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Translations and Extensions of the Nicomachean Identity

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Abstract

We display sequences of interest that do or do not satisfy a Nicomachean identity, and create from them new sequences that do. This is done by introducing translation parameters, adjoining variable parameters, and forming sequential ("Mason-Liouvillian") products. Degeneracies in this ostensibly cubic framework enable us to employ facts about both definite and indefinite quadratic forms. We prove new results about gaps in Nicomachean sequences, and display novel Nicomachean sequences formed from Fibonacci numbers. We also briefly illustrate connections with certain elliptic curves and their geometry. This paper can be viewed as an in-depth study of variations of sequence A000537.

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1 Introduction

Define an operator ν on finite sequences of complex numbers a_i, i.e., $\sigma = (a_1, a_2, \ldots, a_n)$, by

$$\nu(a_1, a_2, \ldots, a_n) = \Big(\sum_{i=1}^{n} a_i\Big)^2 - \sum_{i=1}^{n} a_i^3.$$

The ancient classical Nicomachean identity is

$$\nu(1, 2, 3, \ldots, n) = 0.$$

We call $\nu(a_1, a_2, \ldots, a_n) = 0$ a *Nicomachean-type identity*. The above definition of ν does in fact make sense for the a_i being elements of any ring.

The number and variety of Nicomachean-type identities is very large. Let σ be an arbitrary sequence of positive integers. Then Mason has shown [8], by using bag products, that it can be encoded in a Nicomachean-type identity. In the other direction, Barbeau and Seraj [1] used a simple estimate to show that the number of Nicomachean-type identities for a sequence of length n, with n fixed, is finite. For previous studies of various points of view about Nicomachean-type identities, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Among these, the present paper has a point of view most similar to that of Barbeau and Seraj [1], and, like [1], makes use of Pell's equations.

A natural question is "for what sequences of positive integers or positive rational numbers does $\nu = 0$?". Note that we allow repetitions in our sequences.

We immediately note that aside from the trivial $a_i = 0$ for all i case, there are three very simple types of sequences with $\nu = 0$.

(1) The case $a_i = n$ for $1 \leq i \leq n$.

(2) The case in which, for some re-ordering of the a_i, each a_i is followed by $-a_i$.

(3) The case $\sigma = (\sigma_1; \sigma_2)$ where σ_1 belongs to the case (2) and σ_2 consists of some integer m repeated m times.

Here (3) is a combination of (1) and (2). We also note that $\nu(\sigma)$ is the same for all re-orderings of σ.

Our approach here is to work in a more general setting in which some or all of the a_i are variable parameters, to compute the variable or multivariable function $\nu(\sigma)$, and then to find values of the parameters for which $\nu(\sigma) = 0$. Ideally the resulting a_i will be positive rational numbers, but other outcomes may nonetheless be of interest.

For an example, take

$$\sigma = (1, 2, 2, 3, 4, x, 6, 8).$$

Then

$$\nu(\sigma) = (5 - x)(x - 4)(8 + x)$$

and we have three non-trivial examples of $\nu(\sigma) = 0$. Here the pair $(-8, 8)$ may be removed.

Another problem is to determine what extensions of a sequence with $\nu = 0$ also have this property. For an example, let

$$\sigma = (1, 2, 3, \ldots, 16, a, b).$$

To avoid trivialities, assume $a + b \neq 0$. Then $\nu(\sigma) = 0$ is equivalent to

$$a^2 - a + b^2 - b - ab = 272.$$

A brief search yields the solutions

$$(a, b) = (2, 18), (17, 18), (9, 20) \text{ and } (12, 20).$$

Each leads to a specific non-trivial solution in positive integers of $\nu(\sigma) = 0$, with $(17, 18)$ corresponding to the classical identity. In general, for

$$\sigma = (1, 2, 3, \ldots, n, a, b)$$

with $n \geq 5$, $a + b \neq 0$ and $\{a, b\} \neq \{n + 1, n + 2\}$, the equation $\nu(\sigma) = 0$ implies that one of a or b is less than $n - 1$. Hence this can produce only sequences σ with repetitions. In fact, the equation $\nu(\sigma) = 0$ can be put into a more attractive form by means of the change of variable

$$a = u + v + 1, \quad b = u - v + 1 \text{ or } b = -u + v + 1.$$

This transforms the equation to

$$u^2 + 3v^2 = \frac{n^3 - 1}{n - 1} \quad \text{or} \quad 3u^2 + v^2 = \frac{n^3 - 1}{n - 1},$$

respectively. For $n \geq 5$, it follows from the above that one of u or v is less than $n - 2$, which in turn implies that one of a or b is less than $n - 1$.

The above shows that part of our search for Nicomachean-type identities is tied to the Diophantine analysis of positive definite quadratic forms. We shall see in Section 2 that the solutions of indefinite quadratic forms ("Pell's equation") is also a natural part of this search.

Further pursuit of this "adjoin (a, b) procedure" leads many general Nicomachean-type identities. Of particular interest is

$$\Big(\sum_{j=1}^{7n-12} j + (8n - 12) + (3n - 3) \Big)^2 = \sum_{j=1}^{7n-12} j^3 + (8n - 12)^3 + (3n - 3)^3. \tag{1}$$

Note the growing gap here between the two largest cubes $(7n - 12)^3$ and $(8n - 12)^3$. Several natural questions about the gap will be answered in Section 6.

A further main theme is the introduction of translation parameters, especially for sequences σ with $\nu(\sigma) = 0$. For $\sigma = (a_1, a_2, \ldots, a_n)$, let $\sigma + t$ denote the translation of σ by t, defined by

$$(a_1 + t, a_2 + t, \ldots, a_n + t).$$

The simplest example here is

$$\nu(1+t, 2+t, \ldots, n+t) = -\frac{n}{2}t(t+1)(2t+n+1).$$

This generalizes the classical identity to which it reduces upon letting $t = 0$, the first root of the right-hand side.

Various translations of sequences will be considered in this paper. Some translation examples are from the sequences satisfying $\nu = 0$ such as

$$(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, n, n, \ldots, n),$$

where each integer 1 to n is repeated same number of times. Another example is

$$(1, 2, 2, 3, 4, 5, \ldots, m-1, m, m+2),$$

where 2 is adjoined and $m+1$ is deleted in $(1, 2, \ldots, m+2)$. Also of interest is the multiset

$$AB = \big(a_i b_j : 1 \le i \le p, 1 \le j \le q\big),$$

where $A = (a_i)_{i=1}^{p}$ and $B = (b_j)_{j=1}^{q}$ satisfy $\nu = 0$. This sequential product is called the "bag product" in [8]. These translations will be studied in Section 2 through Section 5.

