$$S_{N,m} = 2^m \sum_{k=1}^{m} \binom{N-m+k-1}{k} \frac{1}{2^k} H_k + 2^m \sum_{k=1}^{m} \frac{1}{2^k} \sum_{j=0}^{k-1} \binom{N-m+k-1}{j} \frac{1}{j+1}.$$

Incomplete binomial sums were also studied by Worsch [19], Dence [7], and Katsuura [12]. Gould's eight-volume online book [10] features a comprehensive list of various types of both complete and incomplete finite binomial sums but lacks detailed proofs and references.

Our research is motivated by applications in which the binomial coefficient encodes the probability distribution of states in a system, and the system measure exhibits a symmetry $k \leftrightarrow (N-k)$. Specifically, we focus on problems related to the mean time of self-organization in abstract model systems, where the measured quantity is proportional to the harmonic numbers H_k.

For proofs, we adapted techniques from Gonzalez [9], modifying them for incomplete finite sums. Our method is based on a general recursive formula for finite binomial series of an arbitrary sequence of real numbers, yielding a second-order linear recurrence relation for S_N and an explicit expression for $I_{N,m}$.

The derived formulae enable the asymptotic analysis of S_N and I_N as $N \to \infty$. Note that such approximations cannot be simply deduced from the asymptotic behavior of H_k for $k \gg 1$, as the sums S_N and I_N include terms H_k for small values of k. We show that as $N \to \infty$,

$$\frac{S_N}{2^{N-1}} = \ln\left(\frac{N}{2}\right) + \gamma + o(1),$$

and

$$\frac{I_N}{2^{N-1}} \sqrt{\frac{\pi N}{2}} = (-1)^{\lfloor \frac{N}{2} \rfloor + 1} \left(\ln\left(\frac{N}{2}\right) + \gamma + o(1) \right).$$

Here, γ denotes the Euler-Mascheroni constant, $\gamma \doteq 0.577$.

2 Recursive combinatorial identity

Analogously to Gonzalez [9], we first derive a recursive combinatorial identity.

Lemma 1. *Let m and N be positive integers, with $m \leq N$, and let $(a_k)_{k=0}^{m}$ be a sequence of real numbers. Then, the following equation holds:*

$$\sum_{k=0}^{m} \binom{N}{k} a_k = \sum_{k=0}^{m-1} \binom{N-1}{k} (a_k + a_{k+1}) + \binom{N-1}{m} a_m. \tag{7}$$

Here, we define $\binom{n}{k} = 0$ for all integers n and k, where $n < k$.

Proof. The claim is valid for all $N \geq 1$ and is demonstrated by the following calculation:

$$\begin{aligned}
\sum_{k=0}^{m} \binom{N}{k} a_k &= \binom{N}{0} a_0 + \sum_{k=1}^{m} \binom{N}{k} a_k \\
&= \binom{N-1}{0} a_0 + \sum_{k=1}^{m} \left(\binom{N-1}{k} + \binom{N-1}{k-1} \right) a_k \\
&= \sum_{k=0}^{m} \binom{N-1}{k} a_k + \sum_{k=0}^{m-1} \binom{N-1}{k} a_{k+1} \\
&= \sum_{k=0}^{m-1} \binom{N-1}{k} (a_k + a_{k+1}) + \binom{N-1}{m} a_m.
\end{aligned}$$

□

Note that Lemma 1 generalizes the identity derived in [9, Theorem 2.1] for the full sum:

$$\sum_{k=0}^{N} \binom{N}{k} a_k = \sum_{k=0}^{N-1} \binom{N-1}{k} (a_k + a_{k+1}),$$

which holds for all $N \geq 1$ and for all sequences $(a_k)_{k=0}^{N}$. This result follows from Lemma 1 by setting $m = N$.

In part of our work, we utilize a slight modification of the identity (7).

