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On the Finiteness of Bernoulli Polynomials Whose Derivative Has Only Integral Coefficients

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Abstract

It is well known that the Bernoulli polynomials $\mathbf{B}_n(x)$ have nonintegral coefficients for $n \geq 1$. However, ten cases are known so far in which the derivative $\mathbf{B}'_n(x)$ has only integral coefficients. One may assume that the number of those derivatives is finite. We can link this conjecture to a recent conjecture about the properties of a product of primes satisfying certain p-adic conditions. Using a related result of Bordellès, Luca, Moree, and Shparlinski, we then show that the number of those derivatives is indeed finite. Furthermore, we derive other characterizations of the primary conjecture. Subsequently, we extend the results to higher derivatives of the Bernoulli polynomials. This provides a product formula for these denominators, and we show similar finiteness results.

1 Introduction

The Bernoulli polynomials $\mathbf{B}_n(x)$ are defined by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \mathbf{B}_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi) \tag{1}$$

and explicitly given by the formula

$$\mathbf{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathbf{B}_{n-k}\, x^k \quad (n \geq 0), \tag{2}$$

where $\mathbf{B}_n = \mathbf{B}_n(0) \in \mathbb{Q}$ is the nth Bernoulli number. It easily follows from (1) that $\mathbf{B}_n = 0$ for odd $n \geq 3$. For more properties see Cohen [4, Chapter 9]. The Bernoulli polynomials $\mathbf{B}_n(x) \in \mathbb{Q}[x]$ are Appell polynomials [1]. Therefore, they satisfy the rule

$$\mathbf{B}_n'(x) = n\mathbf{B}_{n-1}(x) \quad (n \geq 1). \tag{3}$$

While $\mathbf{B}_n(x) \notin \mathbb{Z}[x]$ for $n \geq 1$, which is equivalent to $\operatorname{denom}(\mathbf{B}_n(x)) > 1$ for $n \geq 1$ (the denominators are discussed in the next section), it turns out that

$$\mathbf{B}_n'(x) \in \mathbb{Z}[x] \quad \text{for } n \in \mathcal{S} := \{1, 2, 4, 6, 10, 12, 28, 30, 36, 60\}.$$

The elements of $\mathcal{S}$, viewed as an ordered sequence, equal the finite sequence A094960 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [11] as published in 2004. So far, no further terms have been found. It is mainly assumed that sequence A094960 is indeed finite and completely determined by $\mathcal{S}$. Note that we implicitly omit the trivial case for $n = 0$, since $\mathbf{B}_0(x) = \mathbf{B}_0 = 1$. Define

$$\overline{\mathcal{S}} := \{n \geq 1 : \mathbf{B}_n'(x) \in \mathbb{Z}[x]\}.$$

For our purposes, we split the conjecture into two parts as follows.

Conjecture 1. We have the following statements.

(i) The set $\overline{\mathcal{S}}$ is finite.

(ii) We have $\overline{\mathcal{S}} = \mathcal{S}$.

We link the above conjecture to a more recent conjecture of the author [5] in a p-adic context, where p always denotes a prime. The function $s_p(n)$ gives the sum of the base-p digits of an integer $n \geq 0$. Let $\omega(n)$ be the additive function that counts the distinct prime divisors of n. As usual, an empty product is defined to be 1, and an empty sum is defined to be 0. We consider the product

$$\mathbb{D}_n^+ := \prod_{\substack{p > \sqrt{n} \\ s_p(n) \geq p}} p \quad (n \geq 1), \tag{4}$$

where p runs over the primes. Note that the above product is always finite, since $s_p(n) = n$ for $p > n$. By Kellner [5, Theorem 4], we have a further relation that

$$\omega(\mathbb{D}_n^+) = \sum_{\substack{p > \sqrt{n} \\ \lfloor \frac{n-1}{p-1} \rfloor > \lfloor \frac{n}{p} \rfloor}} 1 < \sqrt{n} \quad (n \geq 1). \tag{5}$$

We shall clarify the notation of $\mathbb{D}_n^+$ in a more general setting in the next section.