One could also translate a subsequence σ^* of σ. For example, if σ is $(1, 2, 3, \ldots, 23)$ and σ^* is the subsequence of elements of σ of the form $3k+1$, then σ with σ^* translated by t is

$$\sigma(t) = (1+t, 2, 3, 4+t, 5, 6, \ldots, 20, 21, 22+t, 23)$$

and

$$\nu(\sigma(t)) = -4t(t+27)(2t-1).$$

For $t = 1/2$, we have a sequence of positive rational numbers, that includes non-integers, for which $\nu(\sigma) = 0$. We know of no previous such example. See the front part of Section 6 for another example.

In Section 7, we study the case of adjoining numbers to finite sequences of numbers so that each resulting sequence satisfies $\nu = 0$. From the study of the sequence $(n, n, \ldots, n)$, where the first n appears n times, adjoining an arithmetic progression of length n leads to an interesting sequence with Fibonacci numbers for which $\nu = 0$.

In Section 8, we begin with an example of a certain sequence σ of positive integers. This sequence possesses the property that when we add one element at a time to σ, the resulting sequence factors completely over the integers, with two of its roots being consecutive positive integers. Next, we consider the following cases where we simultaneously adjoin two elements:

$$(1, 2, 3, \ldots, m, a, b) \text{ and } (1, 2, 3, \ldots, m-1, m+1, a, b),$$

where the latter is obtained by omitting m from $1, 2, 3, \ldots, m+1$ and then adding a, b.

If $\nu(\sigma) = 0$ and we adjoin a, b, where $a + b \neq 0$, to σ, then $\nu(\sigma; a, b)$ must have the quadratic factor

$$-a + a^2 - b - ab + b^2 - 2c,$$

where c is the sum of elements of σ. We will study such quadratics and their geometry in Section 9.

2 Translations of Nicomachean identities and their repetitions

We know of no clear reference but it is not hard to show that

$$\nu(1+t, 2+t, \ldots, n+t) = -\frac{n}{2}t(2t+n+1)(t+1). \tag{2}$$

Here $t = 0$ is the original identity, $t = -(n+1)/2$ leads to an example of the second simple case in the introduction, and $t = -1$ reduces to the original identity with n replaced by $n-1$. The corresponding integral version is

$$\Big(\int_0^x (z+t)\,dz\Big)^2 - \int_0^x (z+t)^3\,dz = -\frac{1}{2}t^2x(x+2t).$$

What can be said if each integer from 1 to n is repeated $u+1$ times? Let

$$R = R(u) = R(u)[\ ]$$

be the operator that replaces any argument x by $u+1$ copies of x. For example

$$R(3)[x] = (x, x, x, x).$$

Then

$$\begin{aligned}&\nu\big(\big(R(u)[1], R(u)[2], \ldots, R(u)[n]\big) + t\big)\\ =&-\frac{n}{2}(u+1)(2t+n+1)\Big(t(t+1) - u\Big(\frac{n(n+1)}{2} + nt\Big)\Big).\end{aligned}$$

Here the case $u = 0$ reduces to the original identity (2), and the case $t = -(n+1)/2$ to the simple case (2) in the introduction as before. The effect of the vanishing of the last factor is less transparent.

The case of $n = 2(u+1)$ is of special interest. The last factor becomes

$$(t+u+1)(t-u(2u+3)).$$

The case $t = -(u+1)$ causes σ to be of the form described in case (3) in the introduction. The case $t = u(2u+3)$ is more interesting. For example, if $u = 6$, then $n = 14$ and $t = 90$. The translation of $(1, 2, 3, \ldots, 14)$ by 90 after a 7-fold repetition gives us a new sequence for which $\nu = 0$.

The search for other cases in which the last factor can be factored leads to Pell's equation. The factorization of the last factor hinges on whether the discriminant with respect to t is a square, i.e., whether

$$1 + n^2(u^2+2u) = y^2$$

has a solution in integers n, u and y. This can be written in the form

$$\frac{y^2-1}{m^2-1} = n^2, \quad m = u+1$$

or in Pell's equation form

$$y^2 = (m^2-1)n^2 + 1.$$

For example, when $m = 6$ we have $y^2 = 35n^2 + 1$. By standard methods, we quickly find a formula for infinitely many solutions here. For example,

$$846^2 = 35 \cdot 143^2 + 1.$$

Here $u = 5$ and $n = 143$, so the last factor is

$$(t - 780)(t + 66).$$

The $t = -66$ root tells us that $(-65, -64, \ldots, -1, 0, 1, \ldots, 76, 77)$ with each number repeated 6 times satisfies $\nu = 0$, while the meaning of $t = 780$ is straightforward. In this $m = 6$ case, the $y^2 = 35x^2 + 1$ solutions have

$$x = \frac{(6+\sqrt{35})^b - (6-\sqrt{35})^b}{2\sqrt{35}}$$

and $x = 143$ when $b = 3$.

In summary, there are an infinity of sequences satisfying $\nu = 0$ that are generated by solutions to an infinite number of Pell's equations. We add that the integral analogue here is

$$\begin{aligned}&\Big(\int_0^x (u+1)(z+t)\,dz\Big)^2 - (u+1)\int_0^x (z+t)^3\,dz\\ =& -\frac{x}{2}(u+1)(2t+x)\Big(t^2 - u\Big(\frac{x^2}{2} + xt\Big)\Big).\end{aligned}$$

3 Translations of other Nicomachean-type identities

There are σ with $\nu(\sigma) = 0$ that have both arbitrarily small and arbitrarily large translations to other $\nu = 0$ sequences. A family of such sequences is given by

$$\sigma = (1, 2, 2, 3, 4, 5, 7, 8, 8, 10, 10, 12, 12, \ldots, 2s, 2s),$$

where the first seven entries are as above and the remaining follow a simple pattern of twice repeated even numbers. It is not hard to show that $\nu = 0$ here. Now

$$\nu(\sigma + t) = -t(-12 + (2s^2 + 2s - 1)t + (2s+1)t^2).$$

The above quadratic in t has the root (num)/(den) where

$$\text{num} = 1 - 2s - 2s^2 + \sqrt{49 + 92s + 8s^3 + 4s^4} \text{ and } \text{den} = 2 + 4s.$$

Thus $t \to 0$ as $s \to \infty$. The product $-12/(1+2s)$ of the quadratic's roots has order $1/s$, and the other root tends to $-\infty$ as $s \to \infty$.