Corollary 2. *Let n be a positive integer and let m be a nonnegative integer with $m \leq n$. Suppose $(a_k)_{k=0}^{m+1}$ is a sequence of real numbers. Then,*

$$\sum_{k=0}^{m} \binom{2n-1}{k} a_k = \sum_{k=0}^{m} \binom{2n-2}{k} (a_k + a_{k+1}) - \binom{2n-2}{m} a_{m+1}. \tag{8}$$

Similarly, let m and n be positive integers with $m \leq n$, and let $(a_k)_{k=0}^{m}$ be a sequence of real numbers. Then,

$$\sum_{k=0}^{m} \binom{2n}{k} a_k = \sum_{k=0}^{m-1} \binom{2n-1}{k} (a_k + a_{k+1}) + \binom{2n-1}{m} a_m. \tag{9}$$

Proof. To prove equation (8) for all m and n where $n \geq 1$ and $1 \leq m \leq n$, we set $N = 2n-1$ in Lemma 1:

$$\begin{aligned}\sum_{k=0}^{m} \binom{2n-1}{k} a_k &= \sum_{k=0}^{m-1} \binom{2n-2}{k} (a_k + a_{k+1}) + \binom{2n-2}{m} a_m \\ &= \sum_{k=0}^{m} \binom{2n-2}{k} (a_k + a_{k+1}) - \binom{2n-2}{m} a_{m+1}.\end{aligned}$$

For $m = 0$, equation (8) simplifies to $a_0 = (a_0 + a_1) - a_1$. Equation (9) follows directly from Lemma 1 by setting $N = 2n$. $\square$

3 Incomplete binomial sum of harmonic numbers

In this section, we use the recurrence relations (8)–(9) to derive formulae for the incomplete binomial sum of harmonic numbers, where the summation is over half of the full index range.

Theorem 3. *Let n be a positive integer. Then,*

$$S_{2n-1} = \sum_{i=0}^{n-1} \binom{2n-1}{i} H_i = 2^{2n-2} \left(H_{2n-1} - 2C_{2n-1} + D_{n-1} \right), \tag{10}$$

$$S_{2n} = \sum_{i=0}^{n} \binom{2n}{i} H_i = 2^{2n-1} \left(H_{2n} - 2C_{2n} + D_n \right) + \frac{1}{2} \binom{2n}{n} H_n. \tag{11}$$

Proof. First, we define $A_n = S_{2n-1}$ and $B_n = S_{2n}$. Then, using equation (8) for $m = n-1$ and $a_k = H_k$ for $k = 0, \ldots, n$, we obtain

$$\begin{aligned}A_n &= \sum_{i=0}^{n-1} \binom{2n-2}{i} \left(H_i + H_{i+1} \right) - \binom{2n-2}{n-1} H_n \\ &= \sum_{i=0}^{n-1} \binom{2n-2}{i} \left(2H_i + \frac{1}{i+1} \right) - \binom{2n-2}{n-1} H_n.\end{aligned}$$

Therefore,

$$\begin{aligned}A_n &= 2\sum_{i=0}^{n-1} \binom{2n-2}{i} H_i + \sum_{i=0}^{n-1} \binom{2n-2}{i} \frac{1}{i+1} - \binom{2n-2}{n-1} H_n \\ &= 2\sum_{i=0}^{n-1} \binom{2n-2}{i} H_i + \frac{1}{2n-1} \sum_{i=0}^{n-1} \binom{2n-1}{i+1} - \binom{2n-2}{n-1} H_n.\end{aligned}$$

Continuing, we find

$$A_n = 2\sum_{i=0}^{n-1}\binom{2n-2}{i}H_i - \binom{2n-2}{n-1}H_n + \binom{2n-2}{n-1}\frac{1}{n} + \frac{1}{2n-1}\left(-1+\sum_{i=0}^{n-1}\binom{2n-1}{i}\right).$$

Hence,

$$A_n = 2B_{n-1} - \binom{2n-2}{n-1}H_{n-1} + \frac{2^{2n-2}-1}{2n-1}. \tag{12}$$

Here, we used the identity

$$\sum_{i=0}^{n-1}\binom{2n-1}{i} = \sum_{i=n}^{2n-1}\binom{2n-1}{i} = \frac{1}{2}\sum_{i=0}^{2n-1}\binom{2n-1}{i} = 2^{2n-2}.$$

Similarly, using the identity (9) for $m = n$ and the same choice of $(a_k)_{k=1}^n$ as in the first part, we obtain the following expression for B_n:

$$\begin{aligned} B_n &= \sum_{i=0}^{n-1}\binom{2n-1}{i}(H_i + H_{i+1}) + \binom{2n-1}{n}H_n \\ &= \sum_{i=0}^{n-1}\binom{2n-1}{i}\left(2H_i + \frac{1}{i+1}\right) + \binom{2n-1}{n}H_n. \end{aligned}$$