Conjecture 2 (Kellner [5, Conjectures 1, 2])**.** We have the following statements.

(i) We have $\mathbb{D}_n^+ > 1$, respectively, $\omega(\mathbb{D}_n^+) > 0$ for $n > 192$.

(ii) There exists a constant $\kappa > 1$ such that

$$\omega(\mathbb{D}_n^+) \sim \kappa \frac{\sqrt{n}}{\log n} \quad \text{as } n \to \infty. \tag{6}$$

At first glance, Conjectures 1 and 2 seem to be incompatible. However, we can establish the following connection.

Theorem 3. *Conjecture 2(i) and (ii) imply Conjecture 1(ii) and (i), respectively.*

Meanwhile, Conjecture 2(ii) with $\kappa = 2$ was proven by Bordellès et al. [2] for sufficiently large $n > n_0$. This result was achieved by exploiting (5), since the condition $s_p(n) \geq p$ as in (4) is replaced by $\lfloor \frac{n-1}{p-1} \rfloor > \lfloor \frac{n}{p} \rfloor$ in (5), which enabled them to use powerful analytic tools. Unfortunately, their methods do not lead to an explicit or computable bound n_0. Using their results, we arrive at the following corollary.

Corollary 4 (Bordellès, Luca, Moree, and Shparlinski [2, Corollary 1.6])**.** *Conjecture 2(ii) is true, so Conjecture 1(i) is true.*

Theorem 5. *We have the following statements.*

(i) *If $n \in \overline{\mathcal{S}}$, then $n+1$ is prime.*

(ii) *If $\overline{\mathcal{S}} \neq \mathcal{S}$ and $n \in \overline{\mathcal{S}} \setminus \mathcal{S}$, then $n > 10^7$.*

As a consequence, the set

$$\overline{\mathcal{S}} + 1 = \{2, 3, 5, 7, 11, 13, 29, 31, 37, 61, \ldots\}$$

contains only primes. It would be very unlikely that $\mathbb{D}_n^+ = 1$ happens for $n > 192$. See the graph [5, Figure B1] of $\omega(\mathbb{D}_n^+)$ in the range below 10^7 and consider the coincident and proven asymptotic formula of $\omega(\mathbb{D}_n^+)$ for sufficiently large n. However, it is still an open task to establish Conjectures 1(ii) and 2(i).

The paper is organized as follows. In the next section, we give a survey about p-adic properties of the denominators of the Bernoulli polynomials. We also show further characterizations of Conjecture 1. In Section 3, we extend the results to higher derivatives of the Bernoulli polynomials. Section 4 contains the proofs of the theorems.

2 Denominators and p-adic properties

To study the denominators of the Bernoulli polynomials, it is convenient to consider for $n \geq 1$ the related denominators

$$\begin{aligned}
\mathbf{D}_n &:= \operatorname{denom}(\mathbf{B}_n) = 2, 6, 1, 30, 1, 42, 1, 30, 1, 66, \ldots, \\
\mathbb{D}_n &:= \operatorname{denom}(\mathbf{B}_n(x) - \mathbf{B}_n) = 1, 1, 2, 1, 6, 2, 6, 3, 10, 2, \ldots, \\
\mathfrak{D}_n &:= \operatorname{denom}(\mathbf{B}_n(x)) = 2, 6, 2, 30, 6, 42, 6, 30, 10, 66, \ldots,
\end{aligned}$$

which are all squarefree. These are the sequences A027642, A195441, and A144845, respectively. Obviously, we have by definition the relation

$$\mathfrak{D}_n = \operatorname{lcm}(\mathbb{D}_n, \mathbf{D}_n). \tag{7}$$

The denominators $\mathbf{D}_n$ of the Bernoulli numbers are given by the well-known von Staudt–Clausen theorem of 1840 (Clausen [3] and von Staudt [12]), which states for even positive integers n that

$$\mathbf{B}_n + \sum_{p-1 \,|\, n} \frac{1}{p} \in \mathbb{Z}, \quad \text{which implies that} \quad \mathbf{D}_n = \prod_{p-1 \,|\, n} p.$$