The sequence

$$\sigma_m = (1, 2, 2, 3, 4, 5, \ldots, m-1, m, m+2) \tag{3}$$

satisfies $\nu = 0$. Then the translation identity

$$\nu(\sigma_m + t) = t\Big(-2(2+t)^2 - \frac{1}{2}(t-1)(3+2t)m - \frac{1}{2}(t-1)m^2\Big)$$

is a polynomial that is, not surprisingly, cubic in t. Here $t = 0$ returns us to the identity $\nu(\sigma_m) = 0$. We now ask, for what values of t other than 0 do we have $\nu = 0$? The quadratic formula yields the positive root

$$t = t(m) = \frac{-16 - m - m^2 + \sqrt{m(-48 + 73m + 10m^2 + m^3)}}{4(m+2)}.$$

This is easily seen to have the limit 1 as $m \to \infty$. This suggests that for large m,

$$\nu(\sigma_m + 1) = \nu(2, 3, 3, 4, 5, \ldots, m-1, m+1)$$

would be near zero. But the above translation identity shows that

$$\nu(\sigma_m + 1) = -18 \tag{4}$$

for $m \geq 5$. This also follows from the identity

$$\Big(\Big(\frac{m(m+1)}{2} - 1\Big) + (m+2) + 3\Big)^2 - \Big(\frac{m(m+1)}{2}\Big)^2 - 1 + 3^3 + (m+2)^3 = -18.$$

This is reminiscent of the fact that for any $c \neq 0$ there are sequences of smooth functions $f_n(x)$ on $[0,1]$ such that $\lim_{n\to\infty} f_n(x)$ exists, $\int_0^1 f_n(x)dx = 0$ for every integer $n \geq 1$, yet $\int_0^1 \lim_{n\to\infty} f_n(x)dx = c$.

4 Multiset notation and Liouville's identity

The earliest non-trivial generalization of the Nicomachean identity is Liouville's

$$\Big(\sum_{d|n} \tau(d)\Big)^2 = \sum_{d|n} \tau(d)^3, \tag{5}$$

where $\tau(d)$ is the number of positive divisors of d. For n a prime power, this is simply the original identity. The basic principle here is that if two sequences $A = (a_i)_{i=1}^{p}$ and $B = (b_j)_{j=1}^{q}$ satisfy $\nu = 0$, then so does the multiset

$$AB = \big(a_i b_j : 1 \le i \le p, 1 \le j \le q\big).$$

For example, $(1,2)(2,4) = (2,4,4,8)$. We write the multiset as A^n if it is the product of n copies of the sequence A. Then we observe that

$$\nu(A^n) = \Big(\sum_{i=1}^{p} a_i\Big)^{2n} - \Big(\sum_{i=1}^{p} {a_i}^3\Big)^n = c\,\nu(A) \tag{6}$$

for some constant c. For example

$$\nu(A^2) = \Big(\Big(\sum_{i=1}^{p} a_i\Big)^2 + \sum_{i=1}^{p} {a_i}^3\Big)\nu(A).$$

We shall examine translations of such products.

Let $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ be the standard prime factorization of n. Then the positive divisors of n are of the form $p_1^{a_1}\cdots p_r^{a_r}$, where $0 \le a_i \le \alpha_i$ for $1 \le i \le r$, and so (5) implies that

$$\Big(\sum_{\substack{0\le a_i\le \alpha_i\\ 1\le i\le r}} (a_1+1)(a_2+1)\cdots(a_r+1)\Big)^2 = \sum_{\substack{0\le a_i\le \alpha_i\\ 1\le i\le r}} \Big((a_1+1)(a_2+1)\cdots(a_r+1)\Big)^3.$$

Add t to each entry of above and consider the squared sum minus the cube sum. Then a straightforward computation yields that

$$\begin{aligned}&\Big(\sum_{\substack{0\le a_i\le \alpha_i\\ 1\le i\le r}} \Big((a_1+1)(a_2+1)\cdots(a_r+1)+t\Big)\Big)^2 - \sum_{\substack{0\le a_i\le \alpha_i\\ 1\le i\le r}} \Big((a_1+1)(a_2+1)\cdots(a_r+1)+t\Big)^3\\ &= -\,t\prod_{i=1}^{r}(\alpha_i+1) f_r(t),\end{aligned}$$

where

$$\begin{aligned} f_r(t) = t^2 - \Big(\prod_{i=1}^{r}(\alpha_i+1) - 3\prod_{i=1}^{r}\Big(\frac{\alpha_i}{2}+1\Big)\Big)t -\\ 2\prod_{i=1}^{r}(\alpha_i+1)\Big(\frac{\alpha_i}{2}+1\Big) + 3\prod_{i=1}^{r}\frac{1}{6}(\alpha_i+2)(2\alpha_i+3).\end{aligned} \tag{7}$$

We shall consider the roots of the quadratic polynomial $f_r(t)$. This takes us beyond the set of integers. It is easily checked that the discriminant of $f_r(t)$ is positive, and we now create the sequence $t(r,m)$, where $t(r,m)$ is the right-most root of $f_r(t)$. Observe that for a sequence

of numbers generated by successive values of a quadratic polynomial, the second difference is constant. The sequence $t(r,m)$ need not be a sequence of values of some polynomial, but it has the property that the second difference converges. The proof of following proposition can be done by computer algebra.

Proposition 1. *For $r=2$ and $\alpha_1=\alpha_2=m-1$, the sequence of second differences of*

$$t(2,m)=\frac{1}{24}\Big(3(-3-6m+m^2)+\sqrt{3(-1+m)^2(11+34m+35m^2)}\Big)$$

converges to

$$\frac{3+\sqrt{105}}{12} \tag{8}$$

and the smaller root converges to the algebraic conjugate of the above as $m\to\infty$. Hence for $A=(1,2,3,\ldots,m)$, the sequence of second differences of the right-most roots of $\nu\big(A^2+t\big)$ converges to (8).

Remark 2. If $\alpha_1=\alpha_2=\ldots=\alpha_r=k$ for $r\ge 1$, then the rth difference of the sequence of the right-most roots seems to converge to a constant as $k\to\infty$. Those constants are

$$-\frac{1}{2},\ \ \frac{1}{12}\big(3+\sqrt{105}\big),\ \ \frac{1}{8}\big(15+\sqrt{545}\big),\ \ \frac{1}{12}\big(117+\sqrt{20985}\big),$$
$$\frac{5}{24}\big(261+\sqrt{84761}\big),\ \ \frac{5}{24}\big(1647+\sqrt{3036705}\big)$$

for $r=1,2,\ldots,6$, respectively.