Consequently,

$$\begin{aligned} B_n &= 2\sum_{i=0}^{n-1}\binom{2n-1}{i}H_i + \sum_{i=0}^{n-1}\binom{2n-1}{i}\frac{1}{i+1} + \binom{2n-1}{n}H_n \\ &= 2\sum_{i=0}^{n-1}\binom{2n-1}{i}H_i + \frac{1}{2n}\sum_{i=0}^{n-1}\binom{2n}{i+1} + \binom{2n-1}{n}H_n \\ &= 2\sum_{i=0}^{n-1}\binom{2n-1}{i}H_i + \binom{2n-1}{n}H_n + \frac{1}{2n}\left(\sum_{i=0}^{n}\binom{2n}{i} - 1\right). \end{aligned}$$

Thus,

$$B_n = 2A_n + \binom{2n-1}{n}H_n + \frac{2^{2n-1}-1}{2n} + \frac{1}{4n}\binom{2n}{n}. \tag{13}$$

Here, we used the identity

$$\sum_{i=0}^{n}\binom{2n}{i} = \sum_{i=n}^{2n}\binom{2n}{i} = \frac{1}{2}\left(\sum_{i=0}^{2n}\binom{2n}{i} + \binom{2n}{n}\right) = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}.$$

Furthermore, let α_n and β_n, $n \geq 1$, denote the terms

$$\alpha_n = \frac{2^{2n} - 1}{2n+1} - \binom{2n}{n} H_n,$$
$$\beta_n = \frac{2^{2n-1} - 1}{2n} + \binom{2n-1}{n} H_n + \frac{1}{4n}\binom{2n}{n}.$$

The pair of equations (12)–(13) reduces to

$$A_n = 2B_{n-1} + \alpha_{n-1}, \qquad B_n = 2A_n + \beta_n. \tag{14}$$

By combining these two recurrence relations, we obtain

$$A_n = 4A_{n-1} + \alpha_{n-1} + 2\beta_{n-1}, \tag{15}$$
$$B_n = 4B_{n-1} + \beta_n + 2\alpha_{n-1}. \tag{16}$$

We set $\omega_k = \alpha_k + 2\beta_k$ for $1 \leq k \leq n-1$. Then, for all k, $1 \leq k \leq n-1$, we can express ω_k as follows:

$$\begin{aligned}
\omega_k &= \frac{2^{2k} - 1}{2k+1} - \binom{2k}{k} H_k + \frac{2(2^{2k-1} - 1)}{2k} + 2\binom{2k-1}{k} H_k + \frac{1}{2k}\binom{2k}{k} \\
&= \frac{2^{2k} - 1}{2k+1} - \binom{2k}{k} H_k + \frac{2^{2k} - 2}{2k} + \left(\binom{2k-1}{k} + \binom{2k-1}{k-1}\right) H_k + \frac{1}{2k}\binom{2k}{k} \\
&= \frac{2^{2k}}{2k+1} - \frac{1}{2k+1} - \binom{2k}{k} H_k + \frac{2^{2k}}{2k} - \frac{2}{2k} + \binom{2k}{k} H_k + \frac{1}{2k}\binom{2k}{k} \\
&= 2^{2k}\left(\frac{1}{2k+1} + \frac{1}{2k}\right) - \left(\frac{1}{2k+1} + \frac{2}{2k}\right) + \frac{1}{2k}\binom{2k}{k}.
\end{aligned} \tag{17}$$

By a recursive application of equation (15), we derive

$$\begin{aligned}
A_n &= 2^2 A_{n-1} + \omega_{n-1} = 2^4 A_{n-2} + \omega_{n-1} + 4\omega_{n-2} = \cdots \\
&= 2^{2n-2} A_1 + \sum_{k=1}^{n-1} 2^{2k-2}\omega_{n-k} = \sum_{k=1}^{n-1} 2^{2k-2}\omega_{n-k}.
\end{aligned}$$