However, the denominators $\mathbf{D}_n$ do not play a role here, since we follow the approaches of Kellner [5] and Kellner and Sondow [8, 9, 10], which are concerned with the p-adic properties of the denominators $\mathbb{D}_n$. For $n \geq 1$, these denominators are given by the remarkable formula

$$\mathbb{D}_n = \prod_{s_p(n) \,\geq\, p} p, \tag{8}$$

which arises from the p-adic product formula; see Kellner [5, Section 5]. The decomposition

$$\mathbb{D}_n = \mathbb{D}_n^- \cdot \mathbb{D}_n^+, \tag{9}$$

where $\mathbb{D}_n^+$ is defined as in (4) and

$$\mathbb{D}_n^- := \prod_{\substack{p < \sqrt{n} \\ s_p(n) \,\geq\, p}} p,$$

leads to Conjecture 2. Note that the decomposition (9) omits the possible term for $p = \sqrt{n}$, but then we would have $p^2 = n$ and so $s_p(n) = 1$.

For computational purposes, those products, which run over the primes p and contain the condition $s_p(n) \geq p$, are trivially bounded by $p < n$. Moreover, the following bounds [5, Lemmas 1, 2] are self-induced by properties of $s_p(n)$. Namely, we have for $n \geq 1$ that

$$s_p(n) < p, \quad \text{if } p > \frac{n+1}{\lambda} \text{ where } \lambda = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

For the sake of completeness, we show that the polynomials $\mathbf{B}_n(x) - \mathbf{B}_n$, which have no constant term, arise in a natural context. For $n \geq 0$, define the sum-of-powers function

$$S_n(m) := \sum_{\nu=0}^{m-1} \nu^n \quad (m \geq 0).$$

It is well known that

$$S_n(x) = \int_0^x \mathbf{B}_n(t)\,dt = \frac{1}{n+1}(\mathbf{B}_{n+1}(x) - \mathbf{B}_{n+1}). \tag{10}$$

As a result of Kellner [5, Theorem 5] and Kellner and Sondow [8, Theorems 1, 2], we then have for $n \geq 0$ that

$$\mathcal{D}_n := \operatorname{denom}(S_n(x)) = (n+1)\,\mathbb{D}_{n+1} = 1, 2, 6, 4, 30, 12, 42, 24, 90, 20, \ldots, \tag{11}$$

which is sequence A064538.

Remark 6. Since the Bernoulli polynomials $\mathbf{B}_n(x)$ are Appell polynomials satisfying the reflection relation $\mathbf{B}_n(1-x) = (-1)^n\,\mathbf{B}_n(x)$, the integral in (10) can be reinterpreted by Faulhaber-type polynomials that are connected with certain reciprocal Bernoulli polynomials, as recently shown by the author; see [6, Example 5.6] and [7, Section 11].

Let $n \geq 1$, and let $\operatorname{rad}(n)$ be the squarefree kernel of n. As introduced in [10], define the decompositions

$$\mathbb{D}_n = \mathbb{D}_n^{\top} \cdot \mathbb{D}_n^{\perp} \quad \text{and} \quad \operatorname{rad}(n) = \mathbb{D}_n^{\top} \cdot \mathbb{D}_n^{\top\star}, \tag{12}$$

where

$$\mathbb{D}_n^{\top} := \prod_{\substack{p \,|\, n \\ s_p(n) \geq p}} p, \quad \mathbb{D}_n^{\perp} := \prod_{\substack{p \,\nmid\, n \\ s_p(n) \geq p}} p, \quad \text{and} \quad \mathbb{D}_n^{\top\star} := \prod_{\substack{p \,|\, n \\ s_p(n) < p}} p. \tag{13}$$

The sequences of $\mathbb{D}_n^{\top}$, $\mathbb{D}_n^{\perp}$, and $\mathbb{D}_n^{\top\star}$ are sequences A324369, A324370, and A324371, respectively. We arrive at the following theorem.