5 The AB case, B an A-truncation

Consider the case $r=2$ and $\alpha_1=m-1$, $\alpha_2=r-1$ with $r<n$. Then $f_r(t)$ in (7) equals

$$f_r(t)=\frac{1}{12}\Big((m+1)(r+1)(2mr-2n-2r-1)+3(mr-3m-3r-3)t-12t^2\Big),$$

i.e., $\nu(AB+t)=-mrtf_r(t)$, where $A=(1,2,3,\ldots,m)$ and $B=(1,2,3,\ldots,r)$. As before, create the sequence of the right-most roots of the above quadratic. Here the first difference of the sequence converges to

$$\frac{1}{24}\Big(3r+\sqrt{3}\sqrt{(5r+1)(7r-5)}-9\Big) \tag{9}$$

as $m\to\infty$. For examples, if $r=2,3,4$, (9) becomes

$$\frac{1}{8}\big(\sqrt{33}-1\big)=0.593070\cdots,\ \ \frac{2}{\sqrt{3}}=1.15470\cdots,\ \ \frac{1}{8}\big(1+\sqrt{161}\big)=1.71107\cdots.$$

These have minimal polynomials

$$-2+x+4x^2,\ \ -4+3x^2,\ \ -10-x+4x^2,$$

respectively. We now rescale the minimal polynomials of the various limiting values of the first differences. For example, replace $-4+3x^2$ and $-10-x+4x^2$ by

$$-8+6x^2 \quad \text{and} \quad -20-2x+8x^2.$$

By doing this to get successive even coefficients for the x^2 term, we find a general formula:

$$-\frac{m(m+1)(m+2)}{3}-\frac{1}{2}(m-2)(m+1)x+2(m+1)x^2. \tag{10}$$

This (10) has rather nice discriminants:

$$\Delta_x=\frac{1}{12}(m+1)^2(6+5m)(2+7m),$$
$$\Delta_m=\frac{1}{6^4}(2+9x+12x^2)^2(16+72x+105x^2).$$

Here the first difference of the sequence of the right-most roots of (10) converges to

$$0.551956\cdots=\frac{3+\sqrt{105}}{24}$$

which is same as (8) divided by 2. This seems to not be a coincidence.

6 Fractions, large denominators, and gaps

By translation of subsequences, we can find additional solutions to $\nu(\sigma)=0$ with the elements of σ being positive rational numbers not all integers. Here are two examples involving arithmetic progressions modulo 7. For σ^* a subsequence of σ, let $\sigma\setminus\sigma^*$ be the sequence obtained from σ by removing all elements of σ^*. Write

$$(\sigma;\sigma^*\to\sigma^*+t)=(\sigma\setminus\sigma^*)\cup(\sigma^*+t).$$

If $\sigma=(1,2,\ldots,67)$ and $\sigma^*=(3,10,17,\ldots,66)$, we have

$$\nu(\sigma;\sigma^*\to\sigma^*+t)=-5t(t+91)(2t+5)$$

and $t=-5/2$ gives an example. For $\sigma=(1,2,\ldots,19)$ and $\sigma^*=(6,13)$, we have a shorter example. Here

$$\nu(\sigma;\sigma^*\to\sigma^*+t)=-t(t+29)(2t-5)$$

and $t=5/2$ gives

$$\tau=\left(1,2,3,4,5,7,8,\frac{17}{2},9,10,11,12,14,15,\frac{31}{2},16,17,18,19\right) \tag{11}$$

with $\nu(\tau)=0$.

Proposition 3. *There exist a sequence σ with positive rational elements and non-integers p/q, with $(p,q)=1$ and q arbitrarily large, such that $\nu(\sigma)=0$.*

Proof. Iterate $w \to \tau w$ with initial condition $w=\tau$ in (11). Then the nth iteration has a positive rational element $p/2^{n+1}$ where p is odd, e.g., $p=17^{n+1}$. Next

$$\nu(\tau^{n+1}) = 195^2 \nu(\tau^n),$$

where 195 is the sum of all elements of the sequence τ. In fact, the sum of all elements of τ^{n+1} is the sum of all elements of τ^n multiplied by 195 and the sum of cubes of all elements of τ^{n+1} is the sum of cubes of all elements of τ^n by $38025 = 195^2$, where 38025 is the sum of cubes of all elements of τ. Since $\nu(\tau)=0$, the result follows. □

How large is the gap of the elements of the sequence with positive integer entries such that $\nu=0$? By iterating $w \to (1,2)w$ with initial condition $w=(1,2)$, we produce sequences having the largest gaps that are halves of their lengths. The proof of Proposition 6 will demonstrate this. There is a sequence that has the largest gap exceeding half of its length, namely

$$(6,6,7,7,8,8,8,9,9,10,10,17).$$

We examine three types of sequences of interest because of the nature of their gaps:

(1) for any $\epsilon > 0$, the sequences where the ratio of the largest gap to the length is greater than $1-\epsilon$,

(2) sequences notable for large gaps, and

(3) sequences having the property that the gap between the two largest integers can be arbitrarily large.

We first show that sequences satisfying (1) exist.

Proposition 4. *For any $\epsilon > 0$, there are sequences having only two distinct elements such that*

$$\frac{\text{the largest gap}}{\text{length}} > 1-\epsilon.$$

Proof. Consider the sequence $(n,n,n,\ldots,n)$ with length n. Also consider

$$(n^2, n^2, \ldots, n^2)$$

with length n^2-2. Joining these sequences results in a sequence with length n^2+n-2, and the largest gap in this sequence is n^2-n. Therefore, the ratio of the largest gap to the length is given by

$$\frac{n^2-n}{n^2+n-2}$$

which tends to 1 as $n \to \infty$. However $\nu=0$ for this sequence. In fact, the sum of the elements in the sequence is given by

$$n^2 + (n^2-2)n^2 = n^4 - n^2,$$

while the sum of the cubes of the elements is $n^4 + (n^2-2)n^6 = n^8 - 2n^6 + n^4$. Since the latter is the square of the former, the result follows. □

We provide an example related to the sequences mentioned in case (2). Consider the sequence:

$$(1, 2, 3, \ldots, 94, 94+1, m, m, \ldots, m),$$

where m is repeated $2 \cdot 94$ times, and we choose $m = 228$. This sequence satisfies $\nu = 0$, and the largest gap in this sequence is $228 - 95 = 133$. The general pattern for such sequences is given by:

$$(1, 2, 3, \ldots, j, j+1, m, m, \ldots, m), \tag{12}$$

where positive integer m is repeated $2j$ times, for appropriate j and m. This implies that

$$m = j + \sqrt{2j^2 + 3j + 2},$$

and it is related to the Pell-type equation

$$2x^2 + 3x + 2 = r^2.$$

It is interesting that a positive integer r satisfying the above equation is connected to the coefficients of certain Chebyshev polynomials. References for this can be found in [10] and in the OEIS integer sequences A056161 and A055979. By elementary methods, we obtain (proof omitted) the following.