Using the identity $A_1 = 0$ in the last equality, we then have

$$\begin{aligned}
A_n = &\left(2^{2n-2}\left(\frac{1}{2n-1} + \frac{1}{2n-2}\right) - \left(\frac{1}{2n-1} + \frac{2}{2n-2}\right) + \frac{1}{2n-2}\binom{2n-2}{n-1}\right) \\
&+ 2^2\left(2^{2n-4}\left(\frac{1}{2n-3} + \frac{1}{2n-4}\right) - \left(\frac{1}{2n-3} + \frac{2}{2n-4}\right) + \frac{1}{2n-4}\binom{2n-4}{n-2}\right) \\
&+ \cdots + 2^{2n-4}\left(2^2\left(\frac{1}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{2}{2}\right) + \frac{1}{2}\binom{2}{1}\right) \\
= &\, 2^{2n-2}\sum_{k=2}^{2n-1}\frac{1}{k} - 2^{2n-1}\sum_{k=2}^{2n-1}\frac{1}{k2^k} + 2^{2n-2}\sum_{k=1}^{n-1}\frac{1}{(2k)2^{2k}}\binom{2k}{k}.
\end{aligned}$$

Using the notation H_n, C_n, D_n introduced in equations (1) and (6), the last identity yields the claim (10) of the Theorem:

$$\begin{aligned} A_n &= 2^{2n-2}\left(H_{2n-1}-1\right) - 2^{2n-1}\left(C_{2n-1}-\frac{1}{2}\right) + 2^{2n-2}D_{n-1} \\ &= 2^{2n-2}H_{2n-1} - 2^{2n-1}C_{2n-1} + 2^{2n-2}D_{n-1}. \end{aligned} \tag{18}$$

Combining equations (14) and (18), we obtain the identity (11):

$$\begin{aligned} B_n = 2A_n + \beta_n &= 2\left(2^{2n-2}H_{2n-1} - 2^{2n-1}C_{2n-1} + 2^{2n-2}D_{n-1}\right) \\ &\quad + \frac{2^{2n-1}-1}{2n} + \binom{2n-1}{n}H_n + \frac{1}{4n}\binom{2n}{n} \\ &= 2^{2n-1}H_{2n-1} - 2^{2n}\sum_{k=1}^{2n-1}\frac{1}{k2^k} + 2^{2n-1}\sum_{k=1}^{n-1}\frac{1}{(2k)2^{2k}}\binom{2k}{k} \\ &\quad + 2^{2n-1}\frac{1}{2n} - \frac{1}{2n} + \binom{2n-1}{n}H_n + \frac{1}{4n}\binom{2n}{n} \\ &= 2^{2n-1}H_{2n} - \frac{1}{2n} - 2^{2n}\left(\sum_{k=1}^{2n}\frac{1}{k2^k} - \frac{1}{(2n)2^{2n}}\right) \\ &\quad + 2^{2n-1}\left(\sum_{k=1}^{n}\frac{1}{(2k)2^{2k}} - \frac{1}{(2n)2^{2n}}\binom{2n}{n}\right) + \binom{2n-1}{n}H_n + \frac{1}{4n}\binom{2n}{n} \\ &= 2^{2n-1}H_{2n} - 2^{2n}C_{2n} + 2^{2n-1}D_n + \binom{2n-1}{n}H_n \\ &= 2^{2n-1}H_{2n} - 2^{2n}C_{2n} + 2^{2n-1}D_n + \frac{1}{2}\binom{2n}{n}H_n. \end{aligned}$$

□

Let δ_N, for $N \geq 1$, denote

$$\delta_N = \begin{cases} 0, & \text{if } N \text{ odd;} \\ \frac{1}{2}\binom{N}{N/2}H_{N/2}, & \text{if } N \text{ even.} \end{cases}$$

A combination of the identities (2)–(3) and (5) implies

$$\sum_{k=\lfloor N/2\rfloor+1}^{N}\binom{N}{k}H_k = 2^{N-1}(H_N - C_N) - 2^{N-1}\left(C_N - D_{\lfloor N/2\rfloor}\right) - \delta_N.$$

The last identity allows us to quantify an asymmetry between the first and the second half of the binomial sum of harmonic series

$$\sum_{k=0}^{\lfloor N/2\rfloor}\binom{N}{k}H_k - \sum_{k=\lfloor N/2\rfloor+1}^{N}\binom{N}{k}H_k = 2^N\left(C_N - D_{\lfloor N/2\rfloor}\right) + \delta_N.$$

Note that an analogous calculation can be used to prove a generalization of the recurrence relations (15)–(16) for the incomplete binomial sums of harmonic numbers $S_{2n-1,m}$ and $S_{2n,m}$. However, the cancellation of the terms $\binom{2n}{m}H_m$ and $2\binom{2n-1}{m}H_m$ in equation (17) only occurs for $m = n$. In the general case, where $m \neq n$, these terms do not cancel, and the explicit formula (18) contains a binomial sum of harmonic numbers on the right hand side, meaning the double sums are not eliminated in the calculation.