Theorem 7 (Kellner and Sondow [10, Theorem 3.1])**.** *For $n \geq 1$, the denominator $\mathfrak{D}_n$ of the Bernoulli polynomial $\mathbf{B}_n(x)$ splits into the triple product*

$$\mathfrak{D}_n = \mathbb{D}_{n+1}^{\perp} \cdot \mathbb{D}_{n+1}^{\top} \cdot \mathbb{D}_{n+1}^{\top\star}.$$

Consequently, the interplay of the factors of $\mathfrak{D}_n$ yields the relations

$$\mathfrak{D}_n = \mathbb{D}_{n+1}^{\perp} \cdot \operatorname{rad}(n+1) = \mathbb{D}_{n+1} \cdot \mathbb{D}_{n+1}^{\top\star} = \operatorname{lcm}(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)). \tag{14}$$

Compared with the classical relation (7), one may observe that the right-hand sides of the above equations involve the numbers and indices $n+1$ in place of n. To simplify notation, we include the case $\mathfrak{D}_0 = 1$, which coincides with (14). Apart from that, we explicitly avoid the case $n = 0$ of the related symbols of $\mathbb{D}_n$ in view of their product identities (12) and (13).

Corollary 8. *Let $n \geq 1$. The following statements hold.*

(i) *We have that $\mathfrak{D}_n = \operatorname{rad}(\mathcal{D}_n)$.*
(ii) *We have that $\mathfrak{D}_n$ is even, which implies that $\mathbf{B}_n(x) \notin \mathbb{Z}[x]$.*
(iii) *We have that $\mathbb{D}_n^{\perp}$ is even, if $n \geq 3$ is odd; otherwise, $\mathbb{D}_n^{\perp}$ is even.*

A different proof of part (ii) via (7) is given by Kellner and Sondow [8, Theorem 4].

Proof. Let $n \geq 1$. We show three parts. (i). From (11) and (14), we derive that $\mathfrak{D}_n = \operatorname{lcm}(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)) = \operatorname{rad}(\mathbb{D}_{n+1}\,(n+1)) = \operatorname{rad}(\mathcal{D}_n)$. (iii). We have $\mathbb{D}_1^{\perp} = 1$. If $2 \mid n$, then $2 \nmid \mathbb{D}_n^{\perp}$. Otherwise, for odd $n \geq 3$, it follows that $2 \nmid n$ and $s_2(n) \geq 2$. By (13), this shows that $2 \mid \mathbb{D}_n^{\perp}$. (ii). Considering (14), the factor $\operatorname{rad}(n+1)$ is even for odd n, whereas $2 \mid \mathbb{D}_{n+1}^{\perp}$ when n is even using part (iii). Both cases show that $\mathfrak{D}_n$ is even for $n \geq 1$. This completes the proof. ☐

The properties of $\mathbb{D}_n$ and $\mathbb{D}_n^{\perp}$ lead to the following characterizations, which are connected with Conjecture 1. For this purpose, we define the sets

$$\overline{\mathcal{R}} := \{n \geq 1 : \mathbb{D}_n = \operatorname{rad}(n+1)\} \quad \text{and} \quad \mathcal{R} := \{3, 5, 8, 9, 11, 27, 29, 35, 59\}.$$

Theorem 9. *Let $n \geq 1$. We have that $\mathbf{B}'_n(x) \in \mathbb{Z}[x]$ if and only if $\mathbb{D}_n^{\perp} = 1$, or equivalently, $\mathfrak{D}_{n-1} = \operatorname{rad}(n)$. In these cases, the number $n+1$ is prime.*

Theorem 10. *If $n \in \overline{\mathcal{R}}$, then $n+1$ is composite. In particular, if n is odd, then $\mathbb{D}_{n+1}^{\perp} = 1$. Otherwise, we have that $n = 2^e$ for some $e \geq 1$. Moreover, the set $\overline{\mathcal{R}}$ is finite.*

Conjecture 11. We have $\overline{\mathcal{R}} = \mathcal{R}$.