Proposition 5. *If the sequence* (12) *satisfies* $\nu = 0$*, then the* j*'s are generated by*

$$\frac{x^4 + x^3 - 19x^2 - 5x - 2}{(x-1)(1-6x+x^2)(1+6x+x^2)},$$

and the corresponding m*'s are generated by*

$$\frac{6(1+x)^2}{(1-x)(1-6x+x^2)(1+6x+x^2)}.$$

Finally, we give a constructive proof about the existence of the third type of sequences.

Proposition 6. *There are sequences with positive integer entries in which the gap between the largest two integers is arbitrarily large and* $\nu = 0$*.*

Proof. Iterate $w \to (1,2)w$ with initial condition $w = (1,2)$. Then we see that $(1,2)^{n+1}$ is the join of $(1,2)^n$ and $2(1,2)^n$. So the sequence $(1,2)^{n+1}$ ends in 2^n and 2^{n+1} in increasing order and has length 2^{n+1} while $2^{n+1} - 2^n = 2^n$. Now $\nu(1,2) = 0$ and

$$\nu\big((1,2)^{n+1}\big) = 3^2 \nu\big((1,2)^n\big)$$

since the sum of all elements of $(1,2)^{n+1}$ is the sum of all elements of $(1,2)^n$ multiplied by 3 and the sum of cubes of all elements of $(1,2)^{n+1}$ is the sum of cubes of all elements of $(1,2)^n$ by $3^2 (= 1^3 + 2^3)$. Since $\nu\big((1,2)^n\big) = 0$, the result follows. □

Additionally, we introduce a sequence containing some interesting irrational numbers with $\nu = 0$.

Proposition 7. *There exist a sequence σ with positive algebraic integers from $\mathbb{Q}(\sqrt{5})$ such that more than half of its elements are irrational and $\nu(\sigma) = 0$. Let ϕ be the golden ratio. Then given $N > 0$, there exists such a sequence containing all of the numbers $(1/\phi)^n$ for $1 \le n \le N$.*

Proof. Take a sequence $(1, 2, 3, 4, 5, 6)$ and multiply (rather than translate by t) the subsequence $(2, 4, 6)$ by the variable x. Then for $A_x = (1, 2x, 3, 4x, 5, 6x)$,

$$\nu(A_x) = -72(x-1)(4x^2 + 2x - 1) = 0$$

when $x = (1/4)\big(-1 + \sqrt{5}\big)$, and $2x = 1/\phi$. This x gives us positive algebraic integers $2x$, $4x$, $6x$ with minimal polynomials

$$x^2 + x - 1, \;\; x^2 + 2x - 4, \;\; x^2 + 3x - 9,$$

respectively. Iterate $w \to A_x w$ with initial condition A_x, where $x = (1/4)\big(-1 + \sqrt{5}\big)$ a sufficient number of times. Then $\nu = 0$ for the resulting sequence by (6), and since the product of a non-zero rational with a positive irrational is irrational, the irrationals will ultimately dominate. Also, since A_x has both 1 and $1/\phi$, we obtain the powers of $1/\phi$. □

7 Adjoining numbers to sequences

For $\sigma = (1,\, 2,\, 3,\, \ldots\, ,\, m,\, x)$ with $m \ge 2$, we have

$$\nu(\sigma) = -(x - m - 1)x(x + m).$$

This says if we want x to preserve the Nicomachean relation, we can go forward with $x = m + 1$, go back with $x = -m$, or do nothing ($x = 0$). Going forward involves the number $m + 1$, the "positive root".

In the "positive root" direction, we let $(a_1,\, a_2,\, \ldots\, ,\, a_n)$ be a finite sequence of real numbers, and keep adjoining nonzero real numbers $x_1,\, x_2,\, \ldots$ with $x_j \ne -x_{j-1}$ for each j so that each resulting sequence satisfies $\nu = 0$. Then for $l \ge 2$, it follows from

$$\nu\big(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_l\big) = \nu\big(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_l, x_{l+1}\big) = 0$$

that

$$\begin{aligned}
&\Big(\sum_{k=1}^{n} a_k + x_1 + x_2 + \cdots + x_l\Big)^2 + x_{l+1}^3 = \sum_{k=1}^{n} a_k^3 + x_1^3 + x_2^3 + \cdots + x_l^3 + x_{l+1}^3 \\
=&\Big(\Big(\sum_{k=1}^{n} a_k + x_1 + x_2 + \cdots + x_l\Big) + x_{l+1}\Big)^2,
\end{aligned}$$

and so

$$x_{l+1}^2 = x_{l+1} + 2\Big(\sum_{k=1}^{n} a_k + x_1 + \cdots + x_l\Big). \tag{13}$$

Similarly, we get

$$x_l^2 = x_l + 2\Big(\sum_{k=1}^{n} a_k + x_1 + \cdots + x_{l-1}\Big). \tag{14}$$

Subtraction of (14) from (13) gives

$$x_{l+1}^2 - x_l^2 = x_{l+1} - x_l + 2x_l = x_{l+1} + x_l,$$

which implies that the adjoined numbers

$$x_2,\, x_3,\, x_4,\, \ldots$$

form an arithmetic progression with difference 1. Putting the above together, we have the following.

Proposition 8. *Let*

$$(a_1,\, a_2,\, \ldots\, ,\, a_n)$$

be a finite sequence of real numbers, and keep adjoining nonzero real numbers

$$x_1,\, x_2, x_3,\, \ldots$$

with $x_j \neq -x_{j-1}$ for each j so that each resulting sequence satisfies $\nu = 0$. Then the adjoined numbers

$$x_2,\, x_3,\, x_4,\, \ldots$$

form an arithmetic progression with difference 1.