4 Incomplete binomial sum of alternating harmonic numbers

Using Lemma 1, we provide a proof of (4) that differs from the original proof by Batir [2].

Theorem 4. *Consider a positive integer N and a non-negative integer m. Then*

$$I_{N,m} = \frac{1}{N} + (-1)^{m+1}\binom{N-1}{m}\left(H_m + \frac{1}{N}\right). \tag{19}$$

Proof. The claim of the theorem reduces to the equality $0 = 0$ for $m = 0$. Therefore, we assume that $m \geq 1$. The identity (7) for $a_k = (-1)^{k+1}H_k$, with $0 \leq k \leq m$, implies

$$\begin{aligned}
I_{N,m} &= \sum_{i=0}^{m-1}\binom{N-1}{i}\left((-1)^{i+1}H_i + (-1)^{i+2}H_{i+1}\right) + \binom{N-1}{m}(-1)^{m+1}H_m \\
&= \sum_{i=0}^{m-1}\binom{N-1}{i}(-1)^i\left(-H_i + H_{i+1}\right) + (-1)^{m+1}\binom{N-1}{m}H_m \\
&= \sum_{i=0}^{m-1}(-1)^i\binom{N-1}{i}\frac{1}{i+1} + (-1)^{m+1}\binom{N-1}{m}H_m.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{N,m} &= \frac{1}{N}\sum_{i=0}^{m-1}(-1)^i\binom{N}{i+1} + (-1)^{m+1}\binom{N-1}{m}H_m \\
&= \frac{1}{N}\sum_{i=0}^{m-1}(-1)^i\left(\binom{N-1}{i} + \binom{N-1}{i+1}\right) + (-1)^{m+1}\binom{N-1}{m}H_m.
\end{aligned}$$

We conclude the proof of (19) by observing that the telescopic sum on the right-hand side of the previous equation reduces to

$$\sum_{i=0}^{m-1}(-1)^i\left(\binom{N-1}{i} + \binom{N-1}{i+1}\right) = \binom{N-1}{0} + (-1)^{m-1}\binom{N-1}{m}.$$

□

5 Asymptotic approximation

While there is no explicit formula for the finite sum C_n, the infinite sum can be evaluated as

$$\sum_{k=1}^{\infty} \frac{1}{k2^k} = \ln 2.$$

This identity directly follows from the Taylor expansion of $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$, for $|x| < 1$, evaluated at $x = \frac{1}{2}$.

Another straightforward calculation yields the Taylor expansion for the generating function, valid for $|x| \leq \frac{1}{2}$:

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{x^{2k}}{2k} = \ln \frac{2}{\sqrt{1-4x^2}+1}.$$

For $x = \frac{1}{2}$, it implies

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{1}{(2k)2^{2k}} = \ln 2.$$

The calculation of the asymptotic approximation of A_n as $n \to \infty$ is then as follows:

$$\begin{aligned} A_n &= 2^{2n-2} \left(\ln(2n-1) + \gamma\right) - 2^{2n-1} \ln 2 + 2^{2n-2} \ln 2 + 2^{2n-2} o(1) \\ &= 2^{2n-2} \left(\ln n + \gamma + o(1)\right). \end{aligned}$$

Similarly, for B_n, as $n \to \infty$, we have $B_n = 2^{2n-1} \left(\ln n + \gamma + o(1)\right)$. This implies

$$S_N = 2^{N-1} \left(\ln N - \ln 2 + \gamma + o(1)\right).$$

for $N \to \infty$. A similar calculation yields an asymptotic approximation of I_N:

$$I_N = \frac{(-1)^{\lfloor N/2 \rfloor + 1}}{\sqrt{\pi N/2}} 2^{N-1} \left(\ln N - \ln 2 + \gamma + o(1)\right).$$

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