Theorem 12. *Conjecture 11, reduced to odd numbers $n \in \mathcal{R}$, implies Conjecture 1.*

3 Denominators and higher derivatives

In this section, we extend the results to higher derivatives of $\mathbf{B}_n(x)$. Let $(n)_k$ denote the falling factorial such that $\binom{n}{k} = (n)_k/k!$. We define the related denominators by

$$\mathfrak{D}_n^{(k)} := \operatorname{denom}(\mathbf{B}_n^{(k)}(x)) \quad (k, n \geq 1).$$

Theorem 13. *Let $k, n \geq 1$. Then we have*

$$\mathfrak{D}_n^{(k)} = \frac{\mathfrak{D}_{n-k}}{\gcd(\mathfrak{D}_{n-k}, (n)_k)} = \frac{\mathbb{D}_{n-k+1}^{\perp}}{\gcd(\mathbb{D}_{n-k+1}^{\perp}, (n)_{k-1})} = \prod_{\substack{p \nmid (n)_k \\ s_p(n-k+1) \geq p}} p \quad (n \geq k). \tag{15}$$

Otherwise, we have $\mathfrak{D}_n^{(k)} = 1$. Moreover, the denominators $\mathfrak{D}_n^{(k)}$ have the property that $p \nmid \mathfrak{D}_n^{(k)}$ for all primes $p \leq k$. In particular, we have

$$\mathfrak{D}_n^{(1)} = \frac{\mathfrak{D}_{n-1}}{\operatorname{rad}(n)} = \mathbb{D}_n^{\perp} \quad (n \geq 1).$$

Define the related sets

$$\overline{\mathcal{S}}_k := \{n \geq 1 : \mathbf{B}_n^{(k)}(x) \in \mathbb{Z}[x]\} \quad (k \geq 1),$$

where $\overline{\mathcal{S}}_1 = \overline{\mathcal{S}}$. Let $\mathcal{S}_k$ denote the computable subsets of $\overline{\mathcal{S}}_k$ with $\mathcal{S}_1 = \mathcal{S}$.

Theorem 14. *We have that all sets $\overline{\mathcal{S}}_k$ are finite for $k \geq 1$. Moreover, we have that*

$$\overline{\mathcal{S}}_1 \subset \overline{\mathcal{S}}_2 \subset \overline{\mathcal{S}}_3 \subset \cdots .$$

Recall that

$$\mathcal{S}_1 = \{1, 2, 4, 6, 10, 12, 28, 30, 36, 60\}.$$

We use the notation, e.g., $\{a - b\} = \{a, \ldots, b\}$ for ranges. We compute the sets

$$\begin{aligned}
\mathcal{S}_2 &= \{1-7, 9-13, 15, 16, 21, 25, 28-31, 36, 37, 55, 57, 60, 61, 70, 121, 190\},\\
\mathcal{S}_3 &= \{1-18, 20-22, 25, 26, 28-32, 35-38, 42, 50, 52, 55-58, 60-62,\\
&\qquad 66, 70-72, 78, 80, 92, 110, 121, 122, 156, 176, 177, 190, 191, 210, 392\}.
\end{aligned}$$

Conjecture 15. We have $\overline{\mathcal{S}}_2 = \mathcal{S}_2$ and $\overline{\mathcal{S}}_3 = \mathcal{S}_3$.

4 Proofs of the theorems

We give the proofs of the theorems in the order of their dependencies. First, we need some key lemmas.