As an example, consider

$$\sigma = (a_1,\, a_2,\, \ldots\, ,\, a_{m+1}) = (1,\, h,\, h,\, \ldots,\, h),$$

where the real number h is repeated m times, and suppose

$$(1,\, h,\, h,\, \ldots,\, h,\, x_1,\, x_2,\, \ldots),$$

where $x_j \neq -x_{j-1}$ for each j, and each sequence

$$(1,\, h,\, h,\, \ldots,\, h,\, x_1,\, x_2,\, \ldots,\, x_j) \;\; (j \geq 2),$$

satisfies $\nu = 0$. Then by Proposition 8, $(x_2,\, x_3,\, x_4,\, \ldots)$ is an arithmetic progression with difference 1. Then using computer algebra, we can show that the subtraction of $1/2$ from the first element of this arithmetic progression, i.e. $x_2 - 1/2$, is a root of the polynomial

$$P(h, m, x) = c_0 + c_2 x^2 + c_4 x^4 + x^6,$$

where

$$c_0 = \frac{1}{64}\big(-512h^3m^3 - 1728h^2m^2 + 8h(64h^2 - 243)m - 225\big),$$
$$c_2 = \frac{1}{16}\big(192h^2m^2 + 432hm + 259\big),$$
$$c_4 = \frac{1}{4}(-24hm - 35).$$

Moreover

$$P(h, m, x) - P(m, h, x) = 8hm(h - m)(h + m).$$

In another direction, we now obtain a sequence with Fibonacci numbers satisfying $\nu = 0$. Below F_n represents the nth Fibonacci number, where $F_1 = 1$ and $F_2 = 1$. Consider the set

$$S = (F_5 + 1, F_5 + 3, \ldots, F_5 + 15) = (6, 8, 10, 12, 14, 16, 18, 20)$$

which is an arithmetic progression with difference 2 and length $8 = F_6$. Then the join of the set

$$A = (F_6, F_6, \ldots, F_6) = (8, 8, \ldots, 8)$$

with F_6 copies of F_6 and the set S, i.e.,

$$(\underbrace{8, 8, \ldots, 8}, \underbrace{6, 8, 10, 12, 14, 16, 18, 20}) \tag{15}$$

has $\nu = 0$. Moreover, the join of (15) and the set $S + F_6$, i.e.,

$$(\underbrace{8, 8, \ldots, 8}, \underbrace{6, 8, 10, 12, 14, 16, 18, 20}, \underbrace{14, 16, 18, 20, 22, 24, 26, 28})$$

also has $\nu = 0$. We generalize this below. The above will correspond to the case when $n = 3$ in Example (10).

The sequence

$$(n, n, \ldots, n) \tag{16}$$

with n copies of n has $\nu = 0$. It is natural to ask if one can extend the sequence (16) by an arithmetic progression of length n to obtain a sequence with $\nu = 0$. Let

$$k^{(m)} = (k, k, k, \ldots, k),$$

where k is repeated m times, and

$$S = (a, a + s, a + 2s, \ldots, a + (n - 1)s),$$

an arithmetic progression of length n with difference s. We denote the join of sequences A and S by $(A; S)$, and consider

$$S_m = (n^{(n)};\ S;\ S + n;\ S + 2n;\ \ldots\ ;\ S + mn) \quad (m \geq 0).$$

Then a direct computation yields

$$\nu(S_m) = \Big(\frac{1}{2}(1+m)n(2a+mn+(-1+n)s)\Big)\alpha(n,s,a), \tag{17}$$

where

$$\alpha(n,s,a) = 2n^2 + na - a^2 + \frac{1}{2}n(n-1)(s-s^2) - (n-1)as.$$

If the first factor on the right side of (17) is set equal to zero, then

$$a = \frac{1}{2}(-mn + s - ns),$$

and for example, the sequence $S_0 = (n^{(n)};\ S)$ becomes the join of $n^{(n)}$ and

$$\Big(-\frac{1}{2}(n-1)s, -\frac{1}{2}(n-3)s, -\frac{1}{2}(n-5)s, \ldots, \frac{1}{2}(n-5)s, \frac{1}{2}(n-3)s, \frac{1}{2}(n-1)s\Big)$$

whose sum and the sum of cubes are both zero, and so only trivialities emerge. Thus we would like to know what values of a are roots of the second factor $\alpha(n,s,a)$. In other words, we want to know for what n is the second factor divisible by $a-h$, where h is a positive integer.

The second factor $\alpha(n,s,a)$ has the discriminant

$$\mathrm{Disc}_a = (9-s^2)n^2 + s^2.$$

This is negative for $n \geq 2$ and $s \geq 4$. For $s = 3$, the second factor $\alpha(n,s,a)$ becomes

$$-(-3+a+n)(a+n).$$

Hence only $s \leq 2$ is of interest.

Proposition 9. *For $s = 1$, the second factor $\alpha(n,s,a)$ on the right side of (17) is divisible by $a-h$, where h is a positive integer, if and only if n^2 is a triangular number. For $s = 2$, the second factor $\alpha(n,s,a)$ is divisible by $a-h$ where h is a positive integer if and only if $n = F_{2k}$, where k is a positive integer. Moreover, in this case*

$$a = F_{2k-1} + 1.$$

Proof. For $s = 1$, the second factor $\alpha(n,s,a)$ is

$$2n^2 + a - a^2 \tag{18}$$

and its discriminant is

$$\mathrm{Disc}_a = 1 + 8n^2.$$

This is a perfect square exactly when n^2 is a triangular number, and in this case (18) $= 0$ implies that

$$a = \frac{1}{2}(1 + \sqrt{1 + 8n^2})$$

is a positive integer. For $s = 2$, the second factor $\alpha(n, s, a)$ is

$$G = n^2 + (1 - a)n - a^2 + 2a$$

and

$$\mathrm{Disc}_a G = 4 + 5n^2.$$

This is a perfect square exactly when $n = F_{2t}$, where $t \geq 1$. In this case,

$$\begin{aligned} a =& \frac{1}{2}\big(2 - n + \sqrt{5n^2 + 4}\big) = \frac{1}{2}\Big(2 - F_{2t} + \sqrt{5F_{2t}^2 + 4}\Big) \\ =& F_{2t-1} + 1. \end{aligned}$$

□

From above proposition, we obtain interesting sequences with Fibonacci numbers satisfying $\nu = 0$.

Example 10. Let

$$S = (F_{2n-1} + 1, F_{2n-1} + 3, F_{2n-1} + 5, \ldots, F_{2n+2} - 1).$$

Then S is an arithmetic progression with difference 2 and length F_{2n}. Let $A = F_{2n}^{(F_{2n})}$. Then the following sequences have $\nu = 0$:

$$(A;\ S;\ S + F_{2n};\ S + 2F_{2n};\ \ldots\ ;\ S + mF_{2n}) \quad (m \geq 0).$$

8 Adjoining (a, b) to sequences

There is a simple type of sequence of positive integers with the following property: if a single variable is adjoined and the ν operator applies, the resulting cubic factors completely over the integers, and two of the roots are consecutive positive integers.

Example 11. An example is

$$(1, 2, 2, 3, 4, 4, 5, 6, 8, 10, 12, x),$$

where ν of this equals $-(x - 6)(x - 7)(x + 12)$. Another example is the join, say A_n, of

$$(1, 2, 3, \ldots, 2n + 1), \quad (2, 4, 6, \ldots, 2n), \quad (2(n + 1), 2(n + 2), \ldots, 2(n + (n + 2))).$$

Here

$$\nu(A_n; x) = -(2 + 2n - x)(3 + 2n - x)(4 + 4n + x).$$

In another direction, one could search for σ with $\nu(\sigma) = 0$ for many σ of the form

$$(1, 2, 3, \ldots, m, a, b)$$

and look for patterns, especially for m belonging to particular arithmetic progressions. An infinite number of such identities with m of the form $7n - 12$ is given by the identity (1) of the introduction. It has the remarkable feature that if the $3n - 3$ is replaced by $5n - 8$ on each side in (1), the equality remains valid. There is also a "nearby" identity, namely

$$\begin{aligned}&\Big(\frac{(7n-10)(7n-9)}{2} + (8n-10) + (3n-4)\Big)^2\\ =&\Big(\frac{(7n-10)(7n-9)}{2}\Big)^2 + (8n-10)^3 + (3n-4)^3.\end{aligned}$$

Here, if the $3n-4$ terms are replaced by $5n-5$, the equality remains true. Another identity of this type is

$$\begin{aligned}&\Big(\frac{(13n+35)(13n+36)}{2} + (15n+42) + (7n+21)\Big)^2\\ =&\Big(\frac{(13n+35)(13n+36)}{2}\Big)^2 + (15n+42)^3 + (7n+21)^3,\end{aligned}$$

and this remains true with $(7n + 21)$ replaced by $(8n + 22)$. Yet another is

$$\begin{aligned}&\Big(\frac{(13n+42)(13n+43)}{2} + (15n+50) + (7n+3)\Big)^2\\ =&\Big(\frac{(13n+42)(13n+43)}{2}\Big)^2 + (15n+50)^3 + (7n+3)^3,\end{aligned}$$

which remains true if $7n + 3$ is replaced by $(8n + 28)$.

There is another observation to make here. In the identity (1), replace every $7n$ by the variable expression tn. Then the difference of the two sides is

$$n(t-7)(11n-15)(n(t+7)-23).$$

In fact, each side satisfies, as a function of t, the functional equation

$$f(7) = f\Big(\frac{23}{n} - 7\Big).$$

Similar remarks to apply to the other identities.

For an example in which the two sides have the same general form as those displayed here, but whose difference is a non-zero constant, see (4).

Next, here is an "adjoin (a, b) procedure" that starts with a σ such that $\nu(\sigma)$ is not zero. It leads to a new infinite class of sequences satisfying Nicomachean-type identity. Omit m from $(1, 2, 3, \ldots, n)$ and then adjoin (a, b). For the case of $m = 1$, there are two non-integer

rational numbers a and b extending $(2, 3, 4, \ldots, n)$ that satisfy Nicomachean-type identity. To show this, we let

$$p(n) = (1+n)(1+2n+5n^2+n^3), \quad q(n) = (1+2n)(1+n+n^2).$$

Note that

$$p(n) - p(-(n+1)) = (1+2n)^3.$$

Define a sequence $R(n)$ of $n+2$ elements by

$$R(n) = \Big(2, 3, 4, \ldots, n, \frac{-p(-(1+n))}{q(n)}, \frac{p(n)}{q(n)}\Big).$$

Then $R = R(n)$ satisfies Nicomachean-type identity. Note that the sum of the additional two elements is

$$\frac{-p(-(1+n))}{q(n)} + \frac{p(n)}{q(n)} = \frac{(1+2n)^2}{1+n+n^2}.$$

For $m \geq 2$, we have $(1, 2, \ldots, m-1, m+1, m+2, \ldots, n, a, b)$. Take

$$(2n+1)(1-3m+3m^2+n+n^2)$$

for the denominator of both a and b. Then use $(n-(m-1))A$ as the numerator of a and $(n+m)B$ of b, where

$$A = 3m^3 + (-1+3m+3m^2)n + (3m-1)n^2 - n^3$$

and

$$B = 1 - 3m^2 + 3m^3 + (2+3m-3m^2)n + (3m+2)n^2 + n^3.$$

Then adjoin (a, b) to $(1, 2, 3, \ldots, n)$, where the $m(\geq 2)$ has been removed. This sequence satisfies the Nicomachean-type identity. A useful check here is that

$$a + b = \frac{m(2n+1)^2}{3m^2 - 3m + 1 + n + n^2}.$$

9 The geometry of adjoining (a, b) to sequences

It seems peculiar that definite and indefinite quadratic forms play a significant role in a study of a cubic nature. But when $\nu(\sigma) = 0$ and (a, b) is adjoined to σ, the $\nu(\sigma; a, b)$ must have a factor $a + b$. In fact,

$$\begin{aligned} \nu(\sigma; a, b) &= \nu(\sigma) + a^2 + b^2 + 2c(a+b) + 2ab - (a^3 + b^3) \\ &= -(a+b)(-a + a^2 - b - ab + b^2 - 2c), \end{aligned}$$

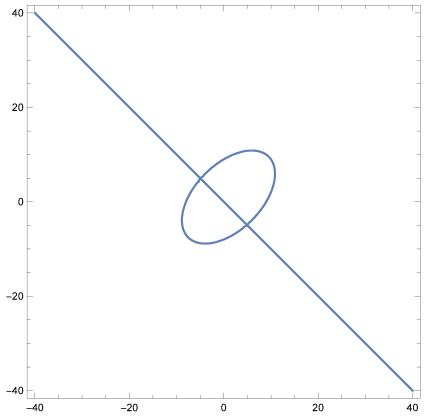

Figure 1: $\nu((1,2,\ldots,n,a,b)) = 0$ $(n = 8)$.

where c is the sum of the elements of σ. Upon discarding the $a+b$ factor, we are left with quadratic polynomials. The ellipse formed by setting any of these quadratic polynomials equal to zero has eccentricity $\sqrt{2/3}$.