Lemma 16. *Let $n \geq 1$. We have the following properties.*

(i) *We have that $\mathbb{D}_n$ is odd if and only if $n = 2^e$ for some $e \geq 0$.*
(ii) *If $n+1$ is composite, then $\mathrm{rad}(n+1) \mid \mathbb{D}_n$ and $\mathrm{rad}(n+1) \mid \mathbb{D}_n^{\perp}$.*
(iii) *If $n \geq 3$ is odd, then $\mathbb{D}_n = \mathrm{lcm}(\mathbb{D}_{n+1}, \mathrm{rad}(n+1))$.*

Proof. (i). See [8, Theorem 1]. (ii). See [5, Theorem 1] and [9, Corollary 2], respectively. (iii). See [10, Theorem 3.2]. □

Lemma 17. *For $n \geq 1$, we have $\mathbb{D}_n^{+} \mid \mathbb{D}_n^{\perp}$.*

Proof. Let $n \geq 1$, and assume that $\mathbb{D}_n^{+} > 1$. Otherwise, we are done. If a prime $p \mid \mathbb{D}_n^{+}$, then by (4) we have $p > \sqrt{n}$ and $s_p(n) \geq p$. Thus, we have $p^2 > n$ implying the p-adic expansion $n = a_0 + a_1 p$. It then follows from $a_0 + a_1 = s_p(n) \geq p$ that $a_0 \neq 0$ and $p \nmid n$. Finally, this shows by (13) that $p \mid \mathbb{D}_n^{\perp}$. □

Lemma 18. *For $k \geq 1$, there exists a number n_k such that $\mathbb{D}_n^{+} > (n+k)_k$ for every $n > n_k$.*

Proof. Let $k \geq 1$. Using the result of Bordellès et al. [2, Corollary 1.6], we have that (6) holds with $\kappa = 2$. Thus, there exists a number n_k such that $\omega(\mathbb{D}_n^+) \geq 2k$ for $n > n_k$. Now consider any set of $2k$ prime divisors $p_1 < p_2 < \cdots < p_{2k}$ of $\mathbb{D}_n^+$. Since $p \mid \mathbb{D}_n^+$ implies $p > \sqrt{n}$, we infer that $n + 1 < p_1 p_2 < p_3 p_4 < \cdots < p_{2k-1} p_{2k}$. Consequently, we get $\mathbb{D}_n^+ > (n+1) \cdots (n+k) = (n+k)_k$ for every $n > n_k$. $\square$

Proof of Theorem 9. Let $n \geq 1$. If $\mathbf{B}_n'(x) \in \mathbb{Z}[x]$, then we have by (3) that

$$\operatorname{denom}(\mathbf{B}_n'(x)) = \operatorname{denom}(n\,\mathbf{B}_{n-1}(x)) = 1.$$

As a consequence, we have that $\mathfrak{D}_{n-1} \mid n$. Applying Theorem 7 and (14), it follows that

$$\mathbb{D}_n^{\perp} \cdot \operatorname{rad}(n) \mid n. \tag{16}$$

Since $\mathbb{D}_n^{\perp}$ is coprime to n, we conclude that $\mathbb{D}_n^{\perp} = 1$. In the other direction, condition (16) is satisfied only if $\mathbb{D}_n^{\perp} = 1$. By (14), this is equivalent to $\mathfrak{D}_{n-1} = \operatorname{rad}(n)$. From Lemma 16(ii), it further follows that $n+1$ must be prime. Otherwise, we would have that $\operatorname{rad}(n+1) \mid \mathbb{D}_n^{\perp}$. This completes the proof. $\square$