These quadratic polynomials depend on c. For example, the two sequences

$$\sigma_1 = (1,2,3,3,3,5,5,6,7,9,10,12), \quad \sigma_2 = (1,2,3,\ldots,10,11)$$

satisfy $\nu = 0$ and $c = 66$ is the sum of elements of both sequences, and the corresponding "adjoined" quadratic polynomial is $a - a^2 + b + ab - b^2 + 132$. The three sequential products arising from σ_1 and σ_2 are

$$\sigma_1\sigma_1, \ \ \sigma_1\sigma_2, \ \text{ and } \ \sigma_2\sigma_2.$$

These induce the same "adjoined" quadratic polynomial with $c = 66^2$. Here the "three" is from $\binom{2+2-1}{2} = 3$, the number of combinations with repetition. Continuing to triple products, we have $\binom{2+3-1}{3} = 4$

$$(\sigma_1\sigma_1)\sigma_1, \ \ (\sigma_1\sigma_1)\sigma_2, \ \ (\sigma_1\sigma_2)\sigma_2, \ \ (\sigma_2\sigma_2)\sigma_2,$$

that induce the same quadratic polynomial with $c = 66^3$. In general, we have $\binom{2+N-1}{N} = N+1$ sequential products with the same quadratic polynomial with $c = 66^N$, and we obtain the following.

Proposition 12. *For any positive integer N, there are at least N distinct sequences σ with $\nu(\sigma) = 0$ such that each $\nu(\sigma; a, b)$ has the same quadratic polynomial factor.*

If we retain the $a+b$ factor and plot $\nu(\sigma; a, b) = 0$ in the (a, b) plane, we have a line cutting through an ellipse. Since a line is a degree 1 curve and an ellipse is a degree 2 curve, we have $(1 + 2 = 3)$ a degenerate cubic. For example, Figure 1 shows the locations of (a, b), where $\nu(1, 2, \ldots, 8, a, b) = 0$.

Now

$$w_n = (1,2,2,3,4,5,\ldots,n), \ \ n \geq 3,$$

has

$$\nu(w_n) = 2(-2 + n + n^2) \neq 0,$$

and the locus of $\nu(w_n; a, b) = 0$ is such a typical cubic curve. Here we have

$$\begin{aligned}\nu(w_n; a, b) &= \frac{1}{4}(-32 - n^2 - 2n^3 - n^4) - a^3 - b^3 + \Big(\frac{1}{2}(4 + n + n^2) + a + b\Big)^2 \\ &= 2(-1 + n)(2 + n) + (4 + n + n^2)a + a^2 - a^3 + (4 + n + n^2)b + 2ab + b^2 - b^3.\end{aligned}$$

A natural way to search for a and b with $\nu(w_n; a, b) = 0$ is to put the cubic into Weierstrass normal form. Consider

$$h_n(u, y) = -3y^2 + \frac{1}{16}(2u + n + 2)P_n(u),$$

where

$$P_n(u) = 3(n + 2)^3 + 2(n^2 + 20)u - 4(3n + 2)u^2 - 8u^3$$

is a cubic polynomial in u. Then $h_n(u, y) = 0$ is essentially in Weierstrass normal form. With

$$u = \frac{a + b}{2} - \frac{n + 2}{2}, \quad y = \frac{a^2 - b^2}{4},$$

we may compute that $h_n(u, y)$ becomes

$$\frac{1}{4}(a + b)\nu(w_n; a, b).$$

In the above particular example with $n = 8$, the curve $\nu(w_8; a, b) = 0$ is given by

$$140 + 76a + a^2 - a^3 + 76b + b^2 - b^3 + 2ab = 0.$$

As shown in Figure 2, the locus of this consists of a smooth finite oval disjoint from a smooth infinite curve that is a configuration reminiscent of a textbook plot of certain typical real loci of cubic (elliptic) curves. Moreover, as n increases, the corresponding sequence of cubic curves displays a nested sequence of smooth ovals. For this, see Figure 3. For the case $n = 8$ again, a numerical search gives the points $(0, 10)$, $(10, 0)$, $(9, 11)$ and $(11, 9)$ on the infinite curve and many integer points with coordinates at most 0 on the finite oval. For example, $(0, -2)$, $(0 - 7)$, $(-2, -8)$ and $(-8, -2)$ are there. For $(a, b) = (11, 9)$, we have

$$\tau = (1, 2, 2, 3, 4, \ldots, 9, 11),$$

an example of one of the σ_m sequences studied in Section 3.

Note that for all n, we have $\nu(w_n; -1, -1) = -6$ and every finite oval is tangent to the line

$$a + b = -2$$

at $(0, -2)$ and $(-2, 0)$. The infinite curve is asymptotic to the line

$$a + b = 0.$$

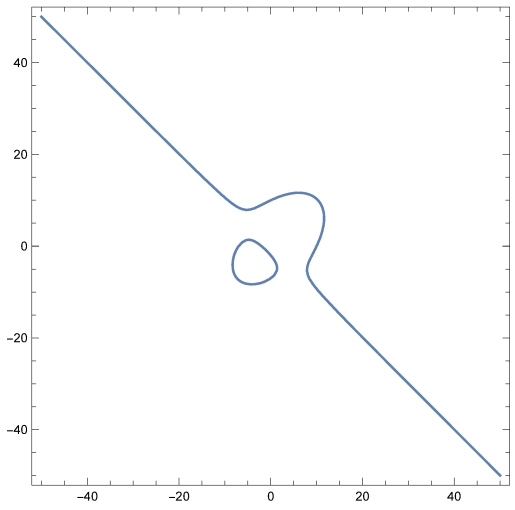

Figure 2: $\nu(w_8; a, b) = 0$.

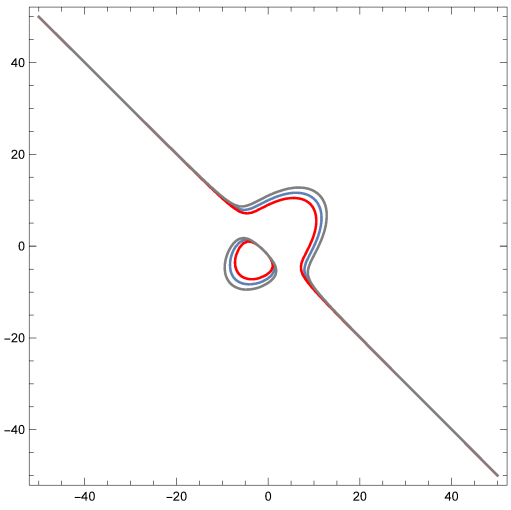

Figure 3: $\nu(w_n; a, b) = 0$ $(n = 7, 8, 9)$.

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