Proof of Theorem 3. First, we assume Conjecture 2(i). Let $n > 192$. We then have that $\mathbb{D}_n^+ > 1$. By Lemma 17, this property transfers to $\mathbb{D}_n^{\perp} > 1$. From Theorem 9, it follows that $\mathbf{B}_n'(x) \notin \mathbb{Z}[x]$ for $n > 192$. We have to check the remaining cases $1 \leq n \leq 192$. By Theorem 9, it then suffices to check the numbers $\mathbb{D}_n^{\perp}$ when $n+1$ is prime. Finally, this confirms that $\mathbf{B}_n'(x) \in \mathbb{Z}[x]$ if and only if $n \in \mathcal{S}$, implying Conjecture 1(ii). Secondly, we now assume Conjecture 2(ii). It follows from (6) that there exists a number n_0 such that $\mathbb{D}_n^+ > 1$ for $n > n_0$. Similar to the first part above, this implies that $\mathbf{B}_n'(x) \notin \mathbb{Z}[x]$ for $n > n_0$. Hence, the set $\overline{\mathcal{S}}$ is finite, which is Conjecture 1(i). $\square$

Proof of Theorem 5. We have to show two parts. (i). By Theorem 9, we have that $n \in \overline{\mathcal{S}}$ implies that $n+1$ is prime. (ii). This follows from computations of the graph [5, Figure B1] of $\omega(\mathbb{D}_n^+)$ in the range below 10^7. $\square$

Proof of Theorem 10. Let $n \in \overline{\mathcal{R}}$. Assume that $\mathbb{D}_n = \operatorname{rad}(n+1)$, where $p = n+1$ is prime. Then $\mathbb{D}_n = p$ contradicts (8), since $s_p(n) = n < p$. Therefore, the number $n+1$ is composite. If n is even, then $\mathbb{D}_n = \operatorname{rad}(n+1)$ implies that $\mathbb{D}_n$ is odd. From Lemma 16(i), it follows that $n = 2^e$ for some $e \geq 1$. Now, we assume that $n \geq 3$ is odd, neglecting the case $\mathbb{D}_1 = 1$. Using Lemma 16(iii), we infer that

$$\mathbb{D}_n = \operatorname{lcm}(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)) = \mathbb{D}_{n+1}^{\perp} \operatorname{lcm}(\mathbb{D}_{n+1}^{\top}, \operatorname{rad}(n+1)) = \operatorname{rad}(n+1),$$

so $\mathbb{D}_{n+1}^{\perp} = 1$ as desired. It remains to show that $\overline{\mathcal{R}}$ is finite. Applying Lemma 18 with $k = 1$ shows that there exists n_1 such that $\mathbb{D}_n^+ > n+1$ for $n > n_1$. Since $\mathbb{D}_n \geq \mathbb{D}_n^+$ by (9), it follows that $\overline{\mathcal{R}}$ is finite. This completes the proof. $\square$

Proof of Theorem 12. Let $\mathcal{R}' = \{3, 5, 9, 11, 27, 29, 35, 59\}$ be the reduced set of $\mathcal{R}$ consisting only of odd numbers. From Theorems 9 and 10, we derive that $n \in \mathcal{R}'$ implies that $\mathbb{D}_{n+1}^{\perp} = 1$ and so $n + 1 \in \mathcal{S}$. Since $\mathcal{R}' + 1 = \mathcal{S} \setminus \{1, 2\}$, the result follows. $\square$

Proof of Theorem 13. First assume that $1 \le n < k$. By (2), the Bernoulli polynomial $\mathbf{B}_n(x)$ is a monic polynomial of degree n. Thus, the kth derivative of $\mathbf{B}_n(x)$ vanishes, yielding $\mathfrak{D}_n^{(k)} = 1$. If $n = k \ge 1$, then $\mathbf{B}_n^{(k)}(x) = n!$ and so $\mathfrak{D}_n^{(k)} = 1$. As $\mathfrak{D}_0 = \mathbb{D}_1^{\perp} = 1$, this case coincides with (15). Now, let $n > k \ge 1$. From (3), it follows that

$$\mathfrak{D}_n^{(k)} = \operatorname{denom}(\mathbf{B}_n^{(k)}(x)) = \operatorname{denom}((n)_k \, \mathbf{B}_{n-k}(x)) = \frac{\mathfrak{D}_{n-k}}{\gcd(\mathfrak{D}_{n-k}, (n)_k)}.$$

Using (14), we have $\mathfrak{D}_{n-k} = \mathbb{D}_{n-k+1}^{\perp} \operatorname{rad}(n-k+1)$. Recall that $\mathfrak{D}_{n-k}$ is squarefree. Together with $(n)_k = (n)_{k-1}(n-k+1)$, we infer that

$$\gcd(\mathfrak{D}_{n-k}, (n)_k) = \gcd(\mathbb{D}_{n-k+1}^{\perp}, (n)_{k-1}) \operatorname{rad}(n-k+1).$$

As a consequence, we obtain

$$\mathfrak{D}_n^{(k)} = \frac{\mathbb{D}_{n-k+1}^{\perp}}{\gcd(\mathbb{D}_{n-k+1}^{\perp}, (n)_{k-1})}. \tag{17}$$

Note that $p \mid (n)_{k-1}$ implies $p \nmid \mathfrak{D}_n^{(k)}$. By (13), we have

$$\mathbb{D}_{n-k+1}^{\perp} = \prod_{\substack{p \nmid n-k+1 \\ s_p(n-k+1) \ge p}} p. \tag{18}$$

Putting (17) and (18) together, we get

$$\mathfrak{D}_n^{(k)} = \prod_{\substack{p \nmid (n)_k \\ s_p(n-k+1) \ge p}} p.$$

From $k! \mid (n)_k$ and $p \nmid (n)_k$, we derive that $p \nmid \mathfrak{D}_n^{(k)}$ for $p \le k$. At the end, we consider the case $k = 1$ for $n \ge 1$. Recall that $\mathfrak{D}_1^{(1)} = \mathbb{D}_1^{\perp} = 1$. For $n > 1$, we have $\mathfrak{D}_n^{(1)} = \mathbb{D}_n^{\perp}$ by (17). From (14) and $\mathfrak{D}_0 = 1$, it finally follows that $\mathfrak{D}_n^{(1)} = \mathfrak{D}_{n-1} / \operatorname{rad}(n) = \mathbb{D}_n^{\perp}$ for all $n \ge 1$. $\square$

Proof of Theorem 14. Let $k \ge 1$. Combining Lemmas 17 and 18, we see that there exists a number n_k such that

$$\mathbb{D}_n^{\perp} \ge \mathbb{D}_n^{+} > (n+k)_k \quad (n > n_k).$$

By Theorem 13 and shifting the index n to $n + k - 1$ in (15), we obtain

$$\mathfrak{D}_{n+k-1}^{(k)} = \frac{\mathbb{D}_n^{\perp}}{\gcd(\mathbb{D}_n^{\perp}, (n+k-1)_{k-1})} \quad (n \ge 1).$$

Since $\gcd(\mathbb{D}_n^\perp, (n+k-1)_{k-1}) \leq (n+k-1)_{k-1} < (n+k)_k$, we then deduce that

$$\mathfrak{D}_{n+k-1}^{(k)} > \frac{\mathbb{D}_n^\perp}{(n+k)_k} > 1 \quad (n > n_k),$$

showing that $\overline{\mathcal{S}}_k$ is finite. As $\mathbf{B}_n^{(k)}(x) \in \mathbb{Z}[x]$ also implies that $\mathbf{B}_n^{(k+1)}(x) \in \mathbb{Z}[x]$, we infer that $\overline{\mathcal{S}}_k \subset \overline{\mathcal{S}}_{k+1}$. Hence, this yields $\overline{\mathcal{S}}_1 \subset \overline{\mathcal{S}}_2 \subset \overline{\mathcal{S}}_3 \subset \cdots$, completing the proof. $\square$